01

Exact solution of the contact boundary value problem for a waveguide structure made of nonlinear and inhomogeneous optical media

© S.E. Savotchenko

Sergo Ordzhonikidze Russian State University for Geological Prospecting, Moscow, Russia
Moscow Technical University of Communications and Informatics, Moscow, Russia
MIREA — Russian Technological University, Moscow, Russia
E-mail: savotchenkose@mail.ru

Received August 18, 2025
Revised August 18, 2025
Accepted November 6, 2025

A model of a waveguide structure made of a gradient medium with a refraction index decreasing in accordance with a generalized hyperbolic profile, which is in contact with a nonlinear medium with its permittivity varying abruptly as a function of intensity of the electric field, is proposed. The corresponding conjugation boundary value problem is formulated, and its exact solution characterizing a nonlinear surface wave is obtained. It is demonstrated that the wave characteristics may be controlled by varying the optical characteristics of the medium.

Keywords: inhomogeneous optical media, nonlinear media, surface waves, nonlinear waves, optical waveguides.

DOI: 10.21883/0000000000

The identification of new patterns of propagation of surface waves in inhomogeneous optical media [1] (including nonlinear ones [2]) based on mathematical modeling is an important fundamental and applied task [3]. The possibility of finding exact solutions in an explicit analytical form to conjugation boundary value problems within the existing models of waveguide structures is especially important [4]. This possibility rests on the choice of a profile of spatial dependence of the refraction index and a shape of nonlinear response of the medium that allow one to find exact solutions of the stationary wave equation (Helmholtz equation) [5].

In the present study, we consider a specific dependence of permittivity that decreases with distance from the surface of an optically inhomogeneous medium (gradient medium [6]). A new exact solution of the Helmholtz equation with variable coefficients is found for this dependence. A given refraction index profile, which determines the permittivity of non-magnetic materials used widely in photonics [7] and bioengineering [8], may be obtained by injecting ions of specially selected impurities [7] or by laser lithography [9]. The obtained exact solution is used to find a solution to the conjugation boundary value problem in analytical modeling of the propagation of surface waves along a flat surface of a gradient medium in contact with a nonlinear optical medium. A simple model of a nonlinear medium, which was proposed in [10] and used for analytical description of nonlinear surface waves [11] and nonlinear waveguides [12], is proposed as a form of nonlinear response for finding the exact solution.

Let us set axis Ox to be perpendicular to flat interface of media $x = 0$ and axis Oy. Half-space $x > 0$ is occupied by a medium with an inhomogeneous refraction index, and half-space $x < 0$ is occupied by a nonlinear medium. Only transverse waves with electric field strength vector $\mathbf{E} = (0, E_y, 0)$ are considered; $E_y(x, z) = u(x)e^{i(knz-\omega t)}$, $k = 2\pi/\lambda$ is the wave number, λ is the wavelength, $n = ck/\omega$ is the effective refraction index, ω is the frequency, and c is the speed of light in vacuum. Transverse profile of electric field strength $u(x)$ is determined from the scalar equation of waveguide theory [5]:

$$u''(x) + \left\{\varepsilon(x, |u|) - n^2\right\}k^2u(x) = 0, \tag{1}$$

where $\varepsilon(x, |u|)$ is the permittivity of the entire medium, which may be presented as

$$\varepsilon(x, |u|) = \begin{cases} \varepsilon_L(x), & x > 0, \\ \varepsilon_N(|u|), & x < 0. \end{cases} \tag{2}$$

The inhomogeneity of profile (2) is modeled by a dependence on the coordinate perpendicular to the contact surface of the media:

$$\varepsilon_L(x) = e_0 + \frac{e_1}{x + h} + \frac{e_2}{(x + h)^2}, \tag{3}$$

where $e_{0,1,2}$ and h are the profile parameters. This profile shape is a generalization of profiles considered in [13] (at $e_0 = e_2 = 0$) and in [14] (at $e_0 = e_1 = 0$). The nonlinear response of the medium in (2) is characterized by a step nonlinearity model

$$\varepsilon_N(|u|) = \begin{cases} \varepsilon_1, & |u| < u_s, \\ \varepsilon_2, & |u| > u_s, \end{cases} \tag{4}$$

where u_s is the threshold field strength at which the permittivity switches abruptly from one constant value ε_1 to another ε_2 [11].

Combining (1)−(4), one may obtain equations

$$u_L''(x)+\left(e_0+\frac{e_1}{x+h}+\frac{e_2}{(x+h)^2}-n^2\right)k^2u_L(x)=0,\ x>0, \quad (5)$$

$$u_{2N}''(x)+(\varepsilon_2-n^2)k^2u_{2N}(x)=0,\ |u|>u_s,\ -x_s<x<0, \quad (6)$$

$$u_{1N}''(x)+(\varepsilon_1-n^2)k^2u_{1N}(x)=0,\ |u|<u_s,\ x<-x_s, \quad (7)$$

where $u_L(x)$, $u_{2N}(|u|)$, and $u_{1N}(|u|)$ are the transverse profiles of electric field strength in the corresponding regions of gradient and nonlinear media; x_s is the boundary separating the region with $|u|>u_s$ and $\varepsilon=\varepsilon_2$ from the region with $|u|<u_s$ and $\varepsilon=\varepsilon_1$ (i.e., the field at this boundary matches the threshold field: $u=u_s$). It is assumed in the step nonlinearity model that a near-surface layer with its optical properties differing from those of the rest of the medium forms near the interface. This layer has a variable thickness x_s that depends on the optical characteristics of the media and the field amplitude.

Equations (5)−(7) are supplemented by a natural set of boundary conjugation conditions (following from the requirement of continuity of field components)

$$u_L(+0)=u_{2N}(-0),\quad u_L'(+0)=u_{2N}'(-0), \quad (8)$$

$$u_{2N}(-x_s+0)=u_{1N}(-x_s-0)=u_s,$$
$$u_{2N}'(-x_s+0)=u_{1N}'(-x_s-0), \quad (9)$$

and conditions at infinity: $u_L(x)\to 0$, $u_{1N}(x)\to 0$ at $|x|\to\infty$.

Thus, the mathematical formulation of the model of a waveguide structure made of nonlinear and inhomogeneous contacting optical media is reduced to conjugation boundary value problem (5)−(9).

The solution of the Helmholtz equation with variable coefficients (5), which is bounded at the origin and at infinity, may be written as

$$u_L(x)=u_{Lm}W_{\mu,\nu}\big(2n_0k(x+h)\big), \quad (10)$$

where $W_{\mu,\nu}$ is the Whittaker function, $n_0=\sqrt{n^2-e_0}$, $\mu=e_1k/2n_0$, $\nu=\sqrt{1-4k^2e_2}/2$, and constant u_{Lm} is determined from the boundary conditions. Solution (10) exists at $n^2>e_0$ and $k^2>1/4e_2$.

The general solution of Eq. (6) at $n^2<\varepsilon_2$ may be written as

$$u_{2N}(x)=u_{2m}\cos\big(p_2(x-x_m)\big), \quad (11)$$

where $p_2^2=(\varepsilon_2-n^2)k^2$ and constants u_{2m} and x_m are determined from the boundary conditions.

A solution of Eq. (7) bounded at infinity exists at $n^2>\varepsilon_1$, and it may be written as

$$u_{1N}(x)=u_{1m}e^{q_1x}, \quad (12)$$

where $q_1^2=(n^2-\varepsilon_1)k^2$ and constant u_{1m} is determined from the boundary conditions.

Inserting solutions (10)−(12) into boundary conditions (8) and (9), one may obtain an exact solution to conjugation boundary value problem (5)−(9) in an explicit form

$$u(x)=u_s\begin{cases}\left(\frac{p_2^2+q_1^2}{p_2^2+q_G^2}\right)^{1/2}\frac{W_{\mu,\nu}\left(2n_0k(x+h)\right)}{W_{\mu,\nu}(2n_0kh)},\ x>0,\\ \left(1+\frac{q_G^2}{p_2^2}\right)^{1/2}\cos\big(p_2(x-x_m)\big),-x_s<x<0,\\ e^{q_1(x+x_s)},\ x<-x_s,\end{cases} \quad (13)$$

where

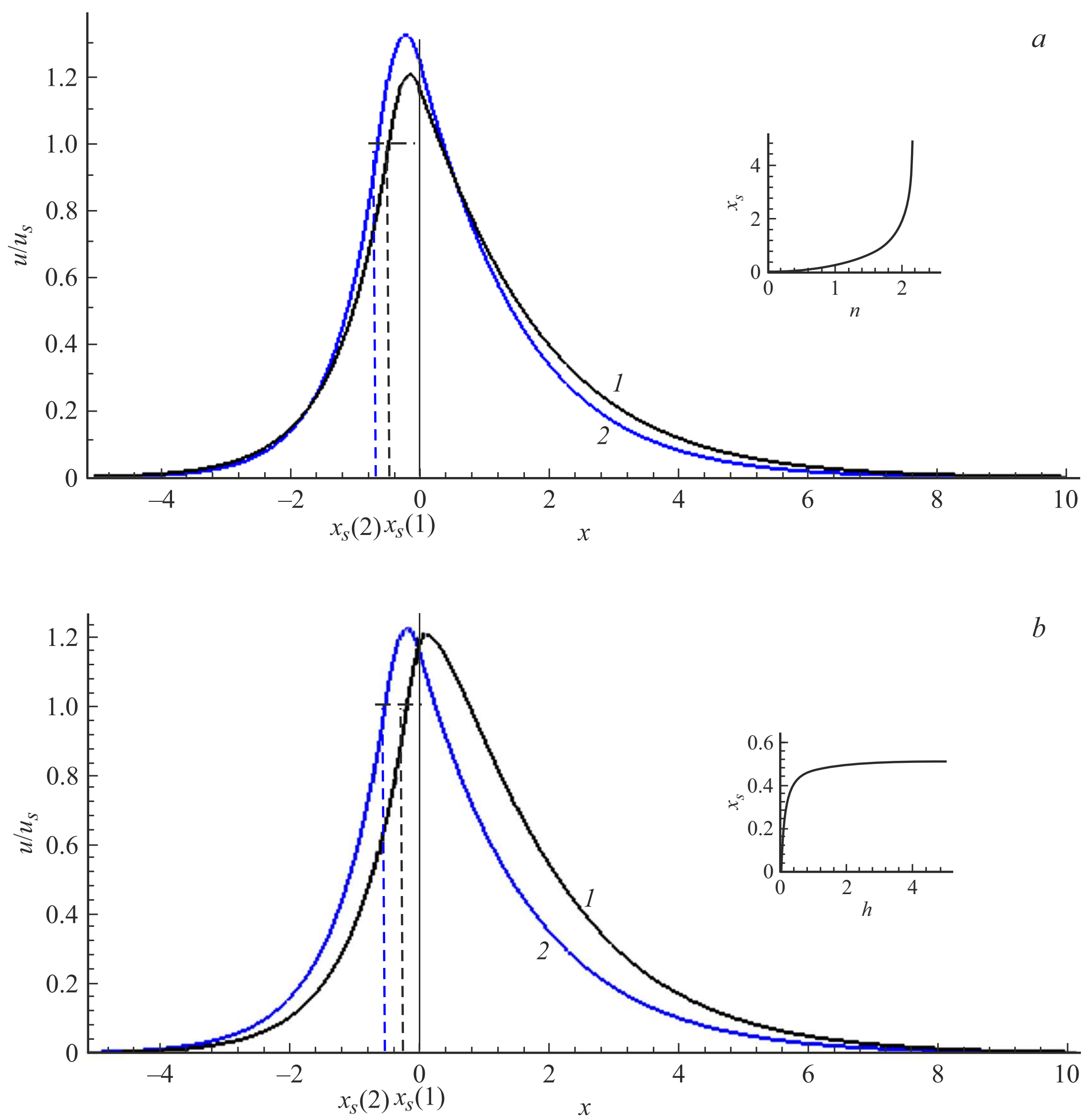
$$x_m=\frac{1}{p_2}\operatorname{arctg}\left(\frac{q_G}{p_2}\right), \quad (14)$$

$$x_s=\frac{1}{p_2}\operatorname{arctg}\left(\frac{q_1}{p_2}\right)-x_m, \quad (15)$$

$$q_G=\frac{k}{n_0}(n_0^2-e_1/2)-\frac{W_{\mu+1,\nu}(2n_0kh)}{hW_{\mu,\nu}(2n_0kh)}. \quad (16)$$

The obtained solution (13) characterizes a nonlinear surface wave propagating along a flat interface between a gradient medium with permittivity profile (3) and a nonlinear medium with step nonlinearity (4). Its characteristic profiles are shown in the figure. The effective refraction index is a free parameter in this model and is not related by a dispersion equation to the system parameters. This implies that a nonlinear surface wave specified by solution (13) may be excited at a continuously varying effective refraction index within the range of permissible values $\max(e_0,\varepsilon_1)<n^2<\varepsilon_2$, which, in turn, is determined by the angle of incidence of a laser beam exciting the surface wave. It was found that an increase in effective refraction index leads to an increase in wave intensity, an expansion of the near-surface layer (an increase in thickness x_s, which is given by expression (15)), and a reduction in depth of field localization in the gradient medium (see panel *a* in the figure). Therefore, it is possible to alter certain characteristics of a nonlinear surface wave, such as its amplitude and localization width near the surface, by varying the angle of incidence during experiments. When parameter h of the gradient profile of permittivity increases, a significant change in the transverse field profile is also observed (see panel *b* in the figure). In this case, the maximum intensity of the nonlinear surface wave shifts from the near-surface layer of the nonlinear medium to the gradient one. The near-surface layer also expands, but the rate of this process decreases with an increase in h, and x_s eventually ceases to increase with h.

Thus, we obtained an exact solution to the conjugation boundary value problem that characterizes a nonlinear surface wave propagating along a gradient medium with a refraction index decreasing in accordance with a generalized hyperbolic profile, which is in contact with a nonlinear medium with its permittivity varying abruptly as a function of intensity of the electric field. The possibility of controlling the characteristics of a surface wave by varying the optical characteristics of the medium was demonstrated.



Transverse profile of the electric field strength of a nonlinear surface wave (13) at $e_0 = -0.1$, $e_1 = 1.5$, $e_2 = 0.2$, $\varepsilon_0 = 0.05$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 5$, $k = 0.5$: *a* — $h = 1$, $n = 1.3$ (*1*) and 1.5 (*2*); *b* — $n = 1.3$, $h = 0.1$ (*1*) and 3 (*2*) (in arbitrary units).

Conflict of interest

The author declares that he has no conflict of interest.

References

[1] D. Mihalache, Rom. Rep. Phys., **76** (2), 402 (2024). DOI: 10.59277/RomRepPhys.2024.76.402
[2] R. Bano, M. Asghar, K. Ayub, T. Mahmood, J. Iqbal, S. Tabassum, R. Zakaria, M. Gilani, Front. Mater., **8**, 783239 (2021). DOI: 10.3389/fmats.2021.783239
[3] S. Savotchenko, Russ. Technol. J., **11** (4), 84 (2023). DOI: 10.32362/2500-316X-2023-11-4-84-93
[4] N.N. Rozanov, *Nelineinaya optika* (Izd. ITMO, SPb., 2008), Vol. 1 (in Russian).
[5] C-L Chen, *Foundations for guided-wave optics* (John Wiley & Sons, Inc., N.Y., 2005). DOI: 10.1002/0470042222
[6] G.P. Agrawal, *Physics and engineering of graded-index media* (Cambridge University Press, N.Y., 2023). DOI: 10.1017/9781009282086
[7] B.K. Singh, V. Bambole, V. Rastogi, P.C. Pandey, Opt. Laser Technol., **129**, 106293 (2020). DOI: 10.1016/j.optlastec.2020.106293
[8] D. Dash, J. Saini, Prog. Electromagn. Res. M, **116**, 165 (2023). DOI: 10.2528/PIERM23032302
[9] M.D. Aparin, T.G. Baluyan, M.I. Sharipova, M.A. Sirotin, E.V. Lyubin, I.V. Soboleva, V.O. Bessonov, A.A. Fedyanin, Bull. Russ. Acad. Sci. Phys., **87**, 710 (2023). DOI: 10.3103/S1062873823701976.
[10] A.E. Kaplan, IEEE J. Quantum Electron., **21**, 1538 (1985). DOI: 10.1109/JQE.1985.1072828
[11] P.I. Khadzhi, L.V. Fedorov, Zh. Tekh. Fiz., **61** (5), 110 (1991) (in Russian).
[12] S.E. Savotchenko, Tech. Phys. Lett., **46**, 823 (2020). DOI: 10.1134/S1063785020080271.
[13] S.E. Savotchenko, Rom. Rep. Phys., **76** (4), 406 (2024). DOI: 10.59277/RomRepPhys.2024.76.406
[14] S.E. Savotchenko, Rom. Rep. Phys., **77** (1), 402 (2025). DOI: 10.59277/RomRepPhys.2025.77.402

Translated by D.Safin