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Singular surface polaritons at interface of identical uniaxial crystals

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Received July 27, 2025
Revised August 18, 2025
Accepted August 19, 2025

It is well known that an interface between media can support the propagation of surface electromagnetic waves. This work investigates a particular configuration involving two identical uniaxial crystals, where the surface polariton exhibits a singular wave characteristics similar to a Voigt wave. We derive complete existence conditions linking the polariton's propagation parameters to the crystals' anisotropy and optic axes orientation. Analytical expressions for the polariton field distributions have been obtained. It has been found that, for a given orientation of the optic axes and degree of anisotropy, up to four pairs of propagation directions are possible. Under certain conditions, the polariton can propagate either parallel or perpendicular to the crystal's optic axis, where it ceases to be singular.

Keywords: surface polariton, surface electromagnetic wave, metamaterials, Dyakonov wave, Dyakonov surface wave, anisotropic medium.

DOI: 10.61011/PSS.2025.09.62367.4-25

1. Introduction

Necessary condition for existence of surface electromagnetic waves in the case of isotropic media requires opposite signs of permittivities of adjacent media. This condition can be fulfilled, for example, for a metal/dielectric interface. The application of these waves called surface plasmon polaritons is limited due to dissipation in a medium with a negative real part of permittivity. Lack of isotropy in at least one of the adjacent media can result in significant variation of properties of the surface wave and their existence conditions. The case of the interface between an uniaxial crystal and an isotropic medium was analyzed by M.I. Dyakonov [1] where shown that the presence of anisotropy leads to the emergence of surface wave propagating along the interface of two transparent dielectrics. The first experimental observation of such waves that are now referred to as Dyakonov surface waves (or dispersion-less surface waves) was carried out in 2008 [2]. These waves can propagate in a limited angular range that is very sensitive to an anisotropy degree [3–5]. The smallness of this range in optical wavelength range in natural minerals makes their observation quiet complicated, but the high sensitivity to variation of permittivity of the adjacent media can be used in sensors [3].

Recently, increasingly complex structures with anisotropic media which supports propagation of surface waves have been considered. For example, structures with an additional thin isotropic dielectric layer between semi-infinite media have an enlarged angular range of propagation [3,6–8]. It was shown in [9] that structures with additional restriction of the two uniaxial crystals by flat interfaces with air (or a conductor) still support propagating of the dispersion-less waves. In this case, the existence of surface waves is possible for the interfaces of crystals that do not support the propagation of waves in an unbounded medium [9,10]. Advances in technology make it possible to create complex structures (metamaterials) with the required electromagnetic properties, including strong anisotropy [11–14]. For a certain wavelength range, the metamaterials can be made to have anisotropic effective permittivity not only with the same signs of principal values, but with different ones as well. In these media that are referred to as hyperbolic media, surface waves similar to the Dyakonov waves are also studied [12,15,16].

From a mathematical point of view on surface waves in anisotropic media, it is interesting to consider a case when a dependence of fields on the distance to the media interface has a term $(\mathbf{ar})\exp(i\mathbf{qr})$, which is a product of the linear function and the exponential function [17–20]. If such expressions are used to describe the field distribution in at least one of the adjacent media, then the wave is referred to as a singular surface wave (polariton) [17,21] or a Dyakonov–Voigt surface wave [18,19]. In particular, if both media are anisotropic, a situation is possible where expressions of the above-mentioned kind describe field distribution in both of them [21,22]. In this case, the wave is referred to as a bisingular surface polariton.

In most cases, the conditions of existence of the surface waves and its characteristics on a propagation direction are studied numerically or approximately. In [23] an analytical solution has obtained in the case of the interface of an isotropic medium with an uniaxial crystal. Also, in [24] an analytical solution in other form has obtained in the case of the interface of an isotropic medium with a biaxial crystal that has the positive principal values of permittivity tensor. The problem is more complicated in the case of two anisotropic crystals and its analytical solution in a general form has not been obtained yet. However, in the case of singular waves, the dispersion equation has a much simpler form. Thus, surface bisingular wave in the case of two different uniaxial crystals with optic axes parallel to the interface has analytically studied in [21]. In this article we consider a similar problem in the case of identical crystals, but the singular form of the solution is chosen only in one of them. We have obtained the conditions of existence that relate mutual orientation of the optic axes or the direction of propagation to the principal values of permittivity. We have also obtained analytical expressions for the field distribution in the singular surface wave.

2. Model

Let us consider a flat interface between the two identical nonmagnetic and nongyrotropic uniaxial crystals, whose optic axes are parallel to this interface. We assume that each crystal is characterized by a permittivity tensor $\hat{\varepsilon}$ with the principal values $\varepsilon_\perp$ and $\varepsilon_\parallel$, where the value of $\varepsilon_\perp$ corresponds to propagation directions of propagation perpendicular to the optic axis, while that of $\varepsilon_\parallel$ corresponds to the propagation direction parallel to the optic axis. We also assume that absorption can be neglected at the frequency ω under consideration, therefore, $\varepsilon_\perp$ and $\varepsilon_\parallel$ are real numbers.

Let us introduce a coordinate system (Figure 1), where $x = 0$ — the interface plane, and Oz — the direction of propagation of the surface wave. Let us use φ and θ to designate angles counted from the axis Oz to the optical axes S and O of the media that fill in half-space $x < 0$ and $x > 0$, respectively. An angle count direction is selected to coincide with the clockwise direction when looking at the plane from the side $x > 0$. The permittivity tensor of each of the crystals in the selected system of coordinates is written as

$$\begin{pmatrix} \varepsilon_\perp & 0 & 0 \\ 0 & \varepsilon_\perp\cos^2(\psi)+\varepsilon_\parallel\sin^2(\psi) & (\varepsilon_\parallel-\varepsilon_\perp)\sin(\psi)\cos(\psi) \\ 0 & (\varepsilon_\parallel-\varepsilon_\perp)\sin(\psi)\cos(\psi) & \varepsilon_\perp\sin^2(\psi)+\varepsilon_\parallel\cos^2(\psi) \end{pmatrix} \tag{1}$$

where $\psi = \varphi, \theta$.

Due to symmetry of the problem, the angles φ and θ are defined with accuracy of up to πn. Mutual orientation of the optic axes S and O in the interface plane can be defined by the angle $\alpha = \theta - \varphi$. At the same time, we assume that $0 < \varphi < \pi$ is fulfilled and the angle θ is selected from a range from 0 to 2π so that $0 \le \alpha < \pi$. The so-defined α is equal to the angle counted from the axis S to the axis O clockwise, if we look from the side of the medium $x > 0$. Since the interface between the crystals disappears when the axes S and O coincide, then we also assume $\alpha \neq 0$. It is convenient to characterize the crystal anisotropy degree by a parameter $\eta \overset{\text{def}}{=} (\varepsilon_\parallel/\varepsilon_\perp) - 1$.

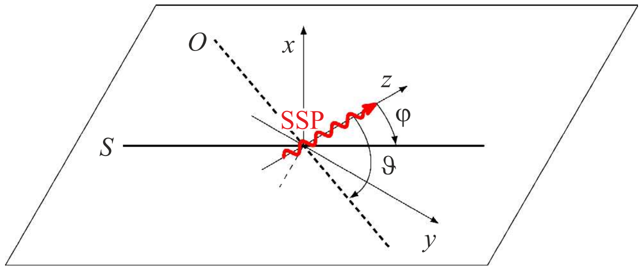

Figure 1. Mutual orientation of the direction of propagation of the wave and the optic axes.

3. Solution

In the chosen coordinate system, the field distributions of the surface wave in the most general form are represented by $\mathbf{E}(\mathbf{r};t) = \mathbf{e}(x)\exp\big(i(qz-\omega t)\big) + \text{c.c.}$, $\mathbf{H}(\mathbf{r};t) = \mathbf{h}(x)\exp\big(i(qz-\omega t)\big) + \text{c.c.}$. After substituting the solution in this form into Maxwell's equations, two projections of the vectors $\mathbf{e}(x)$ and $\mathbf{h}(x)$ can be explicitly expressed through the others. The remaining projections obey a homogeneous system of four linear first-order differential equations. Thus, the dependence of the fields on the normal coordinate x is determined by four roots of the characteristic equation of this system, which are written as follows:

$$\kappa_{o\pm} = \pm\sqrt{q^2 - \varepsilon_\perp\frac{\omega^2}{c^2}},$$

$$\kappa_{e\pm} = \pm\sqrt{(1+\eta\cos^2\psi)q^2 - \varepsilon_\parallel\frac{\omega^2}{c^2}}. \tag{2}$$

If $\kappa_{o\pm} \neq \kappa_{e\pm}$, then the system solution that is bounded when $|x| \to \infty$ is written as a sum of ordinary and extraordinary components $A_o\exp(\kappa_o x)$ and $A_e\exp(\kappa_e x)$. If $\kappa_{o\pm} = \kappa_{e\pm}$, then the solution have to be selected in the singular form $(A + Bx)\exp(\kappa x)$ [25,17].

Here, the case of coinciding roots in one of the crystals and different roots in the other is interesting for us. Let the solution have the singular form in the crystal $x < 0$, then from Eq. (2) we obtain

$$q^2 = \frac{\omega^2}{c^2}\frac{\varepsilon_\perp}{\cos^2\varphi} \tag{3}$$

wherein $\cos^2\theta \neq \cos^2\varphi$. The case $\cos^2\theta = \cos^2\varphi$ corresponds to the bisingular polariton and is realized if $\sin^2(\alpha/2) = 2/|\eta|$ or $\cos^2(\alpha/2) = 2/|\eta|$ [21]. It is clear that the equality (3) can only be satisfied if $\varepsilon_\perp > 0$. The singular solution also requires that the tensor (1) has nonzero off-diagonal components, i.e. $\eta \neq 0$ and $\varphi \neq 0, \pi/2$. Below, these conditions are assumed to be fulfilled.

We express all dimensional units in terms of $\sqrt{\varepsilon_\perp}(\omega/c)$ for convenience. If the positive coinciding roots of the characteristic equation (2) in the crystal $x < 0$ are designated as κ, then the negative roots in the crystal $x > 0$

will be $-\kappa$ and $-\lambda$, where

$$\kappa = |\,\mathrm{tg}\,\varphi|, \tag{4}$$

$$\lambda = \sqrt{\frac{1+\eta\cos^2\theta-(\eta+1)\cos^2\varphi}{\cos^2\varphi}}. \tag{5}$$

The compatibility condition of the standard boundary conditions for a non-zero solution of Maxwell's equations is reduced to

$$\lambda\{\sin^2\theta[\eta\cos^2\varphi+4]+\sin\theta\cos\theta[-2\eta\sin\varphi\cos\varphi]$$
$$+\cos^2\theta\,\mathrm{tg}^2\varphi[\eta\cos^2\varphi-4]\}=\kappa\{\sin^2\theta[\eta\cos^2\varphi-4]$$
$$+\sin\theta\cos\theta[-2\eta\sin\varphi\cos\varphi]+\cos^2\theta\,\mathrm{tg}^2\varphi[\eta\cos^2\varphi+4]\}. \tag{6}$$

If we specify φ and η then Eqs. (5), (6) can be solved relative to θ and λ (see Appendix). By analyzing the solutions, one can obtain the conditions of existence of the singular polariton in the form

$$\begin{cases}\cos^2\varphi > \dfrac{4}{4\sqrt{\eta(\eta+1)}-\eta},\\ \eta\in\left[-\dfrac{4}{3};\dfrac{4}{5}\right].\end{cases} \tag{7}$$

At the same time, for λ we have

$$\lambda = \kappa\,\frac{(\eta^2\cos^4\varphi-16)+4|\eta|\cos^2\varphi\sqrt{D}}{D+4\sin^2(2\varphi)}, \tag{8}$$

where $D=\big(\eta\cos^2\varphi+4\cos(2\varphi)\big)^2+12\sin^2(2\varphi)$, while for the angle θ we have

$$\cos^2\theta = \frac{\lambda^2-\mathrm{tg}^2\varphi+\eta}{\eta}\cos^2\varphi, \tag{9}$$

$$\mathrm{sign}\big(\sin(2\theta)\big)=\mathrm{sign}\{\lambda^2[-4-\eta\cos^2\varphi\cos(2\varphi)]+\lambda\kappa[-8]$$
$$+\kappa^2[-4+2\eta^2\cos^4\varphi+\eta\cos^2\varphi\cos(2\varphi)]\}\mathrm{sign}\big(\sin(2\varphi)\big). \tag{10}$$

If (7) are fulfilled, then the choice of φ within the interval $(0;2\pi]$ determines a pair of angles θ, which differ from each other by π. According to the definition of α, we obtain single-valued function $\alpha(\varphi)$.

It is worth noting interesting cases when the singular polariton propagates along or perpendicular to the optic axis of the crystal in which the solution is nonsingular. The case $\theta=\pi/2$ could be realized in the hyperbolic medium when $\eta<-4$ and

$$\varphi = \pm\arcsin\left(\frac{|\eta|-4}{|\eta|}\right). \tag{11}$$

The case $\theta=0$ could be realized in a strongly anisotropic dielectric if $\eta>8$ and

$$\varphi = \pm\arccos\left(\frac{2\big(1+\sqrt{\eta+1}\big)}{\eta}\right). \tag{12}$$

Let us have mutual orientation of the optic axes S and O (angle α) is specified, rather than the direction of propagation of the singular polariton relative to the optic axis S (angle φ). The direction of counting the angles φ relative to the axis S is counterclockwise if we are looking from the side of the medium $x>0$. By collecting terms at η into one side in Eq. (6), using Eqs. (4)–(5), and cancelling by $(\kappa-\lambda)$, we obtain the compatibility condition of boundary conditions in the form

$$(\kappa+\lambda)^2=\frac{\eta^2}{4}\sin^2\alpha. \tag{13}$$

We note that Eq. (13) can be obtained from Eq. (6) in paper [26] by setting localization constants of ordinary and extraordinary wave in one of the media to be the same, for example, $k_0=k_1=\kappa$ and $k_2=\lambda$. However, the choice of polarizations of partial waves from works jcite24,26 is incorrect if such degeneracy occurs.

From Eqs. (4), (5), and (13) one can obtain the analytical dependence $\varphi(\alpha)$ and existence conditions of singular polariton at fixed η. We give it in the second part of Appendix. Main specific features of $\varphi(\alpha)$ at different η are clearly represented by a family of curves constructed in Figure 2, *a* and Figure 2, *d*. Figure 2, *a* shows the dependences $\varphi(\alpha)$ in hyperbolic media where $\eta<-4/3$, and Figure 2, *d* shows the dependences in dielectric media where $\eta>4/5$. Empty circle marks on the curves correspond to the propagation directions of the bisingular polariton when $\theta=\pi-\varphi$ or $\theta=2\pi-\varphi$, i.e. $\alpha=\pi-2\varphi$ or $\alpha=2\pi-2\varphi$.

At a certain angle between optic axes $\tilde{\alpha}$ one can determine a number of the pairs of SSP propagation directions by a number of intersections of the vertical line $\alpha=\tilde{\alpha}$ with the curves $\varphi(\alpha)$ for specific η. It should be noted that the curves transform into each other when replacing $\varphi\to\pi-\varphi$ and $\alpha\to\pi-\alpha$. Thus, the number of SSP propagation directions does not depend on the choice of the large or the small angle between optic axes as $\tilde{\alpha}$.

It is clear from Figure 2, *a* that up to 4 pairs of the directions of propagation of the singular polariton are possible. When $-4/3>\eta>-(1+\sqrt{17})/2\approx-2.56$ only one pair of the directions is possible. When $\eta\in(-\infty;-5)\cup\big(-4;-(1+\sqrt{17})/2\big)$ two pairs of the directions are possible for some angles between optic axes, wherein when $\eta\in\big(-4;-(1+\sqrt{17})/2\big)$ the direction exists for any α. In crystals where $-5<\eta<-4$ for angles α near $\pi/2$ four pairs of the directions are possible. In the case of dielectric crystals (Figure 2, *d*) two pairs of the directions are possible if $\eta>\big(-1+\sqrt{17}\big)/2\approx1.56$. Within the same interval of the values of η SSP propagation direction exists for any angle α.

When η are negative, the vectors $\mathbf{q}$ are illustrated by selecting a case when there are four pairs of propagation directions and $\theta=\pi/2$ for one of them (Figure 2, *b*). It turns out that in this case one of the three other pairs of the directions necessarily correspond to bisingular polariton. When η are positive, the vectors $\mathbf{q}$ are illustrated by selecting

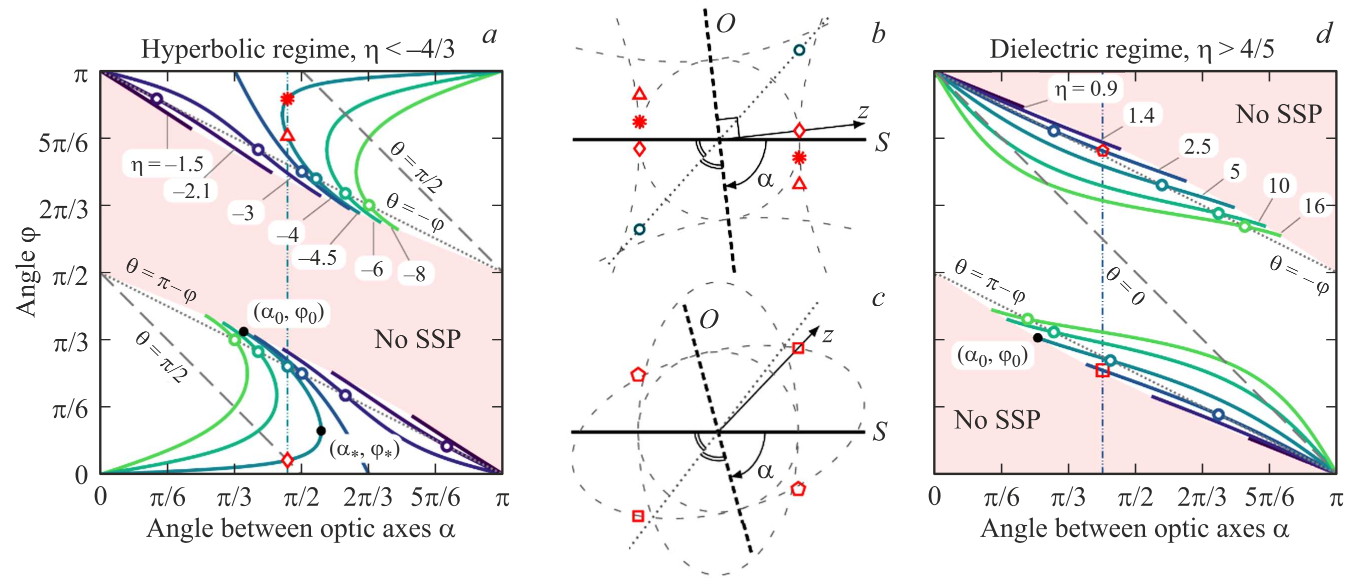

Figure 2. Dependences of the propagation angles of singular surface polariton (SSP) φ on the angle between the optic axes α: a — in the hyperbolic crystals when $\eta < -1$; d — in the dielectric crystals when $\eta > 0$. b — the SSP wave vectors $\mathbf{q}$ for the crystals where $\eta = -4.5$ and angle $\alpha \approx 1.46$. c — the SSP wave vectors $\mathbf{q}$ for the crystals where $\eta = 2.5$, $\alpha = 5\pi/12$. Marks on graphs b and c correspond to the same marks on graphs a and d for the selected angles α.

a case when there are two pairs of propagation directions (Figure 2, c). All vectors $\mathbf{q}$ marked on Figure 2, b and 2, c lie on lines that are perpendicular to the optic axis S and pass through tangent points of light cones boundaries of ordinary and extraordinary waves of the medium $x < 0$ (the curves $\kappa_{o+} = 0$ and $\kappa_{e+} = 0$ in (2)). In [21,23] these lines are called singular axes of surface waves. The directions marked by the circles in Figure 2, b correspond to bisingular polariton and lie on an intersection of singular axes of surface waves of both the media and on a bisector of the angle between the optic axes.

4. Fields

Electric field in the singular polariton is described by the expression

$$\mathbf{E}(\mathbf{r};t) = \mathrm{C}[\mathbf{e}(x)\exp\big(i(qz - \omega t)\big) + \text{c.c.}],$$

where C — the arbitrary constant, $\mathbf{e}(x)$ — the polarization vector that is obtained by direct calculation from the boundary conditions. It is given by the expressions

$$\begin{cases} e_x = iq\Big[\Big(K + \dfrac{\lambda}{\kappa}L\Big)e^{\kappa x} + 2(K+M)\kappa x e^{\kappa x}\Big], \\ e_y = \sigma\Big[\Big(\dfrac{4(K+M)}{\eta\cos^2\varphi} + K + L\Big)e^{\kappa x} - 2(K+M)\kappa x e^{\kappa x}\Big], \\ e_z = \kappa[(K+L)e^{\kappa x} - 2(K+M)\kappa x e^{\kappa x}], \end{cases}$$

when $x < 0$ and

$$\begin{cases} e_x = iq\Big[Ke^{-\kappa x} + \dfrac{\lambda}{\kappa}Le^{-\lambda x}\Big], \\ e_y = \sigma[Me^{-\kappa x} + Ne^{-\lambda x}], \\ e_z = \kappa[Ke^{-\kappa x} + Le^{-\lambda x}], \end{cases}$$

when $x > 0$.

The following notation is introduced here

$$\sigma \stackrel{\text{def}}{=} -\text{sign}(\text{tg}\,\varphi),$$

$$K \stackrel{\text{def}}{=} \big((\kappa+\lambda)/2\kappa\big)\cos\varphi\sin\theta,$$

$$L \stackrel{\text{def}}{=} \sin\varphi\cos\theta,$$

$$M \stackrel{\text{def}}{=} \big((\kappa+\lambda)/2\kappa\big)\sin\varphi\cos\theta,$$

$$N \stackrel{\text{def}}{=} \cos\varphi\sin\theta.$$

The magnetic field dependence

$$\mathbf{H}(\mathbf{r};t) = \mathrm{C}[\mathbf{h}(x)\exp\big(i(qz - \omega t)\big) + \text{c.c.}]$$

can be obtained by calculating rot $\mathbf{E}$. Since Eqs. (3), (4), (8), and also (9), and (10) define q, κ, λ and θ as functions of $\varepsilon_\perp$, $\varepsilon_\parallel$ and φ, therefore, the above-given expressions for fields (14) can also be regarded as functions of these parameters. By virtue of the expressions for $\varphi(\alpha)$ (see Appendix) and Eq. (3), we can choose α or q as one of the given values.

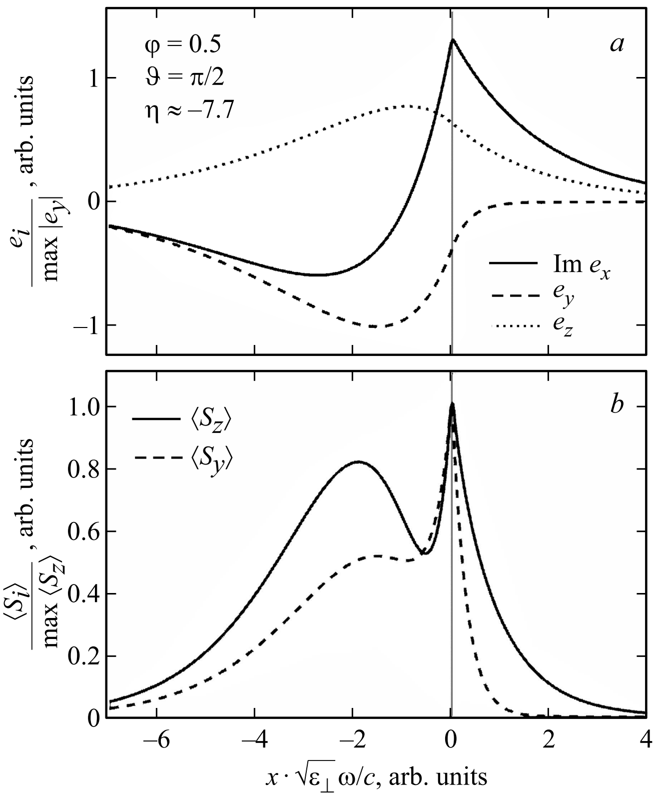

Figure 3. Dependence of the following vectors on the coordinate x in singular surface polariton: a — of electric field polarization e; b — of the averaged Poynting vector $\langle\mathbf{S}\rangle$.

Figure 3 shows dependences of the polarization vector projections $\mathbf{e}(x)$ and the $2\pi/\omega$-period-averaged Poynting vector

$$\mathbf{S}(x) = \frac{c}{4\pi}\,[\mathbf{E}\times\mathbf{H}].$$

Parameters for calculation were selected to be $\varphi = 0.5$ and $\eta = -4/(1-|\sin\varphi|)$, which corresponds to $\theta = \pi/2$. We note that in the medium with the singular solution $(x < 0)$ the projections of $\mathbf{e}$ behave nonmonotonically (Figure 3, a). It is a specific feature of the surface waves in the anisotropic media, whereas in the isotropic ones the fields are localized strictly on the interface. The nonmonotonic dependence of field amplitudes on the distance to the interface can also be observed for nonsingular solutions if contributions by the usual and unusual partial waves are different in a sign. It is clear in Figure 3, b that the dependences $\langle S_y\rangle$ and $\langle S_z\rangle$ are also nonmonotonic, wherein in the medium $x < 0$ each of them has a local maximum (it is weakly noticeable for $\langle S_y\rangle$). We also note that the ratio $\langle S_y\rangle/\langle S_z\rangle$ that determines „the wave energy propagation direction" also depends on x. Since directions of group and phase wave velocities in the anisotropic media often do not coincide the difference of $\langle S_y\rangle$ from zero is not surprising. The unplotted average value $\langle S_x\rangle = 0$ that corresponds to energy transfer by the wave only along the interface.

5. Conclusion

Singular surface polaritons that propagate along the interface of the two identical uniaxial crystals has been investigated. It is predicted that they could exist in such a frequency range where the crystal anisotropy degree

$$\eta = \left(\frac{\varepsilon_\parallel}{\varepsilon_\perp} - 1\right) < -4/3$$

or $\eta > 4/5$ when $\varepsilon_\perp > 0$, where $\varepsilon_\parallel$ and $\varepsilon_\perp$ are principal components of the permittivity tensor. Such values of the anisotropy degree can be observed in natural uniaxial minerals in the infrared or terahertz wavelength range, and they can also be achieved in metamaterials for a wider wavelength range. At the same time, these requirements in the case of two uniaxial crystals are weaker than in the case of the anisotropic medium/isotropic medium interface.

The propagation direction of singular surface polariton is defined by angle φ relative to the optic axis of the crystal, where the solution has singular form, and by a correctly selected angle between the optic axes α. We have obtained the analytical expressions for angles $\alpha(\varphi)$ and $\varphi(\alpha)$ in a dependence on selecting one of them as a predefined parameter. Singular polariton propagates only along special directions, therefore, its existence conditions depend on angles φ or α. These conditions were determined here. At a certain anisotropy degrees and ranges of the angles between the optic axes, up to 4 (2) pairs of propagation directions are possible in a hyperbolic (dielectric) regime. It is interesting to note that when $\eta < -4$ surface polariton propagates perpendicular to the optic axis of the crystal, in which the field distribution has nonsingular form, and when $\eta > 8$ it propagates parallel to that. These directions are forbidden in the crystal with the singular form of the field distribution. We have obtained the analytical expressions for distributions of components of the electromagnetic field in the singular polariton, which are determined as functions of the permittivities $\varepsilon_\perp$, $\varepsilon_\parallel$ and one of angles φ, α or wave vector q. The expressions obtained in the study can be used for constructing solutions of the problem on propagation of the nonsingular waves in the vicinity of the propagation directions of singular surface polariton.

Appendix

Solution with respect to φ and η. Let us find λ, θ and regions of existence of the singular waves at given φ and η. Let us express $cos^2\theta$ and $\sin^2\theta$ from Eq. (5):

$$\cos^2\theta = \frac{\lambda^2 - \kappa^2 + \eta}{\eta}\cos^2\varphi,$$

$$\sin^2\theta = \frac{(1+\eta)\kappa^2 - \lambda^2}{\eta}\cos^2\varphi. \qquad \text{(A1)}$$

Using expressions (A1) we can rewrite Eq. (6) as

$$(\lambda-\kappa)\{\lambda^2[-4-\eta\cos^2\varphi\cos(2\varphi)]+\lambda\kappa[-8] + \kappa^2[-4+2\eta^2\cos^4\varphi+\eta\cos^2\varphi\cos(2\varphi)]\} = (\lambda-\kappa)\operatorname{sign}\big(\sin(2\varphi)\sin(2\theta)\big)2|\eta|\,|\sin\varphi\cos^3\varphi| \times\sqrt{-\lambda^4+\lambda^2[(2+\eta)\kappa^2-\eta]+(1+\eta)(\eta-\kappa^2)\kappa^2}. \quad \text{(A2)}$$

Since $\lambda\neq\kappa$, we can cancel out $(\lambda-\kappa)$. By squaring both the parts of the equation, we obtain the fourth-power equation with respect to λ. Its solutions include a double root $\lambda=-\kappa$ and a pair of roots

$$\lambda=\kappa\,\frac{(\eta^2\cos^4\varphi-16)\pm4|\eta|\cos^2\varphi\sqrt{D}}{D+4\sin^2(2\varphi)},$$

where $D=\big(\eta\cos^2\varphi+4\cos(2\varphi)\big)^2+12\sin^2(2\varphi)$.

It is necessary to determine for which $(\varphi;\eta)$ $\lambda>0$. Boundaries of constant-sign regions of λ can be found from the very equation on λ when $\lambda=0$. By studying the signs of λ inside each of them, one can obtain that the root (8) is positive for

$$\eta\in\left(-\infty;\frac{-4\cos(2\varphi)-4\sqrt{\cos^2(2\varphi)+15}}{15\cos^2\varphi}\right) \cup\left(\frac{-4\cos(2\varphi)+4\sqrt{\cos^2(2\varphi)+15}}{15\cos^2\varphi};+\infty\right),$$

while the other roots $\lambda\le0$ at any $(\varphi;\eta)$. This condition can be rewritten with respect to φ as

$$\begin{cases}\cos^2\varphi>\dfrac{4}{4\sqrt{\eta(\eta+1)}-\eta},\\ \eta\in\left[-\dfrac{4}{3};\dfrac{4}{5}\right].\end{cases}$$

Then, it is required to check that $0\le\cos^2\theta\le1$ is fulfilled, where $\cos^2\theta$ is determined by (A1) and (4), (8). Fulfillment of these inequalities can be verified by studying positive-sign regions of $\cos^2\theta$ and $\sin^2\theta$ similar to the investigation where $\lambda>0$, using Eqs. (A1), and (6). Lack of the negative sign under a root in (A2) follows from $0\le\cos^2\theta\le1$ by virtue of (A1), while matching of the signs of the right-hand and left-hand parts is provided by selecting a sign of $\sin(2\theta)$.

Solution with respect to α and η. Let us find propagation directions of singular waves, which are characterized by angle φ at given α and η. Since values κ and λ are positive, Eq. (13) can be rewritten as

$$\lambda=\frac{|\eta\sin\alpha|}{2}-\kappa. \quad \text{(A3)}$$

Squaring (A3) and using Eqs. (4), (5), we come to the following equation that defines $\operatorname{tg}\varphi$ as a function of α and η:

$$\operatorname{tg}^2\varphi+\left[\frac{\operatorname{sign}(\eta\sin\alpha)\operatorname{sign}(\operatorname{tg}\varphi)-2\cos\alpha}{\sin\alpha}\right]\operatorname{tg}\varphi-\left[\frac{4+\eta}{4}\right]=0. \quad \text{(A4)}$$

Equation (A4) is equivalent to two square equations for $\operatorname{tg}\varphi$ of the different signs. Its roots are required to be real, bound and strictly positive or negative as well as to provide for fulfillment of the condition $\lambda>0$ in (A3) written as $2|\operatorname{tg}\varphi|<|\eta\sin\alpha|$. Since Eq. (A4) and the conditions imposed on its roots are invariant relative to a transformation $\varphi\to\pi-\varphi$, $\alpha\to\pi-\alpha$, and $\operatorname{tg}\varphi$ changes the sign during this transformation, then one can limit oneself by considering when $\varphi\in(0;\pi/2)$. The next positive solutions correspond to this region

$$\operatorname{tg}\varphi_\pm=\frac{2\cos\alpha-\operatorname{sign}(\eta)\pm\sqrt{\big(2\cos\alpha-\operatorname{sign}(\eta)\big)^2+(4+\eta)\sin^2\alpha}}{2\sin\alpha}. \quad \text{(A5)}$$

In media where $-4\le\eta<-4/3$ or $4/5<\eta$, SSP propagation is possible only in the directions that are determined by $\operatorname{tg}\varphi_+$, wherein the angle α shall belong to the interval $(\alpha_0;\pi)$. In media where $\eta<-4$, SSP propagation is possible along the directions $\operatorname{tg}\varphi_+$ when $\alpha\in(\alpha_0;\alpha_*]$ and along the directions $\operatorname{tg}\varphi_-$ when $\alpha\in(0;\alpha_*]$. When $\eta=-4$, the directions $\operatorname{tg}\varphi_+$ are only possible when $\alpha\in(\alpha_0;2\pi/3)$. The boundaries α_0 and α_* are given by

$$\alpha_0=\arccos\left(\frac{\sqrt{\eta(\eta+1)}-2}{|\eta|}\right); \quad \text{(A6)}$$

$$\alpha_*=\arccos\left(\frac{\sqrt{\eta^2+5\eta+4}-2}{|\eta|}\right). \quad \text{(A7)}$$

The boundary α_0 corresponds to $\lambda=0$, while the boundary α_* corresponds to zeroing of the radical in (A5). The condition $\alpha<2\pi/3$ for $\operatorname{tg}\varphi_+$ and lack of the directions $\operatorname{tg}\varphi_-$ when $\eta=-4$ follow from the requirement $\operatorname{tg}\varphi_\pm>0$. By substituting α_0 into (A5), we obtain an expression for the largest angle of propagation $\varphi\in(0;\pi/2)$ in the form

$$\varphi_0=\operatorname{arctg}\left(\sqrt{\sqrt{\eta(\eta+1)}-1-\frac{\eta}{4}}\right). \quad \text{(A8)}$$

Acknowledgments

The authors thank N.S. Averkiev for assistance in the article preparation.

References

[1] M.I. D'yakonov. Sov. Phys. JETP **67**, *4*, 714 (1988). [M.I. D'akonov. ZhETF **94**, *4*, 119 (1988). (in Russian).]

[2] O. Takayama, L. Crasovan, D. Artigas, L. Torner. Phys. Rev. Lett. **102**, *4*, 043903-1 (2009).
[3] O. Takayama, L.C. Crasovan, S.K. Johansen, D. Mihalache, D. Artigas, L. Torner. Electromagnetics **28**, *3*, 126 (2008).
[4] J.A. Polo, A. Lakhtakia. Laser Photonics Rev. **5**, *2*, 234 (2011).
[5] V.I. Alshits, V.N. Lyubimov. Phys. Solid State **44**, *2*, 386 (2002). [V.I. Al'shits, V.N. Lyubimov. FTT **44**, *10*, 1895 (2002). *2*, 371 (2002). (in Russian).]
[6] L. Torner, J.P. Torres, C. Ojeda, D. Mihalache. J. Light. Technol. **13**, *10*, 2027 (1995).
[7] A.N. Furs, V.M. Galynsky, L.M. Barkovsky. Optics and Spectroscopy, **98**, *3*, 454 (2005).
[8] O. Takayama, D. Artigas, L. Torner. Nat. Nanotechnol. **9**, *6*, 419 (2014).
[9] D.A. Chermoshentsev, E.V. Anikin, S.A. Dyakov, N.A. Gippius. Nanophotonics **9**, *16*, 4785 (2020).
[10] D.A. Chermoshentsev, E.V. Anikin, I.M. Fradkin, M.S. Sidorenko, A.A. Dudnikova, A.S. Kalganov, M.F. Limonov, N.A. Gippius, S.A. Dyakov. Nanophotonics, **13**, *16*, 3005 (2024).
[11] D. Artigas, L. Torner. Phys. Rev. Lett. **94**, *1*, 013901 (2005).
[12] O. Takayama, A.A. Bogdanov, A.V. Lavrinenko. J. Phys. Condens. Matter **29**, *46*, 463001 (2017).
[13] K. Korzeb, M. Gajc, D. Pawlak. Opt. Express **23**, *20*, 25406 (2015).
[14] S. Jahani, Z. Jacob. Nat. Nanotechnol. **11**, *1*, 23 (2016).
[15] O. Takayama, E. Shkondin, A. Bodganov, M.E. Aryaee Panah, K. Golenitskii, P. Dmitriev, T. Repaän, R. Malureanu, P. Belov, F. Jensen, A.V. Lavrinenko. ACS Photonics **4**, *11*, 2899 (2017).
[16] M. Moradi. Sci. Rep. **13**, 12353 (2023).
[17] F.N. Marchevskii, V.L. Strizhevskii, S.V. Strizhevskii. Sov. Phys. Solid State, **26**, 901 (1984). [Marchevskii, V.L. Strizhevskii, S.V. Strizhevskii FTT **26**, *5*, 1501 (1984). (in Russian).]
[18] A. Lakhtakia, T.G. Mackay, C. Zhou. Eur. J. Phys. **42**, *1*, 015302-1 (2021).
[19] T.G. Mackay, C. Zhou, A. Lakhtakia. Proc. R. Soc. A **475**, *2228*, 20190317-1 (2019).
[20] L.D. Landau, E.M. Lifshitz. Elektrodinamika sploshnykh sred. Nauka, M. (1992). 664 s. (§ 99). (in Russian).
[21] K.Yu. Golenitskii, N.S. Averkiev. Phys. Rev. A, **111**, *6*, 063508 (2025).
[22] A. Lakhtakia, T.G. Mackay. J. Opt. Soc. Am. B **37**, *8*, 2444 (2020).
[23] K.Yu. Golenitskii. Phys. Rev. B, **110**, *3*, 035301 (2024).
[24] E.E. Narimanov. Phys. Rev. A, **98**, *1*, 013818 (2018).
[25] A.P. Khapalyuk. Optics and Spectroscopy **12**, *1*, 106 (1962).
[26] N.S. Averkiev, M.I. Dyakonov. Optics and Spectroscopy **68**, *3*, 1118 (1990).

Translated by M.Shevelev