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The transferring of a single-photon excitation by a mediator

© A.M. Basharov[1], A.I. Trubilko[2]

[1] National Research Center „Kurchatov Institute“, Moscow, Russia
[2] Saint-Petersburg University of State Fire Service of EMERCOM of Russia, Saint-Petersburg, Russia

e-mail: basharov@gmail.com, trubilko.andrey@gmail.com

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The paper discusses the possibility of transferring the state of a single-photon excitation by a mediator during two oscillators' nonresonance interaction. The single-photon excitation is a broadband packet that nonresonantly affects the mediator oscillator. The packet's central frequency is resonant with that of the other isolated oscillator, to which the excitation is transferred. Under these circumstances the single-photon packet is modeled by a broadband single-photon thermostat. In the process of oscillators' nonresonance interaction the observed effect has been shown to be described by a kinetic equation obtained within the Markov approximation based on algebraic resonance perturbation theory. It is also shown that the case of excitation transfer to an oscillator is possible, which distinguishes the case of an oscillator from the transfer of excitation to an atom. The statistics of the transferred excitation is defined.

Keywords: single-photon packet, transferring the state, isolated oscillator, mediator oscillator.

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1. Introduction

Single-photon wave packets, representing an electromagnetic field state with one excitation, are attractive primarily for quantum information theory purposes [1,2]. It is precisely through such sources that encoding and transfer of information are carried out in most quantum information protocols. At the same time, a single-photon source can be broadband, where, alongside the carrier frequency, the spectral width of the packet plays an important role. In the case where the latter is greater than other characteristic spectral widths in the problem, it can be regarded as a thermostat, with the single-photon state randomly distributed within its spectral width. Such a state can be obtained, for example, via generation of a biphoton light field by a nonlinear crystal, where one photon from the pair enters a detector and is detected, while the other photon reliably enters the optical setup. We emphasize that in this case, the state is prepared via projective measurement, resulting in a purely quantum state of the object. The phase-matching bandwidth of the nonlinear crystal in this case corresponds to the spectral width of the emitted radiation. Within this formulation, radiation with a random distribution of the single excitation across the packet's frequency can be considered a broadband wave packet acting as a thermostat upon subsequent interaction with another physical object, which thereby becomes an open system.

The primary task of quantum theory of open systems is to derive the kinetic equation for the density matrix of the subsystem or subsystems, with the relaxation operator as its main component. It is usually assumed that the subsystem itself, possessing a certain, typically finite, number of degrees of freedom, weakly interacts with the system having a significantly larger number of degrees of freedom, called the thermostat. In works [3,4], from general physical considerations without invoking any model concepts or assumptions except Markovianity, the form of the relaxation operator and the corresponding kinetic equation, known as the Lindblad form, were mathematically established. However, the universal form of the Lindblad relaxation operator in the kinetic equation harbors the danger of its incorrect application. In standard physical situations — when analyzing limiting cases and the like — one often exceeds the applicability limits of the known kinetic equation [5]. In non-standard physical situations, the derivation of the relaxation operator is often neglected, with the Lindblad relaxation operator used, sometimes without justification [6]. For example, in articles [5–7], the relaxation operator in Lindblad form was applied either to the Hamiltonian of the open system in the thermostat field without using the rotating wave approximation, or the rotating wave approximation was used outside the parameter range where it is valid. This led to paradoxical conclusions such as violation of the second law of thermodynamics [6,7]. To avoid such situations in new problems, it is necessary to specify the requirements on the Hamiltonian of the open system in the thermostat under the Markov approximation. Then, the thermostat model is a delta-correlated thermostat, and the effective Hamiltonian should contain no rapidly time-varying terms, so that the thermostat correlation time is minimal in the system [8,9]. Therefore, in many open systems theory problems, a separate issue is the justification of the effective Hamiltonian of the open system with the required properties, which, along with the relaxation operator, determines the kinetic equation.

We emphasize once again — the case of „pure“ photonic (bosonic) systems is special from the viewpoint of kinetic equations. Methods have been developed for exact solutions of multi-particle and multi-mode bosonic problems [10,11,14], which, nevertheless, require numerical modeling. We propose an alternative approach that, in the considered situation and in a number of quantum optics problems, appears simpler and more physically intuitive.

In this study, we consider the transfer of a single-photon broadband packet state with a Gaussian distribution of the photon across its frequency band upon interaction with oscillator systems. Previously, we discussed a similar issue for the interaction of a single-photon broadband packet with atoms. We examine two oscillator systems. One system is a conventional quantum harmonic oscillator resonantly interacting with the thermostat. This same oscillator, which we call the mediator, interacts non-resonantly with the single-photon broadband packet. Its eigenfrequency is assumed to be substantially higher than the central frequency of the single-photon packet, thereby setting the scale for all characteristic evolution times in the problem. Additionally, the mediator oscillator interacts non-resonantly with another oscillator, which is considered „isolated“ both from the thermostat and from the single-photon wave packet. The eigenfrequency of the latter is resonant with the packet's central frequency. Such a two-oscillator scheme is in demand in connection with various architectures of quantum computing systems [15]. The „isolated“ oscillator participates in unitary operations, since its decay into the thermostat is suppressed by the mediator oscillator [16–18]. We emphasize — all the initial field-system interaction processes here are non-resonant.

Another essential feature of this problem setup is the separation of two thermostats—the conventional vacuum thermostat and the „thermostat“ describing the single-photon broadband wave packet. In the case of broadband packet propagation in an atomic medium [19], thermostat separation was achieved based on the narrow angular directivity of the single-photon packet. For two oscillators, we employ algebraic resonance perturbation theory (ARPT) to obtain the effective Hamiltonian, with thermostat separation accomplished by introducing separate interaction constants with each broadband field. In principle, this can be justified by different field interaction conditions with the mediator oscillator.

In the formal solution of the Schrödinger equation with the effective Hamiltonian for the thermostat fields, all orders of the formal solution expansion are accounted for, whereas for the Gaussian broadband single-photon packet, only the two lowest orders are considered. A kinetic equation in the Lindblad–Kossakowski–Gorini–Sudarshan form is obtained for the density matrices of the open system, describing the relaxation and excitation of the oscillator system. It can be used in various problems. In this article, based on the obtained equation, it is shown that the dynamics of both the mediator oscillator and the isolated oscillator factorize. The features of transferring the statistical characteristics of the packet state to the isolated oscillator state are analyzed, and it is proven that under certain conditions, complete transfer of excitation and statistics to the „isolated“ oscillator is possible. This distinguishes the problem of transferring single-photon broadband packet excitation to an oscillator from transfer to an atom. In the atom case, the maximum population of the excited level is 1/3. It should also be emphasized that the obtained equation manifests the nature of bosonic commutation relations and cannot be derived within a classical description of the source. To date, two experimentally demonstrated physical effects of such manifestation exist—the Casimir effect [20] and the Lamb shift [21] of atomic levels.

2. Formulation of the Main Problem

We assume the following problem setup. Let there be two bosonic oscillators with substantially different frequencies, for definiteness $\omega_c \gg \omega_r$ each described by creation operators $c^\dagger$, $r^\dagger$ and annihilation operators c, r satisfying bosonic commutation relations $\left[\alpha, \alpha^\dagger\right] = 1$, $\alpha = c, r$, $\left[c, r\right] = \left[c, r^\dagger\right] = 0$. The oscillator described by operators r will be called isolated, assuming it interacts only with the mediator oscillator c. The interaction of these oscillators is non-resonant, i.e., it occurs in the wing of the absorption line of one of them or in the dispersive limit.

The mediator oscillator interacts with two more subsystems.

First, it interacts with its „own“ thermostat B in the vacuum state. The thermostat B is chosen as a bosonic system described by operators $b_\omega^\dagger, b_\omega$ with similar Heisenberg commutation relations $[b_{\omega'}, b_\omega^\dagger] = \delta_{\omega',\omega}$. In the interaction of the oscillator with the thermostat, the usual conditions of weak subsystem interaction in a broad interaction band with the vacuum environment are assumed, where the back-action of the system on the thermostat can be neglected, considering the latter's state unchanged, which is the standard condition for deriving the kinetic equation.

Secondly, a narrow-directed single-photon broadband packet acts on the mediator oscillator c, whose central frequency ω_0 is substantially lower than the eigenfrequency ω_c of the mediator, but resonant with the eigenfrequency $\omega_0 = \omega_r$ of the isolated oscillator. The creation operators $a^\dagger(\omega)$ and annihilation operators $a(\omega)$ of the quasimodes of the single-photon packet satisfy bosonic commutation relations $\left[a(\omega), a^\dagger(\omega')\right] = \delta(\omega - \omega')$. These same operators define the operators of its intensity, which we write as

$$\epsilon^-(t) = \int\limits_{\omega\in(\omega_0)} d\omega\epsilon(\omega)a(\omega)\exp(-i\omega t),$$

$$\epsilon^+(t) = (\epsilon^-(t))^\dagger. \qquad (1)$$

Here, $\epsilon(\omega)$ characterizes the properties of the single-photon field source, and the integration is performed over its spectral band $S(\omega_0)$ with central frequency ω_0. For convenience,

we assume that $\epsilon(\omega)$ also includes the coupling parameter κ between the oscillator and the single-photon field, which we consider frequency-independent henceforth. Therefore, the dimensionality of this quantity does not match that of the electric field intensity. he free energy of such a quantized packet is given by

$$H_{0,A} = \int_{\omega \in S(\omega_0)} d\omega \frac{|\epsilon(\omega)|^2}{\kappa^2} a^\dagger(\omega) a(\omega). \quad (2)$$

Here the integration is performed over the entire spectral band of the source. We assume that each of the sources can be considered independent. This can be physically realized, for example, as in the case of atoms [19], based on the narrow angular directivity of the single-photon packet upon its incidence on the mirror of a microcavity implementing the mediator oscillator. Note that in this case, the mediator oscillator serves as a source of photon rescattering from one thermostat to another, just as in the case of atoms [19].

Finally, the free energies of the remaining systems under consideration — the oscillators and the thermostat—are written as

$$H_{0,c} = \hbar\omega_c c^\dagger c, H_{0,r} = \hbar\omega_r r^\dagger r, H_{0,B} = \sum_\omega \hbar\omega b_\omega^\dagger b_\omega, \quad (3)$$

and we also write the free energies of the interactions between them described by us, using the Dirac representation with respect to the free Hamiltonian $H_0 = H_{0,c} + +H_{0,r} + H_{0,B} + H_{0,A}$. These interactions are considered electrodipole, and the operators in the Dirac representation will be denoted explicitly by the argument $t$of the following operator-valued functions:

$$V_{c-r}(t) = g\left(ce^{-i\omega_c t} + c^\dagger e^{i\omega_c t}\right)\left(re^{-i\omega_r t} + r^\dagger e^{i\omega_r t}\right),$$

$$V_{c-A}(t) = \left(\epsilon^-(t) + \epsilon^+(t)\right)\left(ce^{-i\omega_c t} + c^\dagger e^{i\omega_c t}\right),$$

$$V_{c-B}(t) = \gamma_{cB} \sum_{\omega \in (\omega_c)} \left(ce^{-i\omega_c t} + c^\dagger e^{i\omega_c t}\right)\left(b_\omega^\dagger e^{i\omega t} + b_\omega e^{-i\omega t}\right), \quad (4)$$

where the constants g, γ_{cB} determine the coupling parameters of the interactions between the oscillators and of the mediator oscillator with its vacuum thermostat. In writing the last expression, it is assumed that the interaction constant value is chosen at the carrier frequency ω_c of the mediator oscillator and remains unchanged across the entire broad frequency band of interaction of this thermostat with the given oscillator. This assumption is already one manifestation of the Markovian nature of the system interaction with the broadband environment and is traditional for problems of deriving kinetic equations for open systems. We recall that the interaction constant of oscillator c with the single-photon broadband packet is already included in the operator-valued expressions for $\epsilon^-(t)$, $\epsilon^+(t)$.

Previously, in [22,23] in the absence of the single-photon broadband packet, we demonstrated the effect of relaxation of the isolated oscillator into the thermostat of the mediator oscillator, which interacts quasi-resonantly with its broadband thermostat in thermodynamic equilibrium. It was shown that the relaxation dynamics in this case factorize as for independent subsystems. Both excitation of the isolated oscillator and its irreversible relaxation are observed. However, the latter occurs at a frequency and in a frequency band substantially different from that into which the mediator oscillator itself dissipates. It was shown that the physical nature of this second-order effect is the emergence of a new quantum interference channel conditioned by the resonant interaction of the mediator oscillator with the thermostat and the non-resonant interaction between the oscillators. In the current problem setup, it is precisely in this frequency range that the narrow-directed single-photon broadband packet is present, which under the stated conditions plays the role of a broadband thermostat. This circumstance plays a substantial role, allowing the dynamics of the entire system to be considered in the superposition of two broadband fields: the single-photon source and the vacuum environment. Therefore, both the question of direct transfer of the single-photon state to the isolated oscillator state and its own relaxation dynamics under these conditions appear relevant.

3. Kinetic Equation for the Density Matrix of the Oscillator System

To derive the kinetic equation for the relaxation dynamics of the open system, we will use ARPT [24,25]. Its basis is obtaining the effective Hamiltonian of the system followed by introducing the Markov approximation to represent the environment as white noise (noise sources of the equilibrium thermostat are defined as delta-correlated random processes in time), which enables constructing the correct equation for the subsystem density matrix. The effective Hamiltonian must satisfy a quite definite and physically intuitive requirement: the absence of rapidly oscillating, or counter-rotating, terms. Typically, these terms are ignored and discarded already at the stage of writing the interaction in traditional kinetic equation derivations. While such neglect may be somewhat justified for resonant and quasi-resonant interactions, it is completely unjustified for non-resonant interactions. Note that these interactions lead, for example, to additional shifts of the fundamental frequency and freezing of the system relaxation rate. This is precisely why a complete derivation of the effective Hamiltonian is required for each new physical situation.

For the wave vector of the entire system Ψ and the total Hamiltonian of the problem $H(t) = V_{c-r}(t) + V_{c-B}(t) + V_{c-A}(t)$ we write the Schrödinger equation in the Dirac representation:

$$\frac{d|\Psi(t)\rangle}{dt} = -i\hbar H(t)|\Psi(t)\rangle. \quad (5)$$

Equation (5), taking into account expressions (4), contains oscillating terms $\exp\Big(\pm i(\omega_c \pm \omega)t\Big)$, $\exp\Big(\pm i(\omega_c \pm \omega_r)t\Big)$, which, depending on the values of frequencyω spanning the entire spectrum, can be either rapidly varying functions of time or slowly varying compared to $\exp\Big(\pm i\omega_c t\Big)$ and $\exp\Big(\pm i\omega_r t\Big)$. Such a situation is characteristic of all problems of optical radiation interaction with a quantum system consisting of two or more energy levels. For open systems, the presence of rapidly varying time functions in the Hamiltonian in the Dirac representation is critical due to the subsequent use of the Markov approximation and representation of thermostat fields as delta-correlated.

In [9] it was shown that to enable the use of the Markov approximation in the theory of open quantum systems, all rapidly time-varying terms must be eliminated from the system Hamiltonian in the Dirac representation. For this purpose, it is convenient to use ARPT, based on the unitary symmetry of quantum theory. The place of ARPT among other methods for obtaining effective Hamiltonians is discussed in work [8].

We perform a unitary transformation of the initial state vector, passing to a new representation:

$$|\tilde{\Psi}(t)\rangle = \exp\Big(-iS(t)\Big)|\Psi(t)\rangle. \tag{6}$$

In this representation, the Schrödinger equation for the transformed vector $|\tilde{\Psi}(t)\rangle$ has the same form as (5), but with the transformed (marked by the „tilde“) sign) Hamiltonian:

$$\tilde{H}(t) = e^{-iS(t)}H(t)e^{iS(t)} - i\hbar e^{-iS(t)}\frac{\partial}{\partial t}e^{iS(t)}. \tag{7}$$

Further transformations are based on expanding the generators $S(t)$ and $\tilde{H}(t)$ in series with respect to the interaction constants present in the problem and accounting for resonance conditions. In our case

$$S(t) = S^{(1,0,0)}(t) + S^{(0,1,0)}(t) + S^{(0,0,1)}(t) + S^{(2,0,0)}(t) + \ldots, \tag{8}$$

$$\tilde{H}(t) = \tilde{H}(t)^{(1,0,0)} + \tilde{H}(t)^{(0,1,0)} + \tilde{H}(t)^{(0,0,1)} + \tilde{H}(t)^{(2,0,0)} + \ldots. \tag{9}$$

The left index of each triple of upper indices describes the order of the term with respect to the coupling constant between the quantized oscillators, the central one — with respect to the interaction of the mediator oscillator with the equilibrium thermostat, and the right one — with respect to the interaction of this same oscillator with the single-photon broadband packet. Expression (9) with a finite number of terms is called the effective Hamiltonian and denoted $\tilde{H}^{Eff}(t)$.

Since the interaction between the oscillators is non-resonant, it begins to manifest only in the second order with respect to their interaction. The interactions of the mediator oscillator with the environment are both resonant when interacting with its own thermostat and non-resonant, manifesting in the second order with respect to interaction with the single-photon source. This is precisely why second-order effects with respect to interactions will manifest in the system under consideration. Alongside trivial effects, there will also be the possibility of new non-trivial relaxation channels arising due to quantum interference of various alternatives described by bilinear combinations of interaction constants.

We will not write out the general equations for the generator terms, as they have been repeatedly presented and their general form remains unchanged for various physical conditions [8,19,24]. The features of describing the action of a single-photon broadband packet together with the thermostat, albeit on atoms, were considered by us in [19]. The technique for obtaining the kinetic equation is also described there. Note that the effective operators entering (9), describing both non-resonant interactions and interactions with broadband fields, disappear in the kinetic equation, leaving only the relaxation operator in Lindblad form and Stark shifts of the energy levels.

As a result of standard ARPT procedures and stochastic differential equations, we obtained the governing kinetic equation for the density matrix ρ^S describing the evolution of the subsystem of the two selected oscillators in the following form:

$$\begin{aligned}\frac{\partial\rho^S}{\partial\tau} &= -\mathcal{K}(\tau)\mu^2\Big(r^\dagger r\rho^S + \rho^S r^\dagger r - -2r\rho^S r^\dagger\Big) \\ &- \frac{1}{2}\mathcal{K}(\tau)\mu^2\Big(rr^\dagger\rho^S + \rho^S rr^\dagger - 2r^\dagger\rho^S r\Big) \\ &-\frac{1}{2}\sigma^2\Big(r^\dagger r\rho^S + \rho^S r^\dagger r - 2r\rho^S r^\dagger\Big) \\ &- \frac{1}{2}\chi^2\Big(c^\dagger c\rho^S + \rho^S c^\dagger c - 2c\rho^S c^\dagger\Big).\end{aligned} \tag{10}$$

Here the dimensionless time$\tau = \omega_c t$ is introduced. The time function$\mathcal{K}(\tau) = 2\big(erf(\delta\tau/2)\big)$ expressed through the error function erf is predetermined by the Gaussian frequency distribution of the single-photon packet. We assume that the single-photon packet has a Gaussian frequency distribution of the photon detection probability density ($\nu = \omega/\omega_c$ — dimensionless frequency, $\nu_0 = \omega_0/\omega_c$):

$$|\epsilon(\nu)|^2 = \frac{1}{\sqrt{\pi}\delta}\exp(-\frac{(\nu-\nu_0)^2}{\delta^2}).$$

The dimensionless spectral width $\delta = \Delta/\omega_c$ is determined by the generation bandwidth Δ of the single-photon radiation source.

The presented equation (10) has the typical form of a Lindblad equation and describes the relaxation and excitation effects of the system. However, it cannot be written heuristically without performing calculations similar to the atomic case [19] and the case of disintegration of the isolated oscillator into the thermostat [22], since

only direct computation can isolate the interference interaction operators, whose reflection is served by the dimensionless parameters $\mu = 2\sqrt{2\pi}g\kappa(\hbar^2\omega_c^{3/2}\sqrt{\omega_0})^{-1}$, $\sigma = 2\sqrt{2\pi}g\gamma_{cB}(\hbar^2\omega_c^{3/2}\sqrt{\omega_0})^{-1}$ and $\chi = \sqrt{2\pi}\gamma_{cB}(\hbar\omega_c)^{-1}$. The interference operators themselves did not enter the final kinetic equation for the oscillator system, since the interference involved broadband fields, which are standardly „excluded" when writing the kinetic equation.

4. Characteristics of the Oscillators

Based on the obtained equation (6), the characteristics of the selected oscillators can be determined. Note that the dynamics of both subsystems — the isolated oscillator onto which the single-photon packet state is transferred, and the mediator oscillator — factorize if there was no initial entanglement in the initial states. Therefore, both subsystems evolve independently of each other. Their nonresonant interaction with each other, as well as interactions with the thermostat and single-photon packet, produce only changes in the eigenfrequencies of the considered subsystems, described by the following formulas:

$$\tilde{\omega}_c = \omega_c - \Pi_c(\omega_r) - \frac{\Pi_r(\omega_0)}{g^2}\int d\omega|\epsilon(\omega)|^2 - \frac{\gamma_{cB}^2}{2\hbar\omega_c},$$

$$\tilde{\omega}_r = \omega_r - \Pi_r(\omega_c).$$

Here the standard ARPT parameters arise:

$$\Pi_c(\omega_r) = \frac{g^2}{\hbar}\Big(\frac{1}{\omega_c+\omega_r} - \frac{1}{\omega_c-\omega_r}\Big),$$

$$\Pi_r(\omega_c) = \frac{g^2}{\hbar}\Big(\frac{1}{\omega_c+\omega_r} + \frac{1}{\omega_c-\omega_r}\Big)$$

The dynamics of the mediator oscillator under the considered conditions reduces directly to its relaxation, caused solely by its resonant interaction with the vacuum thermostat. Thus, the average number of excitations $\langle c^\dagger c\rangle(\tau) = Sp_S\Big(c^\dagger c\rho^S(\tau)\Big)$ is described by the standard expression

$$\langle c^\dagger c\rangle(\tau) = \langle c^\dagger c\rangle_0 \exp\Big(-\chi^2\tau\Big),$$

where $\langle c^\dagger c\rangle_0$ is the average number of excitations of the oscillator at the initial moment of time and, consequently, the initial average decays exponentially or does not develop for the initial vacuum state. It is straightforward to show that in the case of interaction with a hot thermostat whose state follows an equilibrium distribution, the average number of excitations of the mediator oscillator would also be determined by the average photon number density of the thermostat itself.

The dynamics of the isolated oscillator is due to relaxation channels arising exclusively from quantum interference of alternatives of nonlinear interactions of this oscillator with the mediator oscillator, the mediator oscillator with the single-photon packet, and the specifically Gaussian character of the single-photon packet itself. We emphasize that these terms naturally arise when applying ARPT together with quantum stochastic differential equations in deriving the kinetic equation for the density matrix of the open system, thanks to accounting for all rapidly varying terms of the original interaction Hamiltonian [19,24,25]. These factors lead to the following equation for the density matrix ρ^R of the isolated oscillator:

$$\frac{\partial\rho^R}{\partial\tau} = -\mathcal{K}(\tau)\mu^2\Big(r^\dagger r\rho^R + \rho^R r^\dagger r - 2r\rho^R r^\dagger\Big)$$

$$-\frac{1}{2}\mathcal{K}(\tau)\mu^2\Big(rr^\dagger - \frac{1}{2}\sigma^2\Big(r^\dagger r\rho^R + \rho^R r^\dagger r - -2r\rho^R r^\dagger\Big), \tag{11}$$

It is evident that the dynamics of the initially isolated oscillator is determined by interaction with the single-photon packet, which interacts linearly with the mediator, although its coefficients depend on time due to the Gaussian character of the single photon's frequency distribution. Additionally, due to interference, a channel of direct relaxation of the isolated oscillator into the mediator oscillator's thermostat opens up. However, this relaxation occurs in a frequency band centered at the eigenfrequency of the isolated oscillator itself, not the mediator.

From equation (11) follows the kinetic equation for the dynamics of the average number of excitations $n = \langle r^\dagger r\rangle = Sp_R(r^\dagger r\rho^R)$ of the isolated oscillator:

$$\frac{\partial n}{\partial\tau} + \Big(\mathcal{K}(\tau)\mu^2 + \sigma^2\Big)n = \mathcal{K}(\tau)\mu^2. \tag{12}$$

In the considered case, the Gaussian character of the spectral distribution of the broadband single-photon packet leads to the appearance of a nonlinear character in the evolution equation for the average number of excitations. An analytical solution of equation (12) is possible in the case of absence (or possibility of neglecting) interaction with the vacuum thermostat, when $\sigma^2 = 0$. In this case, the average number of excitations of the isolated oscillator is determined by the relation

$$n(\tau) = n_0\mathcal{M}(\tau) + \Big(1 - \mathcal{M}(\tau)\Big), \tag{13}$$

where the function $\mathcal{M}(\tau)$ is determined by the spectral distribution character of the single-photon packet, described by the function $\mathcal{K}(\tau)$, and has the following explicit form:

$$\mathcal{M}(\tau) = \exp\Big(-\frac{4\mu^2}{\delta}\Big[\frac{\delta\tau}{2}erf(\delta\tau/2)$$

$$+\frac{\exp(-\delta^2\tau^2/4)}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}\Big]\Big). \tag{14}$$

Note that in the considered case, the average number of excitations of the isolated oscillator obviously tends to its stationary value and at long times reaches a level equal to unity, just as in the original single-photon packet. Therefore,

in the case of absence of interaction with the vacuum reservoir or the possibility of neglecting this interaction, the average number of excitations is completely transferred from the original single-photon packet to the isolated oscillator without changes.

We show that the state of the isolated oscillator in the considered case has a Gaussian distribution for coordinate and momentum, assuming for simplicity that its initial state is vacuum. This fact can be proven in two different ways.

First, it can be shown that the fourth-order cumulant $k_q^{(4)} = \langle q^4 \rangle - 4\langle q \rangle \langle q^3 \rangle - 3\langle q^2 \rangle^2 + 12\langle q^2 \rangle - -\langle q \rangle^2 - 6\langle q \rangle^4$ for the averages of the canonical coordinate operator q is exactly zero. This follows from the solutions of the system of equations for the averages constructed based on the kinetic equation (11), which for the chosen initial condition have the form

$$\langle q \rangle = 0, \qquad \langle q^3 \rangle = 0,$$

$$\langle q^2 \rangle = \frac{3}{2}(1 - \mathscr{M}(\tau)), \qquad \langle q^4 \rangle = \frac{27}{4}(1 - \mathscr{M}(\tau))^2. \quad (15)$$

Second, introducing the Wigner representation of the density matrix

$$W(q, p, \tau) = \frac{1}{\sqrt{2\pi}} \int \exp\Big(- isp \Big) \langle q + s|\rho|q - s \rangle ,$$

defining the quasiprobability function $W(q, p, \tau)$ of the canonical conjugate coordinate and momentum for the isolated oscillator:

$$\frac{\partial W(q, p, \tau))}{\partial \tau} = \Big[\partial_q \Big(\frac{\mathscr{K}(\tau)}{2}\mu^2 q\Big) + \partial_p \Big(\frac{\mathscr{K}(\tau)}{2}\mu^2 p\Big) + \Big(\partial_q^2 + \partial_p^2\Big)\Big(\frac{3\mathscr{K}(\tau)}{4}\mu^2\Big)\Big] W(q, p, \tau), \quad (16)$$

The presented equation is a Fokker-Planck equation with positive-definite diffusion coefficients and drift coefficients, and without mixed derivatives. This means that the evolution of coordinate and momentum occurs independently, and their distribution functions in the used representation are Gaussian functions. Note that the solutions (15) for the presented averages naturally follow from equation (16).

To determine the statistical characteristics of the number of excitations of the isolated oscillator, we find the normally ordered average $\mathscr{Q} = \langle r^\dagger r^\dagger r r \rangle$, whose simultaneous average obeys the equation following from (12) in the case of absence of interaction with the vacuum thermostat:

$$\frac{\partial \mathscr{Q}}{\partial \tau} + 2\mathscr{K}(\tau)\mu^2 \mathscr{Q} = 4\mathscr{K}(\tau)\mu^2 n(\tau), \quad (17)$$

where the function $n(\tau)$ is defined by solution (13). The solution for the case of vacuum initial state $n_0 = 0$ of the isolated oscillator has the simple form

$$\mathscr{Q}(\tau) = 2\Big(1 - \mathscr{M}(\tau)\Big)^2, \quad (18)$$

where the function $\mathscr{M}(\tau)$ is defined according to (14).

Using the presented expressions, one can trace the evolution over time of the statistics parameter ξ of the number of excitations of the isolated oscillator. This parameter determines the deviation of the statistics from the standard quantum limit set by the vacuum and coherent states of the oscillator, which minimize the uncertainty relation of the conjugate components and for which Poissonian statistics of the registered number of excitations is characteristic, with variance in this case equal to $\Delta n^2 = \langle r^\dagger r^\dagger r r \rangle - \langle r^\dagger r \rangle^2 = n$. The presented statistics parameter characterizes the deviation from the normalized standard quantum limit level in the variance of the number of excitations:

$$\frac{\Delta n^2}{n} = 1 + \xi. \quad (19)$$

Here, the value $\xi = 0$ corresponds to the coherent state of the oscillator, for $\xi > 0$ the statistics turns out to be super-Poissonian, while the case $\xi < 0$ corresponds to sub-Poissonian statistics of the number of excitations. In the considered case, the variance of the average number of excitations of the oscillator $\Delta n^2 = n^2 + n$ and the statistics parameter takes the value $\xi = n$in this case.

The situation of transferring excitation and properties of the single-photon packet to the isolated oscillator, presented by analytical solutions, is ideal; the presence of interaction of the oscillator with the vacuum thermostat of the mediator oscillator leads to the need to account for its irreversible relaxation, which ultimately results in non-ideal transfer of both the excitation itself and the statistical properties of the original packet. This is clearly seen from Figs. 1 and 2, where the dynamics of development and reaching stationary solutions of the average number of excitations of the isolated oscillator and the normally ordered average $\mathscr{Q}$ are presented, respectively. The presented graphs show solutions both for the case of complete absence of relaxation into the vacuum thermostat — solid curves, and taking into account such interaction — graphs presented by points and dashes. For all graphs, the spectral width of the original single-photon packet is normalized to a value equal to unity. It is evident that accounting for the interaction of the systems with the vacuum thermostat leads to incomplete transfer of both the average excitation number and changes in the statistical properties of the original Gaussian packet.

5. Conclusion

It should be noted that consideration of a broadband single-photon packet, whose single excitation is distributed according to a Gaussian law, as a broadband thermostat for other systems upon which it acts, is unknown to the authors neither in terms of formulation nor solution. The following circumstances should be noted in this regard.

1. In the truest sense, the broadband single-photon packet is not a thermostat, since complete photon absorption is possible. However, the fact of its broadband nature, as is

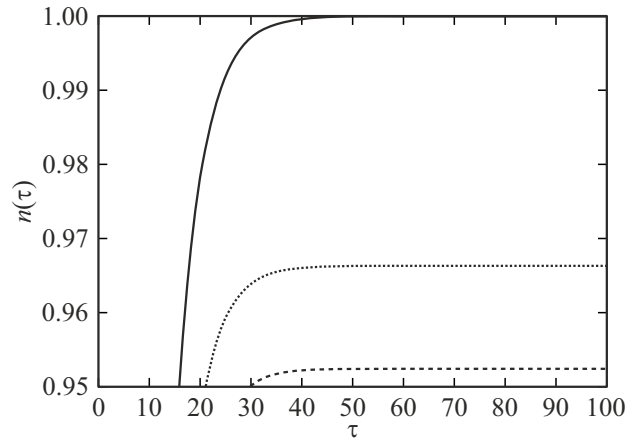

Figure 1. Dependence of the average number of excitations $n(\tau)$ of the isolated oscillator on dimensionless time τ. The following parameter values are chosen: $\delta = 1$, $\mu^2 = 0.1$, solid curve — $\sigma^2 = 0$, points — $\sigma^2 = 0.005$, dashes — $\sigma^2 = 0.01$. The inset shows the behavior of these curves at the initial stage of evolution.

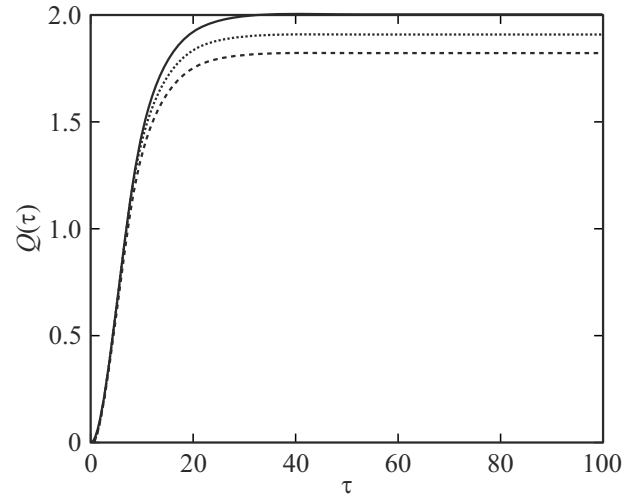

Figure 2. Normally ordered average $Q = \langle r^\dagger r^\dagger r r \rangle$ of the isolated oscillator versus dimensionless time τ. The following parameter values are chosen: $\delta = 1$, $\mu^2 = 0.1$, solid curve — $\sigma^2 = 0$, points — $\sigma^2 = 0.005$, dashes — $\sigma^2 = 0.01$. The value of the function equal to corresponds to the Gaussian state of the oscillator.

often done in open systems theory when interacting with a thermostat, was used in the calculations.

2. Such broadband packets represent single-photon states upon generation of broadband biphoton fields [26,27]. Performing a projective measurement on one photon of the pair unambiguously indicates the action of the other on the selected open system. With this method of state preparation, the state of the analyzed photon is essentially quantum (necessarily accounting for its commutation relations) and cannot be interpreted by any other classical sources. In [27] an example is given of generating broadband biphoton fields in a lithium niobate crystal, where a signal wave is generated at a wavelength of $\lambda = 495\,\text{nm}$ with a spectral width of $\Delta\nu \approx 2.5 \cdot 10^{14}\,\text{Hz}$, which is much larger than the field relaxation rate in optical resonators.

3. The obtained dynamic equation for the open system in the form of (10) is an equation describing both linear absorption and linear amplification of the system based on the Lindblad relaxation operator. It was derived for the quantum source state, since the equation itself manifests the commutation relations of the single-photon source. The Gaussian envelope of the source produces nonlinear effects arising in the system. A closely related problem, where classical fields serve as the source, leads to an equation and solution of a fundamentally different form. Ordinary laser radiation with attenuated photon number, whose average is reduced to unity, serves as the source of such fields. Commutation relations are unimportant for their description, since their characteristics are described here by ordinary c-number functions. In this case, the kinetic equation for the density matrix of the open system is determined only by the dynamic part (without accounting for interaction with the vacuum component) and has the following form:

$$\frac{\partial \rho}{\partial \tau} = i\eta[(r\mathscr{E}^*(\tau) - r^\dagger \mathscr{E}(\tau)), \rho]$$

where η is the coupling constant, $\mathscr{E}(\tau) = \int d\nu \epsilon(\nu) \langle \alpha(\nu) \rangle \exp\left(i(\nu - 1)\tau\right)$ and angular brackets denote the average value of the dimensionless field amplitude at this frequency. Obviously, both the form of the presented equation itself and its solution differ from the case analyzed in this article.

We have shown that in the absence or neglect of the vacuum thermostat, the state of the single-photon quantum broadband source can be transferred to the state of the isolated oscillator in the considered system. With resonant action on the oscillator, analogous transfer is possible due to the complete coincidence of the form of the kinetic equation (11) in this case. This picture is observed thanks to the description of the oscillator by Bose-type operators. If a single two-level atom model is chosen as the isolated system, analogous results do not arise. This is related to the description of the atomic system based on operators with commutation relations characteristic of angular momentum operators. These lead to a stationary value of the number of excitations limited to 1/3. This value turns out to be the same as the limiting population of excitation transfer in an atomic chain [28], and as in the excitation of a two-level atom by equilibrium radiation with an average photon number at the resonant frequency equal to unity [29]. This latter circumstance can also be regarded as justification for the term „thermostat“ with respect to the broadband single-photon packet.

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Conflict of interest

The authors of this paper declare that they have no conflict of interest.

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