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Spectral model of homogeneous isotropic turbulence of a dusty flow

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Using spectral analysis of balance equations for turbulent energy and turbulent dissipation of particle-containing gas, we have derived a closed two-parameter model for two-phase flow turbulence. A dissipativity criterion for validating the model is proposed. Calculations of the carrier flow turbulence parameters in the presence of particles with varying inertia are presented.

Keywords: homogeneous isotropic turbulence, dispersed admixture, turbulent energy and turbulent dissipation spectra, two-parameter turbulence model.

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Microscale structure of gas-dispersed turbulent flows plays a decisive role in a number of up-to-date technical applications (e.g., mixing reagents in chemical engineering facilities, electromagnetic radiation propagation in a dusty atmosphere, etc.).

There are two ways of theoretically modeling two-phase turbulent flows. The first one is direct numerical simulation (DNS). Particles are described in Lagrange variables, the carrier phase is expressed in Euler variables [1–6]. The DNS method requires significant computer resources and essential restrictions on the turbulence Reynolds number and admixture concentration. For engineering applications, important is another way based on describing both phases in Euler variables. The standard two-parameter $E-\varepsilon$-model (E is the turbulent energy, ε is the turbulent dissipation) is supplemented with terms describing the effect of dispersed admixture on the turbulence parameters (see, e.g. [1,7,8]). The particles' viscous drag force obeys the Stokes law; the dynamic relaxation time is $\tau_U = \text{const}$. To estimate the particle inertia, Stokes criterion $\text{St}^{(0)} = \tau_U/\tau_\eta^{(0)}$ is used ($\tau_\eta^{(0)}$ is the Kolmogorov time microscale for a single-phase flow). In this paper, the calculations are compared with DNS data that does not account for the effect of gravity. Some models (e.g., [1,7,8]) imply an emergence, in the degenerating two-phase turbulence, of a dummy energy source inducing enhancement of the carrier phase turbulence relative to that in a dust-free flow. This paper proposes a correct spectral-analysis-based $E-\varepsilon$-model of the particle-containing gas turbulence. Papers [1–6] demonstrate that the turbulence spectral composition depends only on the admixture mass concentration; therefore, we assume the particle concentration distribution to be constant.

Spectral decompositions of the equations for velocities of the dispersed and carrier phases, which account for the interfacial momentum exchange, are as follows:

$$\frac{d\hat{v}_i(\mathbf{k},t)}{dt} = -ik_n\int\hat{v}_i^*(\mathbf{k}',t)\hat{v}_n(\mathbf{k}+\mathbf{k}',t)d\mathbf{k}' + \frac{\hat{u}_i(\mathbf{k},t)-\hat{v}_i(\mathbf{k},t)}{\tau_U},$$

$$k_i\hat{v}_i(\mathbf{k},t) = 0, \qquad (1)$$

$$\frac{d\hat{u}_i(\mathbf{k},t)}{dt} = -ik_n\left(\delta_{i,l} - \frac{k_ik_l}{k^2}\right)\int\hat{u}_l^*(\mathbf{k}',t)\hat{u}_n(\mathbf{k}+\mathbf{k}',t)d\mathbf{k}'$$

$$-\nu_f k^2\hat{u}_i(\mathbf{k},t) - \langle G\rangle\frac{1}{\tau_U}\left[\hat{u}_i(\mathbf{k},t) - \hat{v}_i(\mathbf{k},t)\right],$$

$$k_i\hat{u}_i(\mathbf{k},t) = 0. \qquad (2)$$

Here $\hat{u}_i(\mathbf{k},t)$, $\hat{v}_i(\mathbf{k},t)$ are the spectral decompositions of random velocity, $\mathbf{k}$ and k are the wave vector and its modulus, $\delta_{i,l}$ is the Kronecker symbol, ν_f is the kinematic viscosity coefficient of the gas, asterisk designates the complex conjugation, d/dt is the substantial derivative.

Equation for the gas turbulent energy spectrum $E(t) = \int\hat{E}(\mathbf{k},t)d\mathbf{k}$ looks as follows:

$$\frac{d}{dt}\hat{E}(\mathbf{k},t) + \text{div}_\mathbf{k}\hat{\mathbf{J}}(\mathbf{k},t) = -\hat{\varepsilon}(\mathbf{k},t) - \langle G\rangle\hat{\varepsilon}_p(\mathbf{k},t). \qquad (3)$$

The second term in the eq. (3) left-hand part is the turbulent energy flux divergence across the spectrum. The first term in the eq. (3) right-hand part represents the turbulent dissipation spectrum $\varepsilon(t) = \int\hat{\varepsilon}(\mathbf{k},t)d\mathbf{k}$, $\hat{\varepsilon}(\mathbf{k},t) = 2\nu_f k^2\hat{E}(\mathbf{k},t)$. The last term in the eq. (3) right-hand part is responsible for the interfacial momentum exchange due to the particles' inertia and friction $\varepsilon_p(t) = \int\hat{\varepsilon}_p(\mathbf{k},t)d\mathbf{k}$:

$$\hat{\varepsilon}_p(\mathbf{k},t) = \frac{1}{\tau_U}\left\{2\hat{E}(\mathbf{k},t) - \frac{\langle\hat{v}(\mathbf{k},t)\hat{u}_i^*(\mathbf{k},t)\rangle + \langle\hat{u}_i(\mathbf{k},t)\hat{v}_i^*(\mathbf{k},t)\rangle}{2}\right\}. \qquad (4)$$

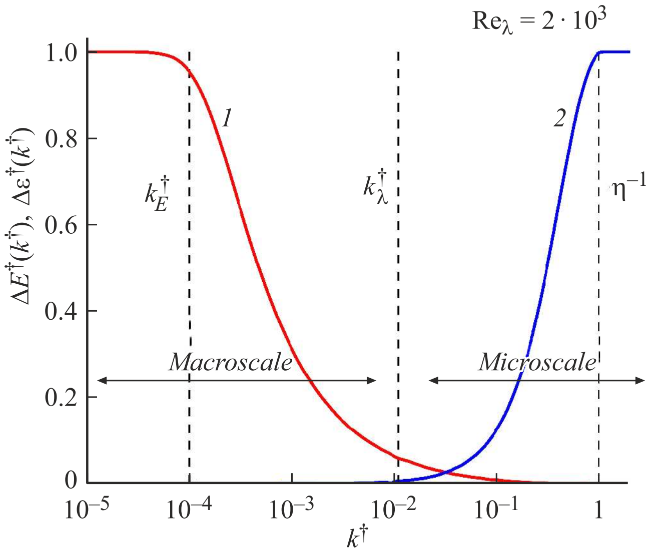

Figure 1. Relative shares of turbulent energy (1) and turbulent dissipation (2) in the spectrum. Crosses indicate the parameters normalized to the Kolmogorov microscale, $k^\dagger = k\eta$. The spectrum was calculated via the model from [9].

Now define the equation for the turbulent dissipation spectrum:

$$\frac{d\hat{\varepsilon}(\mathbf{k}, t)}{dt} = 2\nu_f k^2 \frac{d\hat{E}(\mathbf{k}, t)}{dt}. \quad (5)$$

Taking into account (3), obtain the turbulent dissipation spectrum equation in the following form:

$$\frac{d\hat{\varepsilon}(\mathbf{k},t)}{dt} + 2\nu_f k^2 \mathrm{div}_{\mathbf{k}} \hat{\mathbf{J}}(\mathbf{k}, t) = -2\nu_f k^2 \hat{\varepsilon}(\mathbf{k}, t) - \langle G \rangle 2\nu_f k^2 \hat{\varepsilon}_p(\mathbf{k}, t). \quad (6)$$

Here the second left-hand term originates from microscale velocity fluctuations. The first and second terms in the eq. (6) right-hand part are the dissipation of microscale velocity fluctuations and additional degeneration of microscale fluctuations due to the work of interfacial friction force. To close the interfacial momentum exchange, decomposition of the dispersed phase velocity (1) is used in (3) and (6). In the case of a homogeneous isotropic turbulence, integration of equations (3) and (6) over volume in the wave-vector space provides balance equations of turbulent energy and dissipation

$$\frac{dE(t)}{dt} = -2\nu_f \int_0^\infty k^2 \hat{E}^\circ(k, t) dk - \langle G \rangle \frac{2}{\tau_U} \int_0^\infty [1 - f(k, \tau_U)] \hat{E}^\circ(k, t) dk, \quad (7)$$

$$\frac{d\varepsilon(t)}{dt} = -\int_0^\infty \left\{ 2\nu_f k^2 + \langle G \rangle \frac{2}{\tau_U} \left[1 - f(k, \tau_U)\right] \right\} \hat{\varepsilon}^\circ(k, t) dk = -2\nu_f \int_0^\infty k^2 \hat{\varepsilon}^\circ_{eff}(k, t) dk. \quad (8)$$

Here $\hat{E}^\circ(k, t)$, $\hat{\varepsilon}^\circ(k, t)$ are the results of averaging three-dimensional spectra in the wave space over a sphere with radius k; $f(k, \tau_U)$ is the function of particle response to the gas velocity turbulent fluctuations. Fig. 1 shows that equation (7) describes the admixture effect in the energy-intensive spectrum range $0 < k \leqslant k_E (k_E = L_E^{-1}$, L_E is the integral spatial scale). Equation (8) simulates the dynamics of microscale turbulence $k_\lambda < k \leqslant \eta^{-1}$ $(\eta = (\nu_f^3/\varepsilon)^{1/4}$ is the Kolmogorov spatial microscale, $k_\lambda = \lambda^{-1}$, λ is the Taylor spatial microscale). These scales match three time scales: $T_E = L_E/E^{1/2}$ is the integral scale, $\tau_\lambda = \alpha k_\lambda^{-2/3} \varepsilon^{-1/3}$ is the Taylor microscale ($\alpha \sim 1$ [9]), $\tau_\eta = (\nu_f/\varepsilon)^{1/2}$ is the Kolmogorov microscale. Intensity of the dispersed-phase velocity fluctuations is determined by the response function which depends on autocorrelation function $\Psi(k, t)$ obtained in [10]:

$$f(k, \tau_U) = \frac{1}{\tau_U} \int_0^\infty \exp\left(-\frac{\xi}{\tau_U}\right) \Psi(k, \xi) d\xi, \quad (9)$$

$$\Psi(k, t) = \frac{\exp\bigl(-t/\tau(k)\bigr) - \bigl(\tau_\eta/\tau(k)\bigr) \exp\bigl(-t/\tau_\eta\bigr)}{1 - \bigl(\tau_\eta/\tau(k)\bigr)},$$
$$\tau(k) = \alpha k^{-2/3} \varepsilon^{-1/3}. \quad (10)$$

Relations (9) and (10) give the expression for response function

$$f(k, \tau_U) = 1 - \left[\bigl(1 + \tau(k)/\tau_U\bigr)\bigl(1 + \tau_\eta/\tau_U\bigr)\right]^{-1},$$
$$0 < f(k, \tau_U) < 1. \quad (11)$$

The set of equations for spectra (7), (8) may be approximated by the $E{-}\varepsilon$-model of homogeneous isotropic turbulence

$$\frac{dE}{dt} = -\varepsilon_{eff} - 2\frac{E}{T_E} \langle G \rangle \underbrace{\frac{T_E}{\tau_U} [1 - f(k_E, \tau_U)]}_{macroscale},$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2} \frac{\varepsilon_{eff}}{T_E}, \quad (12)$$

$$\varepsilon_{eff} = \left\{ 1 + \langle G \rangle \underbrace{\frac{T_E}{\tau_U} \frac{2}{C_{\varepsilon 2}} \left[1 - f(k_\lambda, \tau_U)\right]}_{microscale} \right\} \varepsilon. \quad (13)$$

Here $T_E = E/\varepsilon$ is the integral time scale, $C_{\varepsilon 2} = 1.92$ is the constant of standard $E{-}\varepsilon$-model. When $\tau_U \gg T_E$

and $\tau_U \ll \tau_\eta$, the effect of dispersed admixture disappears (see (11)). Reynolds numbers of the energy-intensive and dissipative spectrum ranges are

$$\mathrm{Re}_E = L_E E^{1/2}/\nu_f = E^2/(\nu_f \varepsilon), \qquad \mathrm{Re}_\lambda = (20\mathrm{Re}_E/3)^{1/2}.$$

Variation in the admixture-containing gas turbulent energy follows from (12) and (13)

$$\frac{\Delta \dot{E}_p}{\varepsilon} = 2\frac{T_E}{\tau_U}\left\{\frac{1}{C_{\varepsilon_2}}\left[1 - f\left(k_\lambda, \tau_U\right)\right] + \left[1 - f\left(k_E, \tau_U\right)\right]\right\}. \tag{14}$$

Correctness of model (12), (13) follows from the dissipativity criterion, that is, the gas turbulent energy under homogeneous isotropic turbulence should decrease:

$$\frac{d\mathrm{Re}_E}{dt} = \mathrm{Re}_E\left(\frac{2}{E}\frac{dE}{dt} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dt}\right) < 0. \tag{15}$$

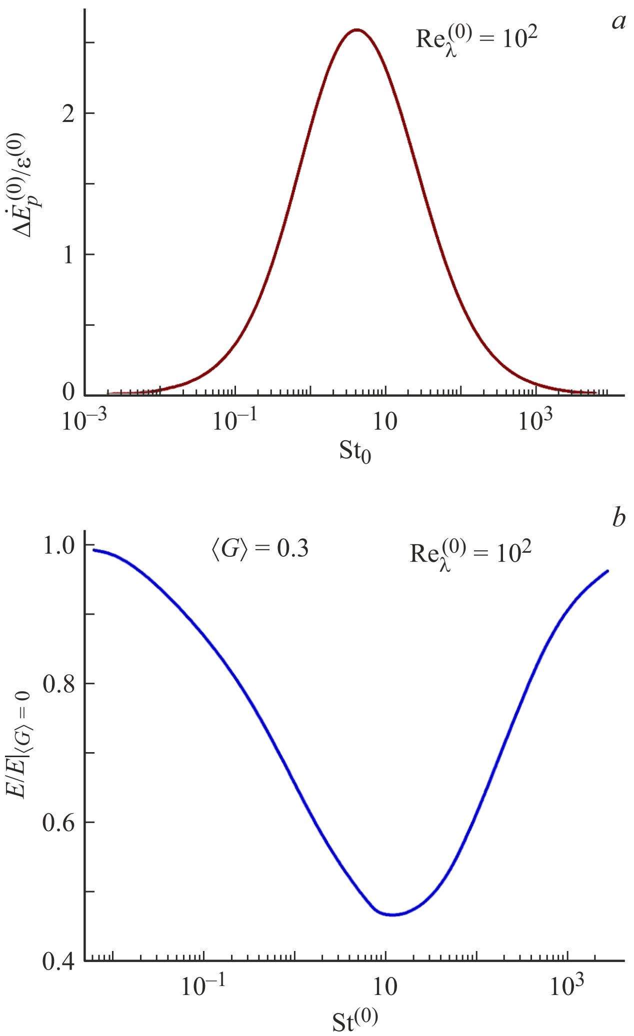

Figure 2. *a* — the share of turbulent energy spent by gas on involving the admixture in turbulent motion (calculated via (14)). *b* — the carrier-phase turbulent energy dependence on the particle inertia parameter.

For equations (12) and (13), inequality (15) takes the following form:

$$\frac{2}{E}\frac{dE}{dt} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dt} = -\frac{\varepsilon_{eff}}{E}\left(2 - C_{\varepsilon 2}\right) - \frac{4}{T_E}\langle G\rangle \frac{T_E}{\tau_U}\left[1 - f\left(k_E, \tau_U\right)\right] < 0. \tag{16}$$

For the standard $E{-}\varepsilon$-model ($(\langle G\rangle = 0)$ $\varepsilon_{eff} = \varepsilon$, inequality (15) is satisfied at $C_{\varepsilon 2} < 2$. Relation (16) shows that model (12), (13) meets criterion (15) for particles of any inertia. A number of two-phase turbulence models [1,7,8] exhibit violation of condition (15). Following [1], let us rewrite equations of the $E{-}\varepsilon$-model taking into account the dispersed phase:

$$\frac{dE}{dt} = -\varepsilon - \alpha_p \frac{E}{T_E},$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2}\frac{\varepsilon}{T_E} - C_{\varepsilon 2}\beta_p \frac{\varepsilon}{T_E}. \tag{17}$$

Here coefficients α_p, β_p represent, respectively, suppression of energy-intensive and microscale velocity fluctuations in dusty gas. Condition (15) takes the following form:

$$\frac{2}{E}\frac{dE}{dt} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dt} = -\left[\left(2 - C_{\varepsilon 2}\right) + \left(2\alpha_p - C_{\varepsilon 2}\beta_p\right)\right]\frac{\varepsilon}{E} < 0.$$

Satisfying this inequality needs $2\alpha_p - C_{\varepsilon 2}\beta_p > 0$. For low-inertia particles ($\tau_\lambda \sim \tau_U \ll T_E$) obtain $\alpha_p \to 0$, $\beta_p > 0$, and condition (15) gets violated. For inertial particles ($\tau_\lambda \ll \tau_U \sim T_E$), obtain $\alpha_p > 0$, $\beta_p \to 0$, and the dissipativity condition is met. Fig. 2, *a* illustrates the selective effect of admixture on the turbulent energy. Fig. 2, *b* shows that significant suppression of turbulent energy by the particles is observed at Stokes numbers $\mathrm{St}^{(0)} \sim 10$. Fig. 3 demonstrates the comparison of calculations obtained via our model with the DNS data for degenerating turbulence [4]. As noted in [2,5], the numerical experiment exhibited considerably slower establishment of the state of isotropy than the physical experiment. This explains the difference between calculations via the standard $E{-}\varepsilon$-model and DNS data for $\langle G\rangle = 0$ (lines *1*). Therefore, we may only speak about qualitative agreement between simulation results. Therefore, it is possible only to speak about qualitative agreement between simulation results. Next publications will present the results of studying the effect of particles on the spectral composition of non-isothermal gas turbulence and their comparison with experiment.

Conflict of interests

The authors declare that they have no conflict of interests.

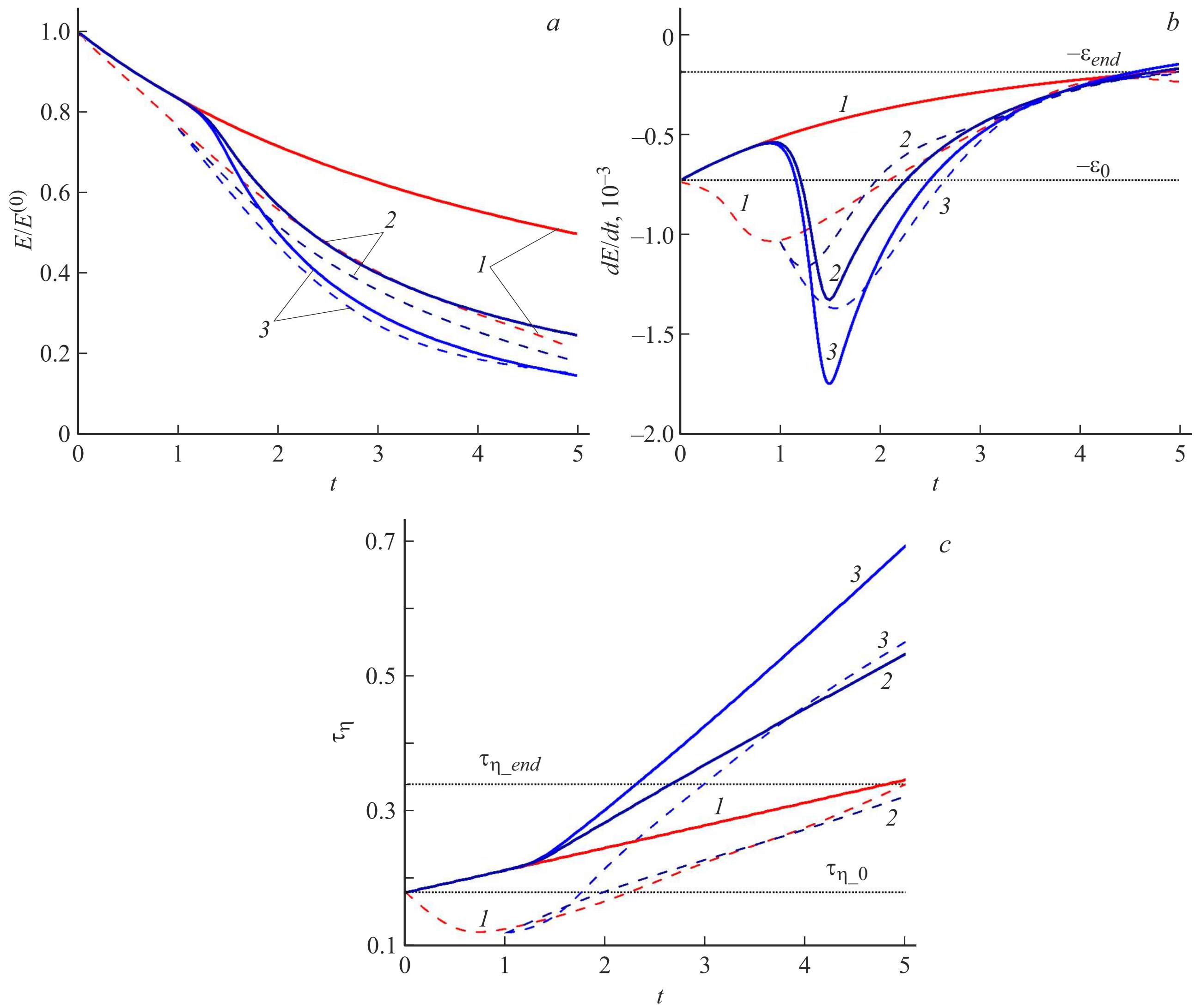

Figure 3. Comparison of calculations via model (12), (13) (solid lines) of turbulent energy (a), turbulent energy variation rate (b), and Kolmogorov time microscale (c) with the DNS data (dashed lines) [4]. Dotted lines indicate the initial and final values of parameters taken from [4]. Curves *1* were obtained at $\langle G\rangle = 0$, curves *2*, *3* at $\langle G\rangle = 1$: $\mathrm{St}^{(0)} = 1$ (*2*) and 5 (*3*).

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