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13

Propagation of finite amplitude waves along a cylindrical jet of magnetic fluid in an axial magnetic field

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The propagation of nonlinear waves along the surface of a cylindrical jet of magnetic fluid in the presence of a strong axial magnetic field is studied. It is demonstrated that, in the case of a liquid with high magnetic permeability, axisymmetric perturbations of the jet boundary of arbitrary amplitude can propagate along it without distortion. The analogy with known solutions describing the propagation of Alfven waves in an unbounded perfectly conducting fluid is discussed.

Keywords: ferromagnetic fluid, axial magnetic field, jet, nonlinear waves, exact solutions.

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Introduction

Magnetic field tangential to the free surface of magnetic fluid has a stabilizing effect on it [1–4] (instead, normal field, leads to destabilization of the system [1,3,5]). This effect can be used to stabilize fluid jets, i.e., to suppress Plateau-Rayleigh capillary instability. The stability of a cylindrical jet of a non-conducting magnetic fluid in the axial magnetic field was first analyzed in [6]; the findings were developed in [7,8] (studies of stability of the conducting fluid jets within MHD model were conducted in [9,10]). The stability of an axisymmetric jet of a magnetic fluid in the presence of all components of an external magnetic field was analyzed in [11] (see also recent paper [12]). Such studies are important for understanding the scenarios of jet decay and subsequent droplet formation [13]. Of considerable interest is the study of the effect of an azimuthal magnetic field on a cylindrical layer of magnetic fluid surrounding a current conductor [14,15]. The possibility of propagation of solitary axisymmetric waves has been experimentally demonstrated for such a system [16]. Nonlinear theory has developed, for example, in [17,18] (small amplitude waves) and [19,20] (strongly nonlinear waves).

This paper studies the propagation of strongly linear axisymmetric waves along the surface of a cylindrical jet of a magnetic fluid in the presence of a strong external axial magnetic field. The law of linear wave dispersion for such a system was obtained in [6,8]. In [21], a weakly nonlinear (i.e., for small-amplitude surface perturbations) theory of wave propagation was proposed for this problem: the nonlinear Schrodinger equation for the envelope of a wave packet was derived, and its modulation instability was described. We demonstrate that in the case of a fluid with high magnetic permeability, one may go beyond considering the waves of small amplitude. It is possible to get exact wave solutions delineating the propagation of perturbations of the cylindrical jet boundary of an arbitrary amplitude (i.e. comparable to both the wavelength and the radius of the jet). Such solutions are similar in a number of properties to the well-known solutions describing the propagation of Alfven waves in an unlimited perfectly conducting fluid [22,23]. Despite the qualitative differences (fundamentally different geometries, media with different physical properties; in fact, only the presence of an external homogeneous magnetic field is common), the equations of motion in both cases admit solutions where arbitrary nonlinear waves can propagate at a constant speed along the field direction.

It should be noted that from a mathematical point of view, the problem of magnetic fluid behavior in external magnetic field is similar to the problem of behavior of a dielectric fluid in external electric field — the corresponding equations coincide after replacing the magnetic field with electric one and magnetic permeability with dielectric permittivity [1]. In studies [24–27] , it was found that nonlinear waves of arbitrary configuration can propagate without distortion along the initially flat free surface of an ideal dielectric fluid along the direction of external tangential electric field. This situation is realized for a fluid with a high value of dielectric constant in case of a sufficiently strong field, when the impact of electrostatic forces becomes dominant. The findings from the present study may be considered as a generalization of the results of [24–27] to another, cylindrical, — system geometry.

1. Dispersion relation

Let us consider the evolution of waves on the free surface of an ideal incompressible non-conductive magnetic fluid having constant magnetic permeability μ. In the undisturbed state, the jet is an infinitely long circular cylinder with a radius of r_0 — the geometry of the problem is shown in Fig. 1. The jet is placed in an external homogeneous magnetic field with a strength of H_0, directed along its axis. Similar to [8] we believe that the field is generated by a solenoid with a radius of R_0 with its axis coinciding with the jet axis.

Let's introduce the function η which determines deformation of the jet surface. The shape of its surface is described by equation $r = r_0 + \eta(\theta, z, t)$, where $\{r, \theta, z\}$ are cylindrical coordinates, t is time. In linear approximation, i.e., for small amplitude deformations $|\partial\eta/\partial z| \ll 1$ and $|\partial\eta/\partial\theta| \ll r_0$, the system behavior is fully described by the dispersion relation corresponding to expression $\eta \propto \exp[i(n\theta + kz - \omega t)]$, where $n = 0, 1, 2\ldots$ is the azimuthal wavenumber, k is axial wavenumber, ω is frequency. Dispersion law is expressed as follows [8]:

$$\omega^2 = \frac{\mu_0 H_0^2(\mu-1)^2 k^2 I_n'(kr_0)}{\rho[\mu I_n'(kr_0) - A I_n(kr_0)]} + \frac{\alpha k[n^2 + (kr_0)^2 - 1] I_n'(kr_0)}{\rho r_0^2 I_n(kr_0)}, \tag{1}$$

where value A is specified as

$$A = \frac{I_n'(kr_0)K_n(kR_0) - I_n(kR_0)K_n'(kr_0)}{I_n(kr_0)K_n(kR_0) - I_n(kR_0)K_n(kr_0)}.$$

Here μ_0 is magnetic constant, ρ is density of fluid, α is surface tension coefficient, I_n and K_n are modified Bessel functions of the first and kind of order n, and L_n' and K_n' are their derivatives with respect to argument (for the environment surrounding the jet, we consider the density to be zero, and the magnetic permeability be equal to unit). The first term on the right-hand side of (1) is responsible for the influence of magnetic field, and the second for the influence of capillary effects.

The jet surface is unstable with respect to perturbations with wave numbers k and n for which the right-hand side is negative and, accordingly, the frequency ω is imaginary. In paper [8], a detailed stability analysis was carried out,

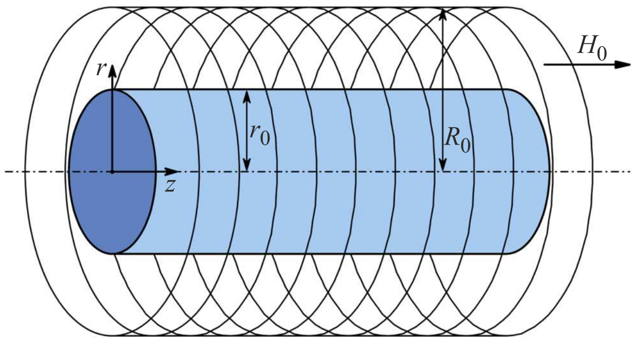

Figure 1. Geometry of the problem (schematic representation).

indicating the stabilizing effect of the magnetic field on the jet. Note that findings of [8] are a generalization of the results of the pioneering work [6], which examined the stability of a magnetic fluid jet in a homogeneous magnetic field in unlimited space. The dispersion law from [6] may be found from (1) by examining the limit $R_0 \gg r_0$. Because the asymptotics are true,

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \;\; K_n(x) \approx \frac{e^{-x}}{\sqrt{2x/\pi}}, \;\; x \to \infty,$$

in this limit we have $A \to K_n'(kr_0)/K_n(kr_0)$, i.e. in (1) the dependence on the solenoid radius fades away.

Let λ be a characteristic spatial scale of the jet surface perturbations. From general considerations, it is clear that for small-scale perturbations ($\lambda \ll r_0$), the term responsible for capillary effects will prevail in the dispersion law (1). Indeed, in this limit, the geometry of the problem changes from cylindrical to flat. Then the first term on the right side (1) will have an order of λ^{-2}, and the second term of λ^{-3} [1–3]. As a result, at sufficiently small λ, capillary forces prevail over magnetic ones.

We will consider large-scale surface perturbations comparable to the radius of the jet, $\lambda \sim r_0$, or, in the terms of wave numbers k and n, $\sqrt{(n/r_0)^2 + k^2} = O(r_0^{-1})$. For such perturbations in a sufficiently strong magnetic field, the term responsible for its influence in the law of dispersion (1) will dominate. The characteristic magnetic pressure is estimated as $p_m = \mu_0\mu H_0^2/2$; the capillary pressure as $p_\alpha = \alpha/r_0$. We introduce the dimensionless parameter $\delta = p_m/p_\alpha$, which characterizes the relative contribution of magnetic and capillary pressures. We will assume the magnetic field to be strong if $\delta \gg 1$ and, as a result, capillary effects can be ignored: wave propagation will be entirely determined by the influence of the magnetic field.

In this paper, we will consider a medium with high magnetic permeability, $\mu \gg 1$, which is quite feasible for ferromagnetic fluids [28]. As a result of expansion of the right-hand side of (1) in small parameter $1/\mu$, the dispersion law takes a simple algebraic form

$$\omega^2 = \frac{\mu_0\mu H_0^2 k^2}{\rho}\left(1 + O(\mu^{-1})\right).$$

When introducing the Alfven velocity $V_A = H_0\sqrt{\mu_0\mu/\rho}$, we obtain

$$\omega^2 \approx V_A^2 k^2. \tag{2}$$

Thus, for the case $\mu \gg 1$, when condition $\delta \gg 1$ (strong magnetic field) is fulfilled, the dispersion relation is radically simplified. For large-scale perturbations of the jet surface ($\lambda \sim r_0$), there's no more any dependence on azimuthal number n, on surface tension α, as well as on geometric parameters r_0 and R_0.

For our subsequent analysis, it is essential to assume that the magnetic permeability μ is constant, i.e., independent of the magnetic field. For ferromagnetic fluids, this approximation is valid if the field strength is less than a certain threshold, which we denote as H_c. Let's consider how such a requirement can be combined with the strong field condition $\delta \gg 1$.

Let's estimate the value of H_c for the ferrofluids used in the experiments. From [3] it is known that for a colloidal ferrofluid, the magnetization M depends linearly on the strength of the applied magnetic field at relatively low H_0. The ratio $M \approx \chi_i H_0$ is valid, where the proportionality coefficient χ_i is the so-called initial magnetic susceptibility. At larger values of H_0 the magnetization approaches the state of saturation: $M \approx M_s = \text{const}$. As the boundary between these limits, it is natural to take the value of the field strength H_c, for which $M = M_s/2$. Then, we'll obtain $H_c = M_s/(2\chi_i)$. For example, in the studies [29] a ferrofluid was used for which $\chi_i = 7.0$, $M_s = 51.4\,\text{kA/m}$, $\alpha = 23.5\,\text{mN/m}$. For the fluid permeability we have $\mu = 1 + \chi_i = 8.0$, i.e. the condition $\mu \gg 1$ is properly fulfilled. For the threshold field strength, the estimate is $H_c = 3.7\,\text{kA/m}$. The value H_0 should not exceed this value so that the magnetic permeability can be considered constant. When $H_0 = H_c$ for $r_0 = 5\,\text{mm}$ we find $\delta = 14.4$, i.e. condition $\delta \gg 1$ for magnetic forces prevalence over the capillary ones is fulfilled. It follows from this that it is possible to select such a magnetic fluid and such values of external field strength that three conditions used in this paper will be fulfilled simultaneously: $\mu \gg 1$, $\delta \gg 1$ and $H_0 \le H_c$.

Let's return to the discussion of the dispersion relation. In terms of perturbation η of the jet boundary, the dispersion law (2) corresponds to the linear wave equation

$$\frac{\partial^2 \eta}{\partial t^2} \approx V_A^2 \frac{\partial^2 \eta}{\partial z^2}. \tag{3}$$

Its solution is

$$\eta \approx \eta^+(z - V_A t, \theta) + \eta^-(z + V_A t, \theta), \tag{4}$$

where $\eta^\pm$ are arbitrary functions (with obvious limitations for linear waves $|\partial\eta^\pm/\partial z| \ll 1$ and $|\partial\eta^\pm/\partial\theta| \ll r_0$) that define the shape of waves propagating without distortion in positive and negative directions of z axis with constant speed V_A.

Below, we will analyze the reasons resulting in such system behavior and also demonstrate that some of our conclusions about the dynamics of the jet boundary linear perturbations may be extended to the general, nonlinear case.

2. Motion equations

As was demonstrated in section 1, it is of interest to consider the behavior of a jet of magnetic fluid under the action of magnetic forces only, i.e., without taking into account the influence of capillary forces. Let's consider this fluid motion as vortex-free. Then we can introduce a scalar velocity potential ϕ so that the velocity vector is defined as

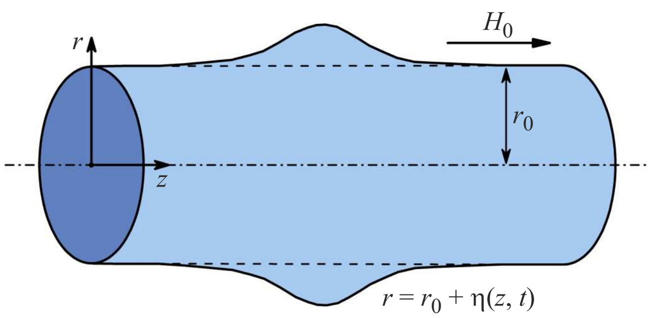

Figure 2. Spatially localized axisymmetric perturbation of the cylindrical jet surface (schematically).

its gradient: $\mathbf{v} = \nabla\phi$. For an incompressible fluid, we have $\nabla \cdot \mathbf{v} \equiv 0$ and, therefore, the potential satisfies the Laplace equation $\nabla^2\phi = 0$.

It was shown above that in the limiting case $\mu \gg 1$, the dispersion law does not contain the azimuthal number n. This gives a reason to neglect azimuthal perturbations of the jet surface, i.e., to consider only axisymmetric perturbations $\eta = \eta(z, t)$ (this corresponds to $n = 0$) (Fig. 2). In the axisymmetric case $\phi = \phi(r, z, t)$, and the Laplace equation for the velocity potential is written as

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{5}$$

It should be solved with the following boundary conditions:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 = -\frac{p}{\rho} + f(t), \quad r = r_0 + \eta(z, t), \tag{6}$$

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial r} - \frac{\partial\eta}{\partial z}\frac{\partial\phi}{\partial z}, \quad r = r_0 + \eta(z, t), \tag{7}$$

$$\frac{\partial\phi}{\partial r} = 0, \quad r = 0, \tag{8}$$

where $p = p(r, z, t)$ is fluid pressure, and f is arbitrary function of time. Condition (6), which is often called dynamic, has the meaning of the unsteady Bernoulli equation taken at the free boundary of the jet. Kinematic condition (7) relates the velocity on the fluid free surface to the function η, which defines its shape. Condition (8) corresponds to the fact that the velocity on the jet axis has only one axial component.

Let's consider only spatially localized perturbations of the jet boundary, i.e. $\eta \to 0$ at $|z| \to \infty$ (Fig. 2). Then, at infinity the following is true

$$\frac{\partial\phi}{\partial r} \to 0, \quad \frac{\partial\phi}{\partial z} \to 0, \quad |z| \to \infty. \tag{9}$$

The pressure in (6) p, according to [30], is set by the expression (here we do not take into account capillary effects)

$$p = p_{\text{atm}} + \frac{\mu_0\mu}{2}(H_n^2 - H_\tau^2) - \frac{\mu_0}{2}(\tilde{H}_n^2 - \tilde{H}_\tau^2), \qquad (10)$$

where p_{atm} is constant external (atmospheric) pressure, H_τ and $\tilde{H}_\tau$ are tangential components, whereas H_n and $\tilde{H}_n$ are components of the magnetic field strength inside the fluid and, respectively, outside it, which are normal to the free boundary. The standard boundary conditions shall be satisfied for the magnetic field components

$$H_\tau = \tilde{H}_\tau, \quad \mu H_n = \tilde{H}_n. \qquad (11)$$

Using them, pressure (10) can be expressed through the components H_τ and H_n of the magnetic field strength inside the fluid:

$$p = p_{\text{atm}} - \frac{\mu_0(\mu - 1)}{2}(\mu H_n^2 - H_\tau^2). \qquad (12)$$

The magnetic field in a jet of a magnetic fluid ($\mathbf{H} = \{H_r, 0, H_z\}$) and in its surrounding space ($\tilde{\mathbf{H}} = \{\tilde{H}_r, 0, \tilde{H}_z\}$) is found from Maxwell's equations for a nonconducting medium in the magnetostatic approximation:

$$\nabla \times \mathbf{H} = 0, \ \nabla \cdot \mathbf{H} = 0, \ \nabla \times \tilde{\mathbf{H}} = 0, \ \nabla \cdot \tilde{\mathbf{H}} = 0. \qquad (13)$$

These equations should be solved together with conditions on the free boundary (11), conditions on the system axis and on the solenoid surface,

$$H_r = 0, \quad r = 0, \qquad (14)$$

$$\tilde{H}_r = 0, \quad r = R_0, \qquad (15)$$

as well as with conditions on infinity

$$H_z \to H_0, \quad H_r \to 0, \quad |z| \to \infty, \qquad (16)$$

$$\tilde{H}_z \to H_0, \quad \tilde{H}_r \to 0, \quad |z| \to \infty. \qquad (17)$$

Since the velocity potential may be selected somewhat arbitrary, instead of a couple of conditions (9) it is possible to use the only condition

$$\phi \to 0, \quad |z| \to \infty. \qquad (18)$$

In this case, the function f included in the Bernoulli equation will be uniquely defined as

$$f = \frac{p_{\text{atm}}}{\rho} - \frac{\mu_0(\mu - 1)H_0^2}{2\rho}.$$

Taken together, the equations given in sec. 2 fully describe the nonlinear evolution of perturbations of the boundary of a magnetic fluid jet under the action of external axial magnetic field.

3. The limit of high magnetic permeability

Let us consider the case of high magnetic permeability $\mu \gg 1$, which can be realized for ferromagnetic fluids. In sec. 1 based on the analysis of the dispersion law (1), we've demonstrated that in this limit linear waves propagate along the jet axis in positive and negative directions without distortion: see solution (4). The condition for the applicability of linear approximation is the smallness of the jet deformations, $|\partial\eta/\partial z| \ll 1$. Here we will consider the propagation of strongly nonlinear waves, i.e., the case of perturbations of the jet boundary with large inclination angles, $\partial\eta/\partial z = O(1)$.

Analyzing the conditions (11) and (14)–(17), we may conclude that the values $\tilde{H}_\tau/H_0$, H_τ/H_0 and $\tilde{H}_n/H_0$ in expansion in small parameter $1/\mu$ relate to $O(1)$ order, and the value H_n/H_0 to the higher order of smallness $O(1/\mu)$. The normal component of the magnetic field strength in a fluid turns out to be much smaller in absolute value than the tangential component, and it can be assumed that the field lines are directed tangentially to the curved boundary. This leads to situation that the problem of finding the distribution of the magnetic field inside the jet is separated from the general problem of finding the distribution of the field in the entire space. For the limiting case $\mu \gg 1$, we can identically put

$$H_n = 0, \qquad (19)$$

and search for the field distribution where the boundary line of force will lie on the fluid free surface.

It should be stresses that the problem of finding the field distribution outside the fluid does is not separated from the original problem for the limit $\mu \gg 1$. To solve it, it is first necessary to find the field distribution inside the jet, and thereby determine H_τ, and only then solve an additional problem outside the jet with the boundary condition $\tilde{H}_\tau = H_\tau$. As a result, the distribution $\tilde{H}_n$ will be found, in particular. This, taking into account the coupling $H_n = \tilde{H}_n/\mu$, will make it possible, with a large but finite μ, to calculate the normal component of the magnetic field inside the fluid, i.e., to actually make the next iteration (in the first iteration $H_n \equiv 0$). However, as it turns out, for the purposes of this work, it is not necessary to implement such an iterative procedure. To describe the fluid motion, we need to find pressure (12) at the boundary of the fluid due to the influence of the magnetic field, and it can be found using only the field distribution inside the jet. Indeed, for the terms included in the right-hand side of (12), we have

$$H_\tau^2/H_0^2 = O(1), \quad \mu H_n^2/H_0^2 = O(1/\mu).$$

Thus, the term in (12) containing the normal component of the magnetic field is small, and it is enough to consider only first approximation where $H_n \equiv 0$. Then, in the basic order of expansion in a small parameter $1/\mu$, we have

$$p \approx -\frac{\mu_0\mu H_\tau^2}{2}, \quad f \approx -\frac{\mu_0\mu H_0^2}{2\rho}. \qquad (20)$$

Taking into account the magnetostatic equations (13), it is possible to reduce the number of unknown functions by introducing an auxiliary scalar potential $\psi(r,z,t)$ so that $\mathbf{H} = \nabla\psi$. This potential satisfies the Laplace equation $\nabla^2\psi = 0$, written in the case of axial symmetry of the problem as

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0. \tag{21}$$

We'll get the boundary conditions for (21) by passing in (14), (16), (19) from the components of magnetic field to the potential ψ via the relations $H_r = \partial\psi/\partial r$ and $H_z = \partial\psi/\partial z$. We find

$$\frac{\partial\psi}{\partial r} = 0, \quad r = 0, \tag{22}$$

$$\psi \to H_0 z, \quad |z| \to \infty, \tag{23}$$

$$\frac{\partial\psi}{\partial r} - \frac{\partial\eta}{\partial z}\frac{\partial\psi}{\partial z} = 0, \quad r = r_0 + \eta(z,t). \tag{24}$$

Here, we used the expression

$$H_n\sqrt{1 + \left(\frac{\partial\eta}{\partial z}\right)^2} = H_r - \frac{\partial\eta}{\partial z}H_z,$$

binding the normal component of the magnetic field strength H_n with axial and radial components H_z and H_r.

Finally, when using the potential ψ and taking into account the relations (19) and (20), the unsteady Bernoulli equation (6) is rewritten as

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 = \frac{\mu_0\mu}{2\rho}\left(\frac{\partial\psi}{\partial r}\right)^2 + \frac{\mu_0\mu}{2\rho}\left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{\mu_0\mu H_0^2}{2\rho}, \quad r = r_0 + \eta(z,t). \tag{25}$$

The remaining equations defining the dynamics of the fluid remain unchanged. Let's again focus on an important feature of the motion equations for the case $\mu \gg 1$: the behavior of a fluid is completely determined by a pair of scalar potentials ϕ and ψ, and their domains of definition ($0 \le r \le r_0 + \eta(z,t)$ and $-\infty < z < \infty$) coinside.

4. Exact wave solutions

In section 4 we'll find the exact solutions of the magnetic fluid motion equations corresponding to the limiting case $\mu \gg 1$. It is clear that the undisturbed state of the system corresponds to a trivial solution

$$\eta = 0, \quad \phi = 0, \quad \psi = H_0 z$$

of motion equations. Let's represent the potential ψ as a sum of the unperturbed solution and perturbation Ψ (in the unperturbed state $\Psi = 0$):

$$\psi(r,z,t) = H_0 z + \Psi(r,z,t).$$

The perturbed potential Ψ, similar as initial ψ, satisfy the Laplace equation

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{2}\frac{\partial\Psi}{\partial r} + \frac{\partial^2\Psi}{\partial z^2} = 0. \tag{26}$$

The boundary conditions for it, as can be easily obtained from (22)–(24), have the form

$$\frac{\partial\Psi}{\partial r} = 0, \quad r = 0, \tag{27}$$

$$\Psi \to 0, \quad |z| \to \infty, \tag{28}$$

$$H_0\frac{\partial\eta}{\partial z} = \frac{\partial\Psi}{\partial r} - \frac{\partial\eta}{\partial z}\frac{\partial\Psi}{\partial z}, \quad r = r_0 + \eta(z,t). \tag{29}$$

The equations describing nonlinear traveling waves are obtained from the above equations of motion by substituting

$$\eta(z,t) = \eta^{\pm}(z \mp Ct), \quad \phi(r,z,t) = \phi^{\pm}(r,z,\mp Ct),$$

$$\Psi(r,z,t) = \Psi^{\pm}(r,z,\mp Ct), \tag{30}$$

where the upper signs correspond to the waves propagating in positive direction of the axis z with a certain velocity $C > 0$, and the lower ones in negative direction. The waves corresponding to the substitution (30) propagate without distortion: their profile does not change in coordinate systems moving with the waves. As seen from (5) and (26), functions $\phi^{\pm}$ and $\Psi^{\pm}$ meet the Laplace equations

$$\frac{\partial^2\phi^{\pm}}{\partial r^2} + \frac{1}{2}\frac{\partial\phi^{\pm}}{\partial r} + \frac{\partial^2\phi^{\pm}}{\partial z^2} = 0,$$

$$\frac{\partial^2\Psi^{\pm}}{\partial r^2} + \frac{1}{2}\frac{\partial\Psi^{\pm}}{\partial r} + \frac{\partial^2\Psi^{\pm}}{\partial z^2} = 0 \tag{31}$$

with conditions arising from (7), (8), (18), (27)–(29):

$$\frac{\partial\phi^{\pm}}{\partial r} = 0, \quad \frac{\partial\Psi^{\pm}}{\partial r} = 0, \quad r = 0, \tag{32}$$

$$\phi^{\pm} \to 0, \quad \Psi^{\pm} \to 0, \quad |z| \to \infty, \tag{33}$$

$$\mp C\frac{\partial\eta^{\pm}}{\partial z} = \frac{\partial\phi^{\pm}}{\partial r} - \frac{\partial\eta^{\pm}}{\partial z}\frac{\partial\phi^{\pm}}{\partial z},$$

$$H_0\frac{\partial\eta^{\pm}}{\partial z} = \frac{\partial\Psi^{\pm}}{\partial r} - \frac{\partial\eta^{\pm}}{\partial z}\frac{\partial\Psi^{\pm}}{\partial z}, \quad r = r_0 + \eta^{\pm}. \tag{34}$$

This shows that the functions $\phi^{\pm}$ and $\Psi^{\pm}$ are defined by the same (up to constants in (34)) equations, and therefore are related by a simple relationship:

$$C\Psi^{\pm} = \mp H_0\phi^{\pm}. \tag{35}$$

Such a simple linear relationship is due to the fact that for the considered traveling waves in coordinate systems moving with the waves, the fluid velocity is directed tangentially to the boundary, i.e. it behaves similarly to the magnetic field in the limit $\mu \gg 1$.

Given the coupling (35), the non-stationary Bernoulli equation (25) will be expressed as

$$\mp C\,\frac{\partial\phi^\pm}{\partial z}+\frac{1}{2}\left(\frac{\partial\phi^\pm}{\partial r}\right)^2+\frac{1}{2}\left(\frac{\partial\phi^\pm}{\partial z}\right)^2=\mp\frac{\mu_0\mu H_0^2}{\rho C}\,\frac{\partial\phi^\pm}{\partial z}$$

$$+\frac{\mu_0\mu H_0^2}{2\rho C^2}\left(\frac{\partial\phi^\pm}{\partial r}\right)^2+\frac{\mu_0\mu H_0^2}{2\rho C^2}\left(\frac{\partial\phi^\pm}{\partial z}\right)^2,\ \ r=r_0+\eta^\pm$$

and, as can be easily noticed, is factorized as

$$\left(\frac{\mu_0\mu H_0^2}{2\rho C^2}-\frac{1}{2}\right)\left[\left(\frac{\partial\phi^\pm}{\partial r}\right)^2+\left(\frac{\partial\phi^\pm}{\partial z}\right)^2\mp 2C\,\frac{\partial\phi^\pm}{\partial z}\right]=0,$$

$$r=r_0+\eta^\pm.$$

Obviously, the last equation turns into an identity when

$$C=H_0\sqrt{\mu_0\mu/\rho}=V_A.$$

As a result, there are no restrictions on the profile of traveling waves $\eta^\pm$: it turns out to be arbitrary. In this case, the potential $\phi^\pm$ (and, consequently, the associated potential $\Psi^\pm$) is found from the given $\eta^\pm$ via the Laplace equation (31) with boundary conditions (32)−(34). We write down the relationships between the potentials ϕ and ψ, as well as the velocity and magnetic field vectors, $\boldsymbol{v}$ and $\mathbf{H}$, that arise when considering traveling waves:

$$\psi=H_0 z\mp\phi\sqrt{\rho/(\mu_0\mu)},\ \ \mathbf{H}=H_0\mathbf{e}_z\mp\boldsymbol{v}\sqrt{\rho/(\mu_0\mu)}.\quad(36)$$

For the solutions found, the velocity of propagation of nonlinear waves along the axis z does not depend on their profile, and in absolute value is equal to Alfven velocity V_A. In fact, this means that the evolution of the free surface of the jet is described by the simplest linear equation

$$\frac{\partial\eta}{\partial t}\pm V_A\,\frac{\partial\eta}{\partial z}=0,\quad(37)$$

to which the wave equation (3) is reduced when considering separately the waves propagating in the direction (upper sign) and against the direction (lower sign) of the axis z.

Thus, non-linearity does not lead to distortion of the profile of arbitrary amplitude waves, which ensures their structural stability.

5. The analogy with Alfven waves

In the exact wave solutions obtained above, there is a certain analogy with Alfven waves propagating in an unlimited space. The equations of magnetic hydrodynamics of an incompressible perfectly conducting non-viscous fluid with $\mu=1$ in the external homogeneous magnetic field have the following form [22]:

$$\frac{\partial\mathbf{H}}{\partial t}+(\boldsymbol{v}\nabla)\mathbf{H}=(\mathbf{H}\nabla)\boldsymbol{v},\quad(38)$$

$$\rho\left[\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\nabla)\boldsymbol{v}\right]=-\nabla\left(p+\frac{\mu_0|\mathbf{H}|^2}{2}\right)+\mu_0(\mathbf{H}\nabla)\mathbf{H},\quad(39)$$

$$\nabla\cdot\mathbf{H}=0,\quad\nabla\cdot\boldsymbol{v}=0,\quad(40)$$

$$\boldsymbol{v}\to 0,\quad\mathbf{H}\to H_0\mathbf{e}_z,\quad|\mathbf{r}|\to\infty.\quad(41)$$

A nontrivial family of exact solutions to these equations can be obtained using the following couplings between unknown functions $\boldsymbol{v}$, $\mathbf{H}$, and p [22]:

$$\mathbf{H}=H_0\mathbf{e}_z\mp\boldsymbol{v}\sqrt{\rho/\mu_0},\quad(42)$$

$$p+\frac{\mu_0|\mathbf{H}|^2}{2}=\text{const}.\quad(43)$$

Coupling (42) coincides with the coupling (36) earlier used for $\mu=1$. The equations (38)−(41) turn out to be compatible with conditions (42) and (43), while the pair of equations (38) and (39) is reduced to a single simplest wave equation

$$\frac{\partial\boldsymbol{v}}{\partial t}\pm V_A\,\frac{\partial\boldsymbol{v}}{\partial z}=0,\ \ V_A=H_0\sqrt{\mu_0/\rho},\quad(44)$$

coinciding in form with (37). The solution (44) is

$$\boldsymbol{v}=\boldsymbol{v}^\pm(x,y,z,\mp V_A t),$$

where $\boldsymbol{v}^\pm$ are arbitrary vector-functions describing the disturbances of the velocity field (as well as related fields $\mathbf{H}$ and p) propagating without distortion with Alfven speed V_A along the direction (upper sign), or against the direction (lower sign) of external magnetic field, i.e. axis z. The only constraints on $\boldsymbol{v}^\pm$ are the following non-burdensome conditions following from equations (40), (41):

$$\nabla\cdot\boldsymbol{v}^\pm=0,$$

$$\boldsymbol{v}^\pm\to 0,\quad|\mathbf{r}|\to\infty.$$

Thus, despite the obvious fundamental differences in the formulation of problems (in our case, the fluid is non-conductive and occupies an area bounded by a free surface; for classical Alfven waves, the fluid is perfectly conducting and occupies the entire space), the dynamics of nonlinear traveling waves in both systems are similar. Nonlinear waves can propagate (individually) in the direction and against the direction of an external magnetic field without distortion.

Conclusion

This paper demonstrates that waves of arbitrary amplitude can propagate without distortion along the surface of a cylindrical jet of a magnetic fluid placed in a strong magnetic field of a coaxial solenoid. This situation is realized for a fluid with a high value of magnetic permeability.

This behavior of waves on the jet surface corresponds to the situation where their propagation is described by a pair of linear equations (37): one equation for waves in the positive and the other in the negative direction of the axis z. However, this does not mean that the influence of nonlinearity may be neglected. The nonlinearity will determine the interaction of oppositely directed waves. For waves on a flat surface of a dielectric/magnetic fluid in a tangential electric/magnetic field, a similar interaction was considered in the works [31–35]. It was found that in a collision the counter-propagating solitary waves conserve momentum and energy [31]. In this case, the waves do not retain their shape. They deform, which eventually leads to the formation of singularities — regions with large gradients of velocities and fields [32]. A similar consideration for a jet is complicated by the fact that the method of dynamic conformal transformations used in [31,32] is not applicable in cylindrical geometry. Nevertheless, it is natural to assume that the counter-propagating waves will interact in a similar way.

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Conflict of interest

The authors declare that they have no conflict of interest.

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