Spectral problems in galactic dynamo theory: asymptotic and numerical approaches

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The origin of galactic magnetic fields is described using magnetohydrodynamics and dynamo theory. Two different scales are considered: the microscopic scale, where equations for magnetic field evolution and transport, directly following from Maxwell's equations, are applicable, and the macroscopic scale, where dynamo equations for mean fields are considered after averaging over turbulence scales. Of particular interest are models that take into account the significant thickness of the galactic disk. In all models, the possibility of magnetic field generation is described by a spectrum of corresponding differential operators.

Keywords: galaxies, spectra, differential operator, magnetism.

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Introduction

Currently, there are unambiguous proofs of presence of magnetic fields in spiral galaxies [1]. They are based on Faraday rotation measures and investigation of a synchrotron radiation spectrum. At the same time, there are issues occurring to theoretically explain origination of the magnetic fields in such objects. An interstellar gas in most of these galaxies is a partially ionized hydrogen. Therefore, it can be expected that evolution of the fields in them is described by magnetic hydrodynamics laws for well-conducting media. Particularly problematic is presence of two fundamentally different scales.

In case of low scales, it is possible to use microscopic equations arising from Maxwell's equations in a continuum. When there are vortices of various, sometimes incommensurable scales, the field can decay, be transferred or enhanced. Then, a problem of excitation of the magnetic field is reduced to studying a spectrum of respective differential operators [2]. Presence of eigenvalues with a positive real part indicates a capability of generation. Such problems are relevant both for theoretical astrophysics as well as for computational mathematics.

In case of global magnetic fields, a much more important role is played by fields averaged over scales that are related to sizes of turbulent vortices. After averaging, the induction equation originates an additional summand related to average helicity of turbulent motions. Together with differential rotation (that is related to a decrease of angular rotational speed of a galaxy with increasing distance from the source), it result in impact of $\alpha\Omega$-dynamo [3]. In the same way as in the previous case, growability of the magnetic field is found by studying the spectrum of the respective differential operator.

For the thin galaxies, D. Moss [4] suggested a thin disc approximation, which can reduce the problem of evolution of the magnetic field to two partial differential equations for radial and azimuthal field components. Although this model is created for computational modeling, it allows finding the spectrum exactly in an axisymmetric case. However, it is not enough for many real objects: thus, although a disc thickness is significantly less than the radius, their value can be quite comparable.

An RZ-model for the magnetic field is called upon to solve this problem [5]. It allows finding the magnetic field with a finite disc thickness (when its ratio to the radius is not a small parameter of the problem) and takes into account a much more complicated vertical structure. One of the difficulties of this approach is much higher complexity of the differential operator that characterizes evolution of the field. It is not self-adjoint and, therefore, application of standard approaches of the theoretical physics is not justified mathematically strictly. We have considered a simplified model, in which transformation of variables results in a problem for Hermitian operators, whence asymptotics for the field can be constructed. These results are related to the case of the discs that have an innegligible thickness. Besides, it was assumed that it varied with increasing distance from the center.

The present study has investigated the spectrum of the operator for the galactic disc that expands with increasing distance from the center (this problem corresponds to real astrophysical objects). Due to complexity of the equations, we solve it numerically, using a reverse iteration method. For this, it is necessary to solve a chain of systems of linear equations.

1. Spectrum problems for space-quasiperiodic fields

When considering the problem of magnetic field generation by a turbulent flow, it is natural to try to simulate this flow by a quasiperiodic field. Then, the dynamo problem can be solved by pseudo-spectral methods with decomposition of desirable magnetic modes into finite Fourier series. If N harmonics are used for discretizing for each of the basic periods, then the mode is described by N^6 harmonics, thereby making the problem quite difficult for computation. In this regard, the said approach to solving the dynamo problem was tested by us on the example of calculating dominating modes of a passive scalar transport operator. Taking into account diffusion of a scalar impurity, it is written as

$$L[c] = k\Delta c - (\mathbf{u}\nabla)c,$$

where k is a diffusion constant, $\mathbf{u}$ is a speed field, and c is a scalar field of the impurity concentration. This operator describes evolution of the scalar field (such as a temperature or concentration of a chemical agent) and is used for statistical analysis of turbulence (see, for example, [6]). Computationally, it is largely similar to a magnetic diffusion operator. However, since in the kinematic dynamo generation is possible only for a three-dimensional flow according to a Zeldovich Ya.B. theorem [7], it is possible to limit ourselves by selecting two-dimensional $\mathbf{u}$ that depends on two Cartesian variables, thereby decreasing the number of calculation Fourier harmonics to N^4 and requiring much less random access memory and a calculation scope.

The simulation flow is pre-defined as a superposition of

$$\mathbf{U} = \alpha\boldsymbol{\nu} + (1-\alpha)\mathbf{w},$$

where space-periodic $\boldsymbol{\nu}(\mathbf{x})$ and $\mathbf{w}(\mathbf{x})$ have incommensurate periods $L_{\boldsymbol{\nu}_j}$ and L_{w_j} along the Cartesian coordinate x_j, while the parameter $0 \le \alpha \le 1$ can be interpreted as a„quasi-periodicity degree“ of the flow $\mathbf{u}$. Solenoidity conditions $\mathrm{div}(\mathbf{u}) = 0$ as termed by the Fourier coefficients $\boldsymbol{\nu}_\mathbf{k}$ and $\mathbf{w}_\mathbf{k}$ are written as $\mathbf{k}_{\boldsymbol{\nu}}\boldsymbol{\nu}_\mathbf{k} = \mathbf{k}_\mathbf{w}\mathbf{w}_\mathbf{k} = 0$, where $\mathbf{k}_{\boldsymbol{\nu}} = 2\pi(n_1/L_{\boldsymbol{\nu}_1}, n_2/L_{\boldsymbol{\nu}_2})$, $\mathbf{k}_\mathbf{w} = 2\pi(m_1/L_{\mathrm{w}_1}, m_2/L_{\mathrm{w}_2})$ are wave vectors, n_j and m_j are integer numbers. Solenoid fields $\boldsymbol{\nu}$ and $\mathbf{w}$ are synthesizes as the finite Fourier series ($N = 32$ harmonics in each Ca directions) with respective periodicities, pseudo-random coefficients and an exponentially-decreasing energy spectrum. The mode c is given as

$$c = \sum_{\mathbf{n},\mathbf{m}} c_{\mathbf{n},\mathbf{m}} \exp[i(\mathbf{k}_{\boldsymbol{\nu}} + \mathbf{k}_\mathbf{w})\mathbf{x}].$$

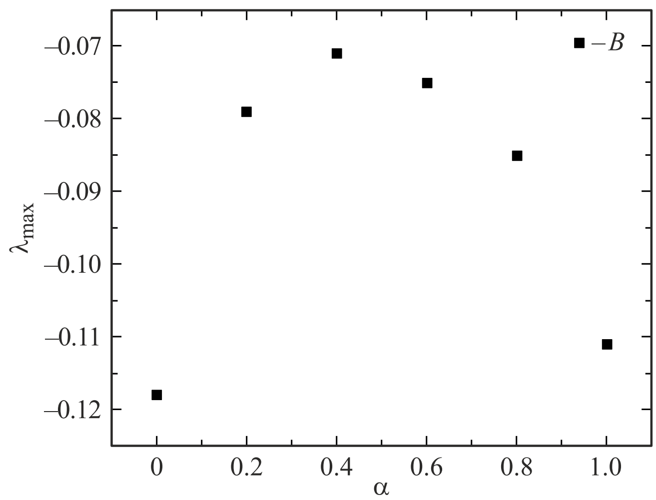

Figure 1. Increment of growth of the dominating mode in the problem of passive scalar transport as a function of the flow quasi-periodicity degree α.

Fig. 1 shows preliminary results of calculations for the periods $L_{\boldsymbol{\nu}} = 2\pi$, $L_\mathbf{w} = 2^{3/2}\pi$ and the diffusion constant $k = 0.1$. Residual errors of eigenmodes do not exceed 10^{-4}. According to Fig. 1, in the problem in question decay of the dominating mode in the average part of the interval $0 \le \alpha \le 1$ is much less than for the periodic flows at boundaries of this interval. In can be interpreted as favorability of flow quasi-periodicity for improving mixability of the transferable scalar.

This experiment demonstrated applicability of the proposed approach to calculating the dominating modes for the quasi-periodic speed fields; further, it is planned to apply it for much complex objects when solving a problem on effects of kinematic dynamo in them.

2. Spectrum problem for the RZ-model in the expanding disc

Evolution of the large-scale magnetic field is described by the mean-field dynamo equation. When using the RZ-model, the main field component is azimuthal. In dimensionless variables (when the distances are measured in units of an object radius), the azimuthal field B obeys the following differential equation [8]:

$$\partial B/\partial t = D^{1/2}B + D^{1/2}z\partial B/\partial z + \lambda^2(\partial^2 B/\partial z^2 + \partial^2 B/\partial r^2 + \partial B/r\partial r - B/r^2),$$

which is pre-defined in the dimensionless variables (the distances are measured in galaxy radii) within the range $0 < r < 1$, $-h(r) < z < h(r)$, $h(r) = h_0(r/r_0)^{9/8}$, at whose boundary the field vanishes. Here, D is dynamo number that characterizes simultaneous impact of the α-effect and differential rotation.

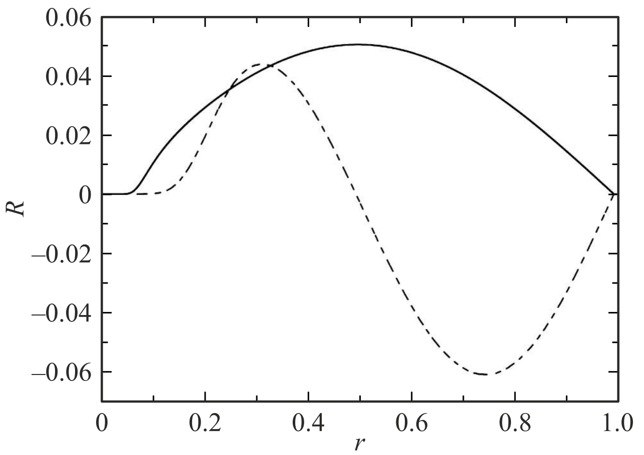

Figure 2. Senior radial eigenfunctions. The solid curve is $R_1(r)$, the dashed curve is $R_2(r)$.

A transition to an eigenvalue problem is carried out by standard substitution

$$B(r, z, t) = B(r, z)\exp(\gamma t).$$

Then, we transit from the variable z to a scaled coordinate

$$X = z/h(r)$$

with subsequent separation of the variables:

$$B(r, x) = R(r)X(x).$$

Then, we solve the problems by the variables x and r. The eigenproblem for $R(r)$ includes derivative up to the second order:

$$\gamma R = \hat{A}R;$$

$$\hat{A} = [D^{1/} - \lambda^2/r^{25/8} + \kappa_m/r^{9/8}]$$

$$+\lambda^2/r^2 R + \lambda^2/rd/dr + \lambda^2 d^2/dr^2;$$

$$R(r_{\min}) = R(1) = 0,$$

where $\kappa_m = \lambda^2\pi^2 m^2/4$ is an eigenvalue that corresponds to the mode $X_m(x)$.

Now, let us numerically estimate the eigenfunctions and the eigenvalues. For this we use the reverse iteration method. It is especially effective for calculating the first few eigenvalues of the non-self-adjoint operator. Using some initial approximation of the eigenvector, the method sequentially improves it with iterations

$$R_{k+1} = \big((\hat{A} - \sigma\hat{E})^{-1}R_k\big)/C_k,$$

where C_k are normalization constants, and σ is approximation of the desired eigenvalue.

By discretizing the equations with second-order finite differences with taking into account boundary conditions $R(r_{\min}) = R(1) = 0$ [9], we have obtained a matrix representation of the equations as three-diagonal systems. These systems were solved by a sweep method. However, in our problem, due to some approximations a diagonal dominance condition is disturbed, thereby making the classical scheme unstable. In order to provide stability of the numerical solution by taking into account possible alternating coefficients, we have used a modified algorithm of non-monotonic sweep.

Fig. 2 shows the found first and second eigenfunctions. For the eigenvalues when $D = 10$ we obtain $\gamma_1 = 3.1$, thereby meaning generatability of the magnetic field. At the same time, for the high-order modes we have to talk about decay.

Conclusions

The study has investigated the eigenproblems, which occur during generation of the magnetic fields of the galaxies. For this, we have studied the simulation equation that describes an increase of the low-scale component. It also included investigation of an increase of the magnetic field in the „thick" disc with a variable thickness and obtaining of the senior eigenfunctions. It is shown that for the realistic values of the dynamo numbers, the increase is possilbe only for the senior radial mode.

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Conflict of interest

The authors declare that they have no conflict of interest.

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