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Relaxation of double quantum dot qutrit

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Relaxation of qutrit in two tunnel-coupled quantum dots is considered. Infrared radiative transitions between three energy levels of spatial quantization used for storage and processing of quantum information are investigated as possible relaxation channels. It is proven that by using earlier proposed metric of quantum processes — maximal deviation metric — in the case of qutrit the contribution of relaxation in the decrease of probability of sought outcome of quantum computations is represented by compact analytic expression.

Keywords: Qutrit, relaxation, metric of quantum processes, decoherence, maximal deviation metric, double quantum dot, semiconductor charge qutrit.

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Introduction

One of intensely developing fields of science is quantum computations. In the future, quantum computers which form the basis of this field will be able to solve practical tasks that can not be solved by the most powerful classic supercomputers [1,2].

The common-architecture quantum computer is a system that consists of a large number of elementary logic quantum cells with several discrete states, most often, two-level ones — qubits or, for example, three-level ones — qutrits that are considered in the present study. In order to process information more effectively than the classical computer, a system of multiple qutrits is required not to lose coherency when performing thousand of logic operations. One of the attractive systems for creating complete scalable quantum processors is systems based on electron states in double semiconductor quantum dots [3–5]. An important problem that prevents fast progress in creating the quantum computers, including quantum computers based on spatial electron states in the semiconductor quantum dots, is errors that originate during performing the quantum logic operations due to noises in the qubits and inaccuracies of execution. As a result, one of the principal tasks in constructing practically useful full-scale quantum information processors is reduction of a value of hardware errors to the level, below which their influence can be neutralized by algorithmic methods by, for example, procedures of quantum error correction. That is why it is absolutely necessary to evaluate the systems that are candidates for quantum information processers by a degree of influence of interaction with the external environment on the level of hardware faults that occur in them.

The present study is aimed at considering decoherence processes in qutrits that use the three lower energy states of spatial quantization of charge carriers in double dot nanostructures.

1. Qutrit design

We have considered a silicon nanostructure that consists of two quantum dots separated by a tunnel barrier, with one electron, and is shown in Fig. 1.

A proposed diagram of manufacturing our studied structure uses technological steps which are similar to stages of manufacturing of the FinFET transistor: off-etching of the so-called fin (rib), which is an active area that will include our electron, is followed by formation of a composite layer and a dielectric with a high permittivity index. After that, thin metal shutters are applied to be used for forming boundaries of quantum dots and a barrier between them and to be capable of being utilized later for adjusting parameters of the qutrit and controlling it. It results in forming a

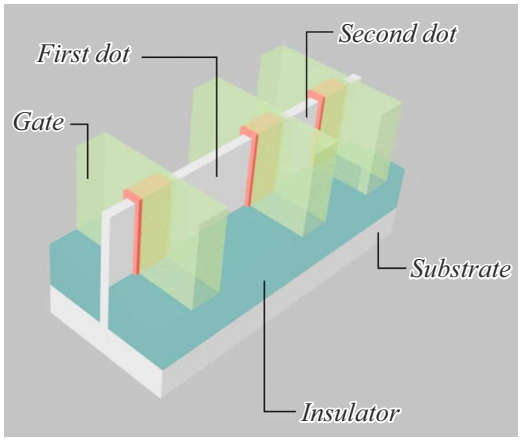

Figure 1. Qutrit diagram.

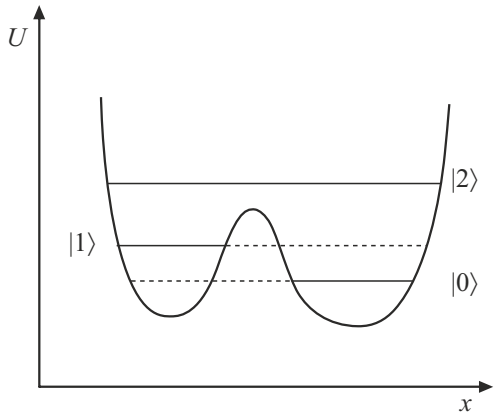

Figure 2. Lower energy levels in the system of two tunnel-coupled semiconductor quantum dots.

potential profile along the structure and electron spatial quantization levels that are qualitatively shown in Fig. 2.

2. Calculation of the relaxation rates

The qutrit relaxation rate was calculated in the low-temperature limit. It was assumed that the energy required for transitions to the higher levels was much higher than the lattice temperature and the energy distance between the three basic levels of spatial electron quantization in the two-well potential of two adjacent quantum dots. The influence of the lattice temperature on the relaxation rates in the structures of the qubits based on double quantum dots was evaluated in the study [6]. It was shown that at the temperatures of at most several Kelvins the influence of the temperature does not result in significant changes of the relaxation rate. In our case, the energies of interlevel transitions by more than an order exceed the energies of interlevel transitions, which are considered in the article [6], and therefore the influence of heating to the helium temperatures will not all the more have a substantial effect.

The electron wave functions in the three lower spatial states $\psi_j(x)$, where $j = 0, 1, 2$, were searched from solving a one-dimensional stationary Schrödinger equation. Localization of the electron in the two transverse directions y and z was considered to be constant and much less than longitudinal distribution along the axis x. The two-well potential used for calculating the wave functions was considered to be rectangular as shown in Fig. 3.

A matrix element of a dipole moment operator d_{ab} between the electron wave functions in the a-th and b-th energy levels of our system $\psi_a(x)$ and $\psi_b(x)$:

$$d_{ab} = \int_{-\text{width}/2}^{\text{width}/2} \psi_a \cdot qx \cdot \psi_b dx, \qquad (1)$$

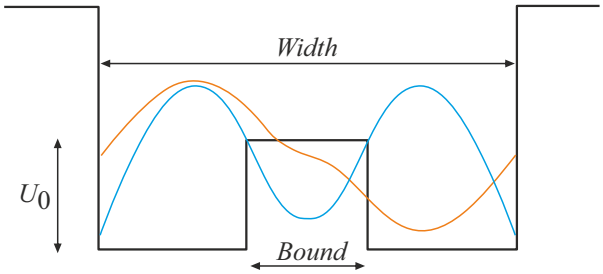

Figure 3. Profile of the potential along the axis x and a progress of the wave functions of the two lower energy states of the electron $\psi_0(x)$ and $\psi_1(x)$.

where width is a width of the quantum dot under our consideration.

Due to the dipole moment, the qutrit, being at the excited levels 1 and 2, can execute a radiative transition and change its state, which at a computational level will cause a fault in quantum information that processed and stored in it.

Probabilities of the radiative transitions were calculated in the Markov approximation. The vector potential $A(r, t)$ of the field emitted by the dipole in a medium with permittivity ε:

$$\hat{A}(r,t) = \frac{1}{\sqrt{\varepsilon\varepsilon_0 V}} \sum_k \sum_s \sqrt{\frac{\hbar}{2\omega}} \left(\hat{a}_{ks} \varepsilon_{ks} e^{ikr} + \hat{a}_{ks}^{\dagger} \varepsilon_{ks}^{*} e^{-ikr} \right). \qquad (2)$$

Here, V — the volume of the medium, k — the wave vector of an emitted flat wave, s — polarization of radiation, ω — the radiation frequency, ε_{ks} — the directing vector of wave polarization.

Interaction of the electron with the radiation field is described by a standard Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} - \frac{e}{2m}(\hat{p}\hat{A} + \hat{A}\hat{p}) + \frac{e^2\hat{A}^2}{2m} + U(r) = \hat{H}_0 + \hat{H}'. \qquad (3)$$

Here, m — the effective mass of the electron in the semiconductor, $U(r)$ — the potential of the double quantum dot. The matrix element of the transition in a dipole approximation

$$\langle j|\hat{H}'|f\rangle = -\frac{e}{m} \sum_{ks} \sqrt{\frac{\hbar}{2\varepsilon_0 \varepsilon V \omega}} \times \left\langle i \left| \hat{a}_{ks}^{\dagger} \boldsymbol{\varepsilon} e^{ikr} p + \hat{a}_{ks} \hat{\boldsymbol{\varepsilon}}_{\boldsymbol{ks}} e^{-ikr} p \right| f \right\rangle. \qquad (4)$$

The complete probability of photon radiation per unit time:

$$\Gamma_{ab} = \frac{\omega_{ab}^3 d_{ab}^2}{3\pi c^3 \hbar \varepsilon_0} \sqrt{\varepsilon}. \qquad (5)$$

Final probabilities of spontaneous relaxation with photon radiation Γ_{ab} were calculated by the Fermi golden rule according to the formula (5).

Let us consider a dependence of the complete probability of photon radiation per unit time with the transitions

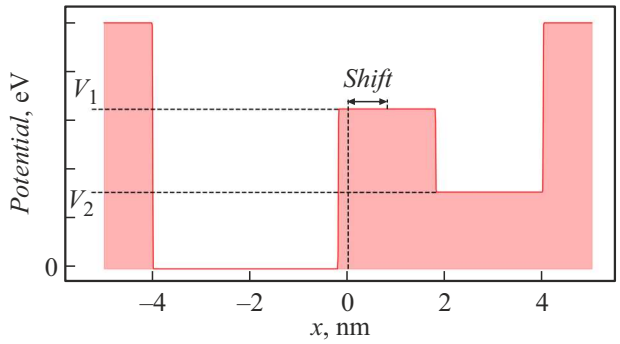

Figure 4. Profile of the two-dot structure with the barrier width of 2 nm with shift in relation to the center and the barrier height of V_1 eV and the second (right) well's height of V_2.

between the three lower energy electron states ψ_0, ψ_1 and ψ_2 from the potential profile along the axis x.

We will shift the potential along the barrier axis and watch variation of Γ_{10}, Γ_{21}, Γ_{20}. We will raise the level of the second well to the height V_2 by assuming that the level of the first well is 0, while V_1 will denote a barrier height. The following parameters of the system are selected: the width of the whole quantum dot is 8 nm, the barrier width is 2 nm, as shown in Fig. 4.

3. Dynamics of the quantum state of the qutrit during relaxation

During relaxation of the qutrit, its state becomes coupled to the environment and formalism of the density matrix shall be used for description [7]. We designate the density matrix of the three-level system as ρ. Let $|0\rangle$, $|1\rangle$, $|2\rangle$ be the ground, first and second excited energy states of the system, respectively. Then, in the first approximation without relaxation the system Hamiltonian is written as follows:

$$H = \varepsilon_0|0\rangle\langle 0| + \varepsilon_1|1\rangle\langle 1| + \varepsilon_2|2\rangle\langle 2|, \tag{6}$$

where ε_0, ε_1, ε_2 are the energies of the states $|0\rangle$, $|1\rangle$ and $|2\rangle$, respectively.

In the Markov approximation, the dynamics of the state is described by the Lindblad equation:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] - \frac{1}{2}\sum_{\alpha=1}\Big(\mathbf{L}^{(\alpha)^\dagger}\mathbf{L}^{(\alpha)}\rho(t) + \rho(t)\mathbf{L}^{(\alpha)^\dagger}\mathbf{L}^{(\alpha)} - 2\mathbf{L}^{(\alpha)}\rho(t)\mathbf{L}^{(\alpha)}\Big), \tag{7}$$

where $\mathbf{L}^{(\alpha)}$ — the Lindblad operators that determine a nonunitary part of evolution, $\hbar$ — the reduced Planck constant, i — the imaginary unit, t — the time.

The density matrix of the qutrit can be expressed for the relaxation process in an arbitrary moment of time in the energy basis when $t > 0$:

$$\rho(t) = \begin{pmatrix} \rho_{00}(t) & \rho_{01}(t) & \rho_{02}(t) \\ \rho_{10}(t) & \rho_{11}(t) & \rho_{12}(t) \\ \rho_{20}(t) & \rho_{21}(t) & \rho_{22}(t) \end{pmatrix} \tag{8}$$

and due to awkwardness of the obtained expressions and impossibility of further simplification we have to separately write out components of the density matrix below:

$$\rho_{00}(t) = \rho_{00}(0) + \rho_{11}(0)(1 - e^{-\Gamma_{10}t}) + \rho_{22}(0)\left(1 + \frac{-e^{(-\Gamma_{20}-\Gamma_{21})t}(\Gamma_{10}-\Gamma_{20}) + \Gamma_{21}e^{-\Gamma_{10}t}}{\Gamma_{10}-\Gamma_{20}-\Gamma_{21}}\right), \tag{9}$$

$$\rho_{01}(t) = \rho_{01}(0)e^{(i\varepsilon_{10}/\hbar - \Gamma_{10}/2)t}, \tag{10}$$

$$\rho_{02}(t) = \rho_{02}(0)e^{(i\varepsilon_{20}/\hbar - (\Gamma_{20}+\Gamma_{21})/2)t}, \tag{11}$$

$$\rho_{10}(t) = \rho_{10}(0)e^{(-i\varepsilon_{10}/\hbar - \Gamma_{10}/2)t}, \tag{12}$$

$$\rho_{11}(t) = \rho_{11}(0)e^{-\Gamma_{10}t} + \rho_{22}(0)\frac{\Gamma_{21}(-e^{-\Gamma_{10}t} + e^{(-\Gamma_{20}-\Gamma_{21})t})}{\Gamma_{10}-\Gamma_{20}-\Gamma_{21}}, \tag{13}$$

$$\rho_{12}(t) = \rho_{12}(0)e^{(i\varepsilon_{21}/\hbar - (\Gamma_{10}+\Gamma_{20}+\Gamma_{21})/2)t}, \tag{14}$$

$$\rho_{20}(t) = \rho_{20}(0)e^{(-i\varepsilon_{20}/\hbar - (\Gamma_{20}+\Gamma_{21})/2)t}, \tag{15}$$

$$\rho_{21}(t) = \rho_{21}(0)e^{(-i\varepsilon_{21}/\hbar - (\Gamma_{10}+\Gamma_{20}+\Gamma_{21})/2)t}, \tag{16}$$

$$\rho_{22}(t) = \rho_{22}(0)e^{-(\Gamma_{20}+\Gamma_{21})t}, \tag{17}$$

where additional designations are introduced for compactness

$$\varepsilon_{ij} = \varepsilon_i - \varepsilon_j, \quad (i, j) \in \{(1, 0), (2, 0), (2, 1)\}. \tag{18}$$

By an ideal process, we mean unitary evolution of the system (that effectively executes the phase operation of the qutrit) under effect of the eigen Hamiltonian (6) without taking into account interaction with an electromagnetic field. We designate the matrix of the density that describes the system in this case as ρ^{ideal}.

Its dynamics complies with the Liouville's quantum equation:

$$\frac{d\rho^{ideal}(t)}{dt} = -\frac{i}{\hbar}[H, \rho^{ideal}(t)]. \tag{19}$$

By repeating the similar actions as in the previous section as well as using a condition for equality of the density matrices at the initial moment of time in both the processes, we obtain the expression for ρ^{ideal} in the energy basis:

$$\rho_{00}^{ideal}(t) = \rho_{00}^{ideal}(0), \tag{20}$$

$$\rho_{01}^{ideal}(t) = \rho_{01}^{ideal}(0)e^{i\varepsilon_{10}t/\hbar}, \tag{21}$$

$$\rho_{02}^{ideal}(t) = \rho_{02}^{ideal}(0)e^{i\varepsilon_{20}t/\hbar}, \tag{22}$$

$$\rho_{10}^{ideal}(t) = \rho_{10}^{ideal}(0)e^{-i\varepsilon_{10}t/\hbar}, \tag{23}$$

$$\rho_{11}^{ideal}(t) = \rho_{11}^{ideal}(0), \tag{24}$$

$$\rho_{12}^{ideal}(t) = \rho_{12}^{ideal}(0)e^{i\varepsilon_{21}t/\hbar}, \tag{25}$$

$$\rho_{20}^{ideal}(t) = \rho_{20}^{ideal}(0)e^{-i\varepsilon_{20}t/\hbar}, \tag{26}$$

$$\rho_{21}^{ideal}(t) = \rho_{21}^{ideal}(0)e^{-i\varepsilon_{21}t/\hbar}, \tag{27}$$

$$\rho_{22}^{ideal}(t) = \rho_{22}^{ideal}(0). \tag{28}$$

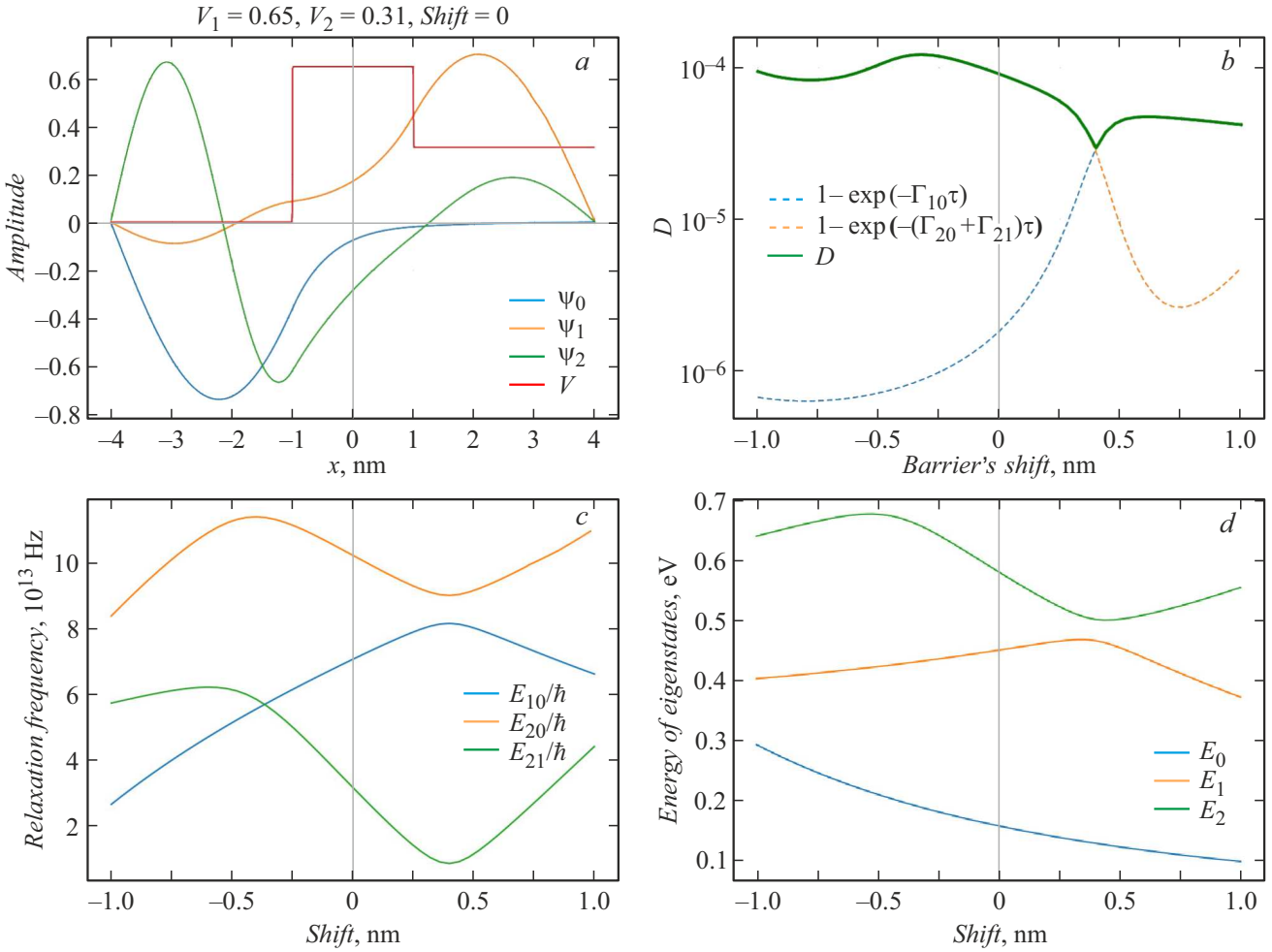

Figure 5. *a* — the graph of the wave functions that correspond to activated discrete levels in the point (the parameters of the point are shown in the figure. *b* — the graph of the dependence D (a qutrit decoherence measure) on the barrier shift for the well of the width of 8 nm and the barrier of the width of 2 nm, $V_2 = 0.31$ eV, $V_1 = 0.65$ eV. The relaxation was calculated for the typical time $\tau = 1000\,\frac{\hbar\pi}{\varepsilon_{10}}$, where ε_{10} — the energy of the transition between 0 and 1 levels. *c* — the graph of the dependence of the relaxation frequencies between the respective levels on the barrier shift. *d* — the graph of the dependence of the energy of the eigenstates on the barrier shift.

4. Measure of maximum deviation of the qutrit

In order to evaluate the measure of the influence of spontaneous relaxation on the quality of the qutrit-stored quantum state, we have used a measure of maximum deviation D, which is also called the decoherence measure in the case if one of the processes is ideal and the other is noisy [8,9]. We note that in our situation no satisfactory results are derived from attempts to write out an analytical expression for evaluating the influence of spontaneous relaxation of the qutrit on accuracy of the phase operations based on alternative approaches that use calculation of the other parameters like, for example, fidelity, randomized benchmarking, cross entropy, by using the expressions for the elements of the density matrix, which are given in Section 3. In the best case, the resultant analytical expressions occupy several pages after maximum simplification. We briefly characterize these parameters as applied to our conditions.

Fidelity quantitatively determines „proximity“ between the two density matrices and is defined as

$$F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right)^2, \qquad (29)$$

where ρ and σ are the density matrices of the two quantum states [10]. Its definition requires, in particular, to calculate square roots from the matrices and usually results in quite bulk expressions.

If randomized benchmarking is applied, an average error of the basic quantum operations in execution of multiple random sequences of the basic operations is evaluated [11]. First, a set of arbitrarily selected sequences of the Clifford gates is executed and they shall return the quantum processor into the initial state. Then, the measurement is performed at the end of each sequence in order to make

sure that the initial state is obtained, and a dependence of fidelity between the obtained ψ_m and initial states ψ on the sequence length m is plotted. In case of the errors that do not depend on the parameters and the time the graph will correspond to

$$F(\psi_m, \psi) = Ap^m + B, \tag{30}$$

where the coefficients A and B take into account the errors of state preparation and the measurement as well as a side effect of the error on a final control element; p is a fidelity decay parameter that characterizes average probability of no error when executing one Clifford gate in the sequence. Relaxation results in exponential decay

$$p \approx e^{-t_g/T_1}, \tag{31}$$

where t_g — duration of the quantum operation, T_1 — the typical decay time. Thus, using randomized benchmarking it is possible to evaluate a contribution by relaxation when applying a randomized set of the operations. This method does not always correctly evaluate the errors even by an order of the magnitude, since in many cases the sequence of the operations in practically useful intelligent algorithms is not quite random for this method.

The cross entropy makes it possible to compare a distribution of the measurement results with the ideal distribution [12]. For this purpose, a set of random quantum circuits is created and they are quite complex so that their output state is close to the random one. Each execution provides a bit sequence that is a state of the qubits after application of the circuit. A frequency of occurrence of these bit sequences forms an empirical distribution of probabilities and then the classic computer simulates ideal behavior of the circuit (without noises and errors).

By the calculated entropy between the real and ideal distributions, it is possible to evaluate a level of the noises and the errors in the system. Cross entropy:

$$F_{\text{XEB}} = 2^n \langle P(x_i) \rangle_i - 1 = \frac{2^n}{k} \sum_{i=1}^{k} |\langle 0^n | C | x_i \rangle|^2 - 1, \tag{32}$$

where n — the number of the qubits, $P(x_i)$ — the probability of obtaining a bit string x_i in the ideal quantum circuit C and averaging is taken across all the measured results. In our case, it is also impossible to obtain a compact analytical expression for the cross entropy.

These considerations led us to necessity of using a maximum deviation metric D. At the same time, D is not just one of the set of abstract mathematical metrics, but it also has a direct physical meaning — it guarantees that probability of extracting a desired value from the quantum processor (the qutrit, in our case) will deteriorate in absolute magnitude due to noise no more than by D, wherein it is strictly proven that generally this estimate can not be improved [9].

We note that the qutrit that is in the initial state predefined by a nine-element density matrix $\rho(0)$ in the ideal (noiseless) case shall evolve by the law $\rho^A(t)$, while in the real situation it state varies by the law $\rho^B(t)$. $D(t)$ between the two processes A and B, which are described by the density matrices $\rho^A(t)$ and $\rho^B(t)$, respectively, and satisfy the initial condition

$$\rho^A(0) = \rho^B(0) = \rho(0) \tag{33}$$

is determined as follows:

$$D(t) = \sup_{\rho(0)} \left\| \rho^A(t) - \rho^B(t) \right\|_\lambda, \tag{34}$$

where an exact upper bound is taken across all the possible initial states of the processes, while $\| \cdot \|_\lambda$ designates an operator norm that is generally determined as

$$\| A \|_\lambda = \sup_{\varphi \neq 0} \sqrt{\frac{\langle \varphi | A^\dagger A | \varphi \rangle}{\langle \varphi | \varphi \rangle}}. \tag{35}$$

In our case, since we calculate the norm of the operator that is a difference between two density operators, the operator norm always exists and is equal to a maximum (in absolute magnitude) eigenvalue of the difference matrix.

$$\| A \|_\lambda = \max_i |\lambda_i(A)|. \tag{36}$$

The supremum for the metric D is achieved when noise action results in maximum deviation of the state of the qutrit from its state in the ideal process.

In our case, the quite cumbersome calculations unexpectedly finally led to a compact and physically clear result — the maximum deviation is achieved at the initial state $|2\rangle$, if $\Gamma_{10} \leq \Gamma_{20} + \Gamma_{21}$ or at the initial state $|1\rangle$, if $\Gamma_{10} > \Gamma_{20} + \Gamma_{21}$.

Thus, the sought decoherence measure of the qutrit:

$$D(t) = 1 - e^{-\max(\Gamma_{10}, \Gamma_{20}+\Gamma_{21})t}. \tag{37}$$

The calculations of the qutrit level positions, the relaxation rates and the decoherence measures are shown in Fig. 5. It is clear that within the selected area of the parameters the quality of the qutrit is better than $2 \cdot 10^{-4}$ everywhere. The time of execution of the phase operation τ is selected to be $1000\,\hbar\pi/\varepsilon_{10}$, since during optical control execution of the qutrit control operations for a shorter time will result in non-resonant effect on the adjacent transitions and the adjacent qutrits and can cause a control error that exceeds about 10^{-3}. The parameters are specially selected not for the most optimal case, but to demonstrate a transition of the maximum from one process to another as the barrier is shifted (a break on the graph of the dependence of the maximum deviation measure on the barrier position).

Conclusion

We have considered the design of the qutrit based on the spatial electron states in the double semiconductor quantum dot. For the processes of spontaneous relaxation of the

qutrit, we have determined the measure of decoherence D that is introduced by these processes. The simple and clear analytical expression that is obtained for the decoherence measure makes it possible to evaluate the quality of the qutrits in the dependence on their relaxation rates.

Conflict of interest

The authors declare that they have no conflict of interest.

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