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Stokes and Navier–Stokes models for describing the instability of the charged boundary of a conducting liquid

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We consider the developed stages of the Tonks–Frenkel instability of the free charged surface of a conducting liquid, when, due to the sharpening of the boundary, the scale of the problem is reduced to the microscopic one, and viscous effects begin to play a decisive role. In such a situation, the Stokes approximation is often used to study the flow of fluids. However, as demonstrated, a feature of the Tonks–Frenkel instability with its characteristic explosive behavior of a number of physical quantities is the incorrectness of its analysis within the framework of this approximation. A correct description of the fluid dynamics during the singularity formation requires the use of the full Navier–Stokes equations.

Keywords: Tonks–Frenkel instability, Taylor cone, Navier–Stokes model, Stokes approximation.

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The free surface of a conducting liquid is unstable in a sufficiently strong external electric field [1–3]. The mechanism of this Tonks–Frenkel (TF) instability is as follows: under the influence of a field with strength E, a free surface electric charge of density $\sigma = \varepsilon_0 E$, where ε_0 is the electric constant, is induced at the liquid boundary. The interaction of the field with this charge leads to the emergence of a force directed outward from the liquid. It is convenient to characterize the action of this force by introducing electrostatic pressure $p_E = \sigma E/2 = \varepsilon_0 E^2/2$. When the boundary is deformed, the field strength at wave crests will exceed the strength at troughs. A p_E pressure gradient arises, leading to further deformation of the boundary. If the influence of the gravity field is neglected, the unstable modes for an unbound flat surface are those with wave numbers $k < k_c = \varepsilon_0 E_0^2/\alpha$, where E_0 is the external field strength (it is assumed that it is uniform and directed along the normal to the unperturbed (flat) boundary) and α is the surface tension coefficient. In the region of $k > k_c$ (i.e., at small-scale perturbations of the boundary), instability is suppressed by capillary forces.

The most important feature of the TF instability is its explosive nature. At nonlinear stages, a formally unlimited sharpening of the surface is observed. As a result, the surface curvature, the local electric field, and the liquid velocity [4–7] become infinite within a finite time. The formation of a conical tip completes the hydrodynamic stage of evolution of the system. Further analysis requires the introduction of emission processes that determine the behavior of the system at local fields on the order of $\sim 10^8$ V/cm and above [8]. Note that one should not confuse stationary conical formations (conventional Taylor cones [9]) with dynamic cones that arise at the final stages of instability development [10].

When a conical tip forms, the characteristic spatial scale of the problem (λ) changes radically. According to estimates [7], this scale decreases by six orders of magnitude (from 10^{-2} to 10^{-8} m) in transition from the onset of instability development to the late stages where emission processes are initiated, while the characteristic time (τ) changes even more significantly, by nine orders of magnitude ($\tau^2 \propto \lambda^3$ scaling). With such a significant change in spatial and temporal scales, numerical methods become ineffective, and the TF instability may be studied only analytically in the $\lambda \to 0$, $\tau \to 0$ limit. The key objectives of such research will be to determine the behavior of fundamental physical quantities near the singularity and to find the limiting angle of the forming cone. The present study is focused on the analysis of correctness of various approaches to problem solving.

While the liquid may be considered ideal (i.e., its viscosity may be neglected) at the initial stages of instability development on the macroscale, viscous effects will inevitably become prominent at the final stages on the microscale [5–7]. In terms of Reynolds number $\mathrm{Re} = \rho\lambda^2/(\eta\tau)$, where ρ is the liquid density and η is the dynamic viscosity, we have $\mathrm{Re} \propto \lambda^{1/2}$ (with the $\tau^2 \propto \lambda^3$ scaling taken into account). The Reynolds number decreases in the course of evolution of the system (i.e., as scales λ and τ grow smaller). The curvature radius of the apex of the forming tip and the time remaining until the moment of singularity formation may be used as estimates for λ and τ, respectively. As is known, viscous effects dominate over dynamic ones at sufficiently small Re values. Naturally, the extreme complexity of analytical study of flows of liquids with a free boundary makes any reasonable simplification of the description of nonlinear stages of the TF instability worthwhile. A question arises whether the conclusion about

the dominance of viscous effects can be used in some way.

The use of the Stokes approximation (i.e., the transition to formal limit $\mathrm{Re} = 0$; see and example of its use in microscale electrohydrodynamics in [11]) appears to be a natural simplification. In the context of instability of the charged surface of a conducting liquid, the Stokes approximation was applied in [12,13]. It was demonstrated in these studies that conical tips form on the surface of charged and neutral droplets in an electric field within a finite time. It was concluded that scaling $E \propto \tau^{-1/2}$, which is established by the balance between electrostatic and viscous forces, holds true for the field strength at the tip apex. Expression $u \propto \lambda/\tau \propto \tau^{\chi-1}$, where χ is a certain exponent, is valid for liquid velocity u and scale λ. It is noteworthy that the value of χ cannot be determined unambiguously within the Stokes approximation used in [12,13]. The angle of the forming cone (β) is also non-universal: it is affected by the initial conditions (in most cases, $50 \leqslant \beta \leqslant 60^\circ$).

Note that the identified uncertainties in the χ exponent and the β angle are interrelated. In terms of scale λ, the local field strength follows scaling $E \propto \lambda^{-1/(2\chi)}$, and the field potential then follows scaling $\varphi \propto E\lambda \propto \lambda^{1-1/(2\chi)}$. It follows from the known solution for the distribution of the field potential around a conducting cone [7,13] that the $\varphi \propto \lambda^\gamma$ dependence, where exponent γ is related to cone angle β as $P_\gamma\big(-\cos(\beta/2)\big) = 0$, is valid. Here, P_γ is the Legendre function of order γ and the exponent increases from 0 to 1 as the angle changes from 0 to π. Since $\gamma = 1 - 1/(2\chi)$ in the examined case, the uncertainty in χ underlies the uncertainties in γ and β.

An approach to the analysis of developed stages of the TF instability was proposed in [5] for an ideal liquid and generalized to the case of a viscous liquid in [7]. It relies on the following technique: the relation between the spatial and temporal scales is found from the dispersion law of linear surface waves, and the influence of nonlinearity is taken into account by replacing external field E_0 with local field E. Despite its simplicity, this approach allows one to take into account the fact that an increase in field strength at the apex of the forming tip leads to a change in the spectrum of unstable modes. This makes it possible to describe the process of variation (reduction) of scales λ and τ during the formation of a singularity. The application of this approach to an ideal liquid revealed the above-mentioned $\tau^2 \propto \lambda^3$ scaling and the fact that the dynamic cone angle matches the static Taylor cone angle of 98.6°. These results are consistent with both experimental data [10] and numerical calculations in the limit of large Re [5,6]. In the case of a viscous liquid within the Navier–Stokes model, the discussed approach predicts a three-fold smaller limiting angle $\beta \approx 33.1^\circ$ with the same scaling corresponding to $\chi = 2/3$. This contradicts the results obtained in the Stokes limit: according to [12,13], quantities β and χ are non-universal and depend on the initial conditions. Let us figure out the reason for this discrepancy and which approach should be trusted.

The dispersion law of electrocapillary waves for a viscous liquid corresponding to the aperiodic regime of motion takes the following form in the Navier–Stokes model [7,14,15]:

$$\rho\big(\gamma + 2\eta k^2/\rho\big)^2 - \varepsilon_0 E_0^2 k^2 + \alpha k^3 = 4\rho^{-1/2}\eta^{3/2}k^3\sqrt{\gamma + \eta k^2/\rho}, \quad (1)$$

where $\gamma > 0$ is the real increment of instability (for comparison, see [16] for gravity and [17] for capillary-gravity waves). Analyzing the limiting case of $\mathrm{Re} = 0$, one may derive the dispersion law for the Stokes approximation from (1). Since $\mathrm{Re} \propto \rho\gamma/(\eta k^2)$, this approximation corresponds to the formal limit $\rho \to 0$. Thus, we obtain $\gamma = \big(\varepsilon_0 E_0^2 - \alpha k\big)/(2\eta) = \alpha(k_c - k)/(2\eta)$.

It can be seen that $\gamma > 0$ (i.e., the surface is unstable) at $k < k_c$ (viscosity does not affect the range of unstable wave numbers). A peculiar feature of the Stokes limit is that the maximum of the instability increment is always found at $k = 0$. It is written as $\gamma_{\mathrm{St}} = \varepsilon_0 E_0^2/(2\eta)$.

Dimensionless variables $\kappa = k/k_c$ and $\Gamma = \gamma/\gamma_{\mathrm{St}}$ are convenient for further analysis. The dispersion law in the Stokes approximation then assumes a compact form

$$\Gamma = 1 - \kappa. \quad (2)$$

The following is obtained for the initial Navier–Stokes model:

$$\big(\Gamma + 2\delta^2\kappa^2\big)^2 - 2\delta^2\kappa^2(1-\kappa) - 4\delta^3\kappa^3\sqrt{\Gamma + \delta^2\kappa^2} = 0, \quad (3)$$

where dimensionless control parameter

$$\delta = 2^{1/2}\varepsilon_0^{1/2}\rho^{-1/2}\alpha^{-1}\eta E_0.$$

This parameter may be interpreted as the ratio of characteristic increments for ideal ($\gamma_{id} \propto \varepsilon_0^{3/2}E_0^3/(\alpha\rho^{1/2})$, see [5,7]) and highly viscous (γ_{St}) liquids; i.e., $\delta \propto \gamma_{id}/\gamma_{\mathrm{St}}$.

Let us consider how relation (3) between increment Γ and wave number κ varies with a change in parameter δ (i.e., with a change in applied field E_0). At $\delta > 1$, dispersion law (3) corresponding to the Navier–Stokes model is approximated with fine accuracy by an expression resolved with respect to the increment:

$$\Gamma \approx 2\delta\kappa\left(\sqrt{3 - 3\kappa + \delta^2\kappa^2} - \delta\kappa\right)/3. \quad (4)$$

The figure presents dependences $\Gamma(\kappa)$ characterized by (4) for $\delta = 2$, 6, 20, and 100 (colored curves). Dependence (2) corresponding to the Stokes approximation (black line) is also shown. It is clear how the limit transition from the Navier–Stokes model to the Stokes approximation proceeds: as δ grows, dependences (4) approach asymptotically linear dependence (2) in the region of $0 < \kappa \leqslant 1$.

It may seem that the illustrated transformation of the $\Gamma(\kappa)$ dependence with variation of parameter δ testifies to the

correctness of application of the Stokes approximation in analysis of developed stages of the TF instability. However, this is not true. The limit transition does not occur at point $\kappa = 0$. At this point, $\Gamma = 1$ in the Stokes approximation; in the Navier–Stokes model, we have $\Gamma = 0$ with any δ. Let us discuss how such a discrepancy will manifest itself during the formation of a singularity.

The decisive role in development of the TF instability is played by the so-called dominant mode with the largest increment (i.e., the one corresponding to the $\delta\Gamma/\partial\kappa = 0$ extremum condition). Let us denote the corresponding increment and wave number as Γ_d and κ_d ($\gamma_d \equiv \Gamma_d\gamma_{\mathrm{St}}$ and $k_d \equiv \kappa_d k_c$ in the dimensional form). Consequently, mode κ_d will „win“ (i.e., reach a higher amplitude) the competition between different modes from the $0 < \kappa < 1$ range already at linear stages of instability. This mode will be the one setting the characteristic scale.

The growth of the tip leads to an enhancement of the field strength at its apex. It is clear that the liquid behavior near the apex at nonlinear stages of instability will be governed not by external field E_0, but by local field E that exceeds it significantly. In effect, this will make control parameter δ variable. If we assume that it is determined by local field E, its value will increase in the process of instability development, and this will transform the dispersion law. The figure may then be interpreted as an illustration of evolution of the dispersion law with an increase in local field at the apex of the protrusion. It is fundamentally important that the position of the maximum of the $\Gamma(\kappa)$ dependence (i.e., the parameters of the dominant instability mode) will change.

At large δ, we derive the following from (4) in the leading order: $\kappa_d \approx (3/2)^{1/3}\delta^{-2/3}$, $\Gamma_d \approx 1 - (3/2)^{4/3}\delta^{-2/3}$ (i.e., $\kappa_d \to 0$ and $\Gamma_d \to 1$ at $\delta \to \infty$, which is what is illustrated in the figure). One might get the wrong impression that a transition to the Stokes regime with dominant mode $\kappa = 0$, which has $\Gamma = 1$, occurs. In actual truth, we obtain the following in the initial dimensional variables (external field E_0 was substituted here with local field E):

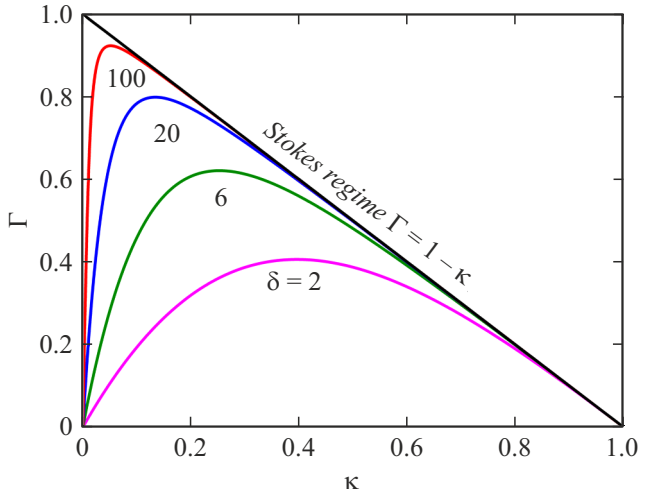
$$k_d \approx \left(\frac{3\rho\varepsilon_0^2 E^4}{4\alpha\eta^2}\right)^{1/3}, \gamma_d \approx \frac{\varepsilon_0 E^2}{2\eta} - \frac{3^{4/3}\varepsilon_0^{2/3}E^{4/3}\rho^{1/3}\alpha^{2/3}}{2^{8/3}\eta^{5/3}}. \quad (5)$$

Thus, $k_d \propto E^{4/3}$ and k_d increases with field strength. Consequently, the maximum in the Navier–Stokes model moves away from its position ($k = 0$) corresponding to the Stokes regime.

Relation $E \propto \lambda^{-3/4}$ between scale $\lambda \propto 1/k_d$ of the problem and the local field follows from (5). Due to the $E \propto \lambda^{-1/2\chi}$ scaling, it corresponds to $\chi = 2/3$. Thus, index χ is determined unambiguously when one uses the Navier–Stokes model. It has been demonstrated recently in [7] that this χ value corresponds to limiting cone angle $\beta \approx 33.1°$: condition $P_\gamma\big(-\cos(\beta/2)\big) = 0$ is satisfied for it at $\gamma = 1 - 1/(2\chi) = 1/4$. Note that sharpening of the apex may be interpreted as a reduction in spatial scale $\lambda \propto 1/k_d$ due to an increase in wave number of the dominant mode. This interpretation is meaningless within the Stokes model, since there is no shift of the maximum in the dispersion law; the maximum always remains at $k = 0$.

Let us summarize our analysis. At the final stages of development of the TF instability when the characteristic spatial scale is reduced infinitely due to sharpening of the surface, viscous effects start to play a significant role. Under their influence, the instability increment in the leading order grows as $\gamma_d \approx \varepsilon_0 E^2/(2\eta)$ (i.e., as in the Stokes regime [12,13]). This expression does not include surface tension coefficient α, suggesting that a dynamic balance is established between viscous and electrostatic forces. A qualitative difference between the Stokes limit and the Navier–Stokes model is in degeneracy of the dominant instability mode. In the case of Navier–Stokes equations, the dependence of the increment on the wave number features a maximum at a finite value $k_d \propto E^{4/3}$, which increases at the apex of the protrusion due to the enhancement of local field E. In the Stokes regime, the maximum increment is always found at $k = 0$ and does not shift. This is the reason why the value of exponent χ in dependence $\lambda \propto \tau^\chi$ cannot be determined unambiguously within the Stokes model. An exact value of $\chi = 2/3$ is obtained in the more general Navier–Stokes model.

It can be concluded that the Stokes approximation does not describe correctly the final stages of development of the TF instability and that the complete Navier–Stokes equations need to be used in this case. Only these equations provide an adequate description of the liquid dynamics at the stage of singularity formation and allow one to determine the limiting angle of the forming conical tip.



Dependences (4) of instability increment Γ on wave number κ corresponding to the Navier–Stokes model for $\delta = 2, 6, 20, 100$ (colored curves) and dependence (2) corresponding to the Stokes regime (black line). A color version of the figure is provided in the online version of the paper.

Conflict of interest

The author declares that he has no conflict of interest.

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