11

Coupled chaotic generator and multi-frequency quasi-periodic system

© A.P. Kuznetsov, L.V. Turukina

Saratov Branch, Kotel'nikov Institute of Radio Engineering and Electronics, Russian Academy of Sciences, Saratov, Russia
E-mail: turukinalv@yandex.ru

Received June 20, 2024
Revised December 23, 2024
Accepted December 23, 2024

The interaction of the system with chaos (Kislov–Dmitriev generator) and multi-frequency quasi-periodic oscillations (ensemble of van der Pol generators) is considered. Bifurcations of doubling of a high-dimensional invariant torus and the emergence of chaos upon its destruction are revealed. A cascade of specific bifurcations of a chaotic attractor has been discovered, corresponding to the appearance of a different number of additional zero Lyapunov exponents. The stability of the Landau–Hopf scenario during interaction with a chaotic subsystem is shown.

Keywords: chaotic generator, quasi-periodicity, Lyapunov exponents, bifurcations.

DOI: 10.61011/TPL.2025.04.61011.20030

The study of coupled generators is one of the key problems in both radiophysics and electronics, as well as nonlinear oscillation theory in general [1,2]. The advancement of computer technology and the theory of dynamic systems and its applications to multidimensional systems [3–8] makes the problem of oscillations relevant both in the classical scenario of two generators and in scenarios with greater numbers of them [9–13]. Periodic, quasi-periodic, and chaotic regimes are all possible for individual generators in this case. The issue of interaction of several generators with different types of oscillations remains understudied. One of the options may be coupling between chaotic and multi-frequency quasi-periodic subsystems. In the present study, the Kislov–Dmitriev generator [14] in a chaotic regime was chosen as the first subsystem. This generator is a closed loop of a nonlinear amplifier, an *RLC* filter, and an inertial element. A system of five coupled van der Pol generators [15] was chosen in order to enable quasi-periodic oscillations with different numbers of incommensurate frequencies in the second subsystem. A similar system was also considered in [16,17]. Note that the van der Pol equation characterizes not only the classical generator, but also a large number of systems of different nature [18]. In system [15], new modes cross successively the excitation threshold as the coupling parameter decreases, so that increasingly high-frequency quasi-periodic oscillations are generated. This pattern corresponds to the well-known Landau–Hopf scenario [19]. The presence of five oscillators allows for several steps of such a scenario to be implemented. Note that a single oscillator coupled to the Kislov–Dmitriev generator was examined in [20]. Specifically, the possible stabilizing influence of the van der Pol generator on the chaotic generator was demonstrated. Such systems may also be of interest in the context of practical applications (e.g., in problems of communication and chaos control).

In accordance with [14,15,20], one may write the following equations for the system under consideration:

$$\ddot{x} + \frac{1}{Q}\dot{x} + \omega_0^2 x + k(\dot{x} - \dot{x}_5) = z,$$

$$T\dot{z} + z = Mx\exp(-x^2),$$

$$\ddot{x}_n - (\lambda_n - x_n^2)\dot{x}_n + \left(1 + \frac{\Delta}{4}(n-1)\right)x_n + \frac{\mu}{4}\sum_{i=1}^{5}(\dot{x}_n - \dot{x}_i) = 0,$$

$$\ddot{x}_5 - (\lambda_5 - x_5^2)\dot{x}_5 + (1+\Delta)x_5 + \frac{\mu}{4}\sum_{i=1}^{5}(\dot{x}_5 - \dot{x}_i) + k(\dot{x}_5 - \dot{x}) = 0. \quad (1)$$

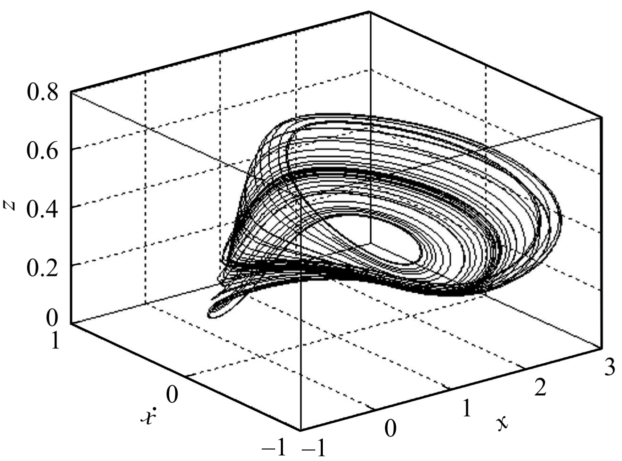

Figure 1. Phase portrait of an individual Kislov–Dmitriev generator in a chaotic regime. $T = 10$, $Q = 20$, $M = 2.75$, and $\omega_0 = 0.5$.

Relation between regime types and Lyapunov exponents spectrum

Designation	Regime type	Spectrum of largest Lyapunov exponents
P	Periodic (limit cycle)	$\Lambda_1 = 0,\ \Lambda_{2,3,4,5,6,7} < 0$
$2T$	Two-frequency quasi-periodic (two-dimensional torus)	$\Lambda_{1,2} = 0,\ \Lambda_{3,4,5,6,7} < 0$
$3T$	Three-frequency quasi-periodic (three-dimensional torus)	$\Lambda_{1,2,3} = 0,\ \Lambda_{4,5,6,7} < 0$
$4T$	Four-frequency quasi-periodic (four-dimensional torus)	$\Lambda_{1,2,3,4} = 0,\ \Lambda_{5,6,7} < 0$
$5T$	Five-frequency quasi-periodic (five-dimensional torus)	$\Lambda_{1,2,3,4,5} = 0,\ \Lambda_{6,7} < 0$
$6T$	Six-frequency quasi-periodic (six-dimensional torus)	$\Lambda_{1,2,3,4,5,6} = 0,\ \Lambda_7 < 0$
C	Chaos	$\Lambda_1 > 0,\ \Lambda_{2,3,4,5,6,7} \leqslant 0$

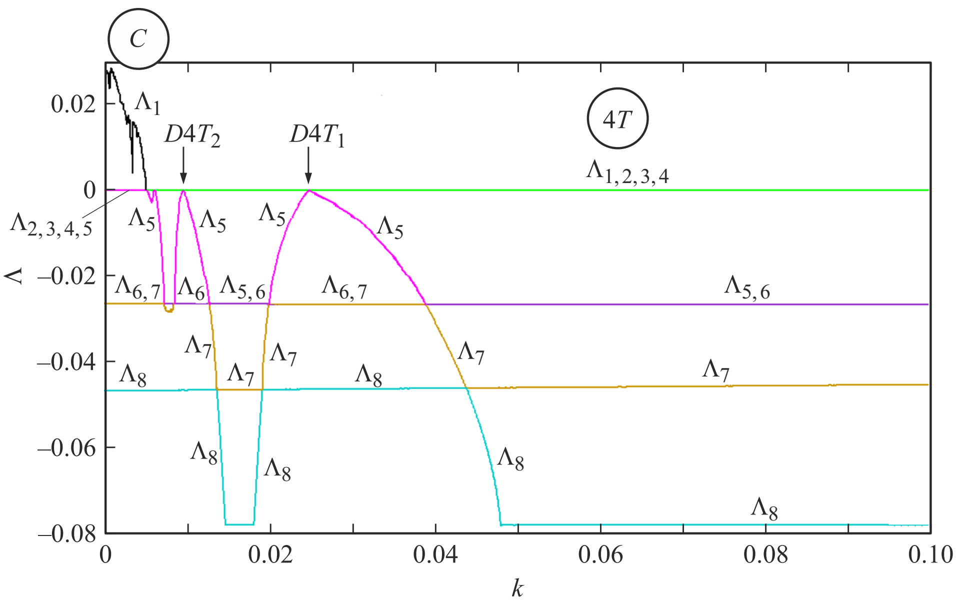

Figure 2. Dependence of the largest Lyapunov exponents of system (1) on parameter k of coupling between the quasi-periodic and chaotic subsystems. $\mu = 0.25$ and $\Delta = 3$. The Kislov–Dmitriev generator parameters are as follows: $M = 2.75$, $T = 10$, $Q = 20$, and $\omega_0 = 0.5$.

Here, x and z are the Kislov–Dmitriev generator variables, x_n are the variables of van der Pol generators, and n varies from 1 to 4. The Δ parameter controls the mutual frequency detuning of van der Pol generators (with the frequency of the first one taken as unity). Parameter k characterizes the coupling between the Kislov–Dmitriev generator and the quasi-periodic subsystem. This coupling is established through the fifth van der Pol generator and is dissipative.

Let us use the following set of Kislov–Dmitriev generator parameters: $T = 10$, $Q = 20$, $M = 2.75$, and $\omega_0 = 0.5$. It corresponds to the chaotic oscillation regime, which is illustrated by the phase portrait in Fig. 1. Frequency value $\omega_0 = 0.5$ provides frequency detuning from all van der Pol generators. Following [15], we set van der Pol generator excitation parameters $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$, $\lambda_4 = 0.4$, $\lambda_5 = 0.5$, and $\Delta = 3$. In this case, as „internal“ coupling

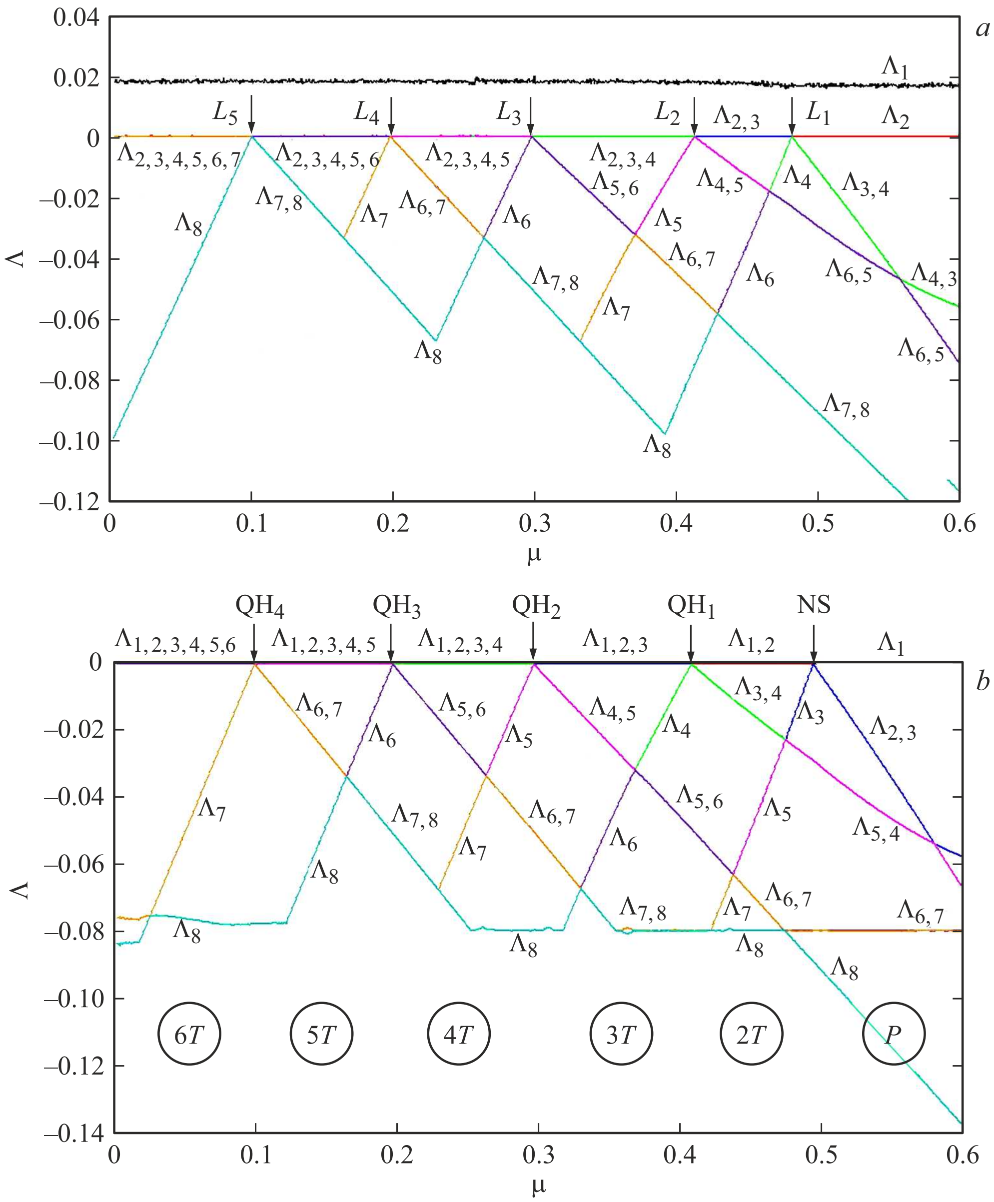

Figure 3. Dependence of the largest Lyapunov exponents of system (1) on parameter μ of coupling between the van der Pol generators. $k = 0.0025$ (*a*) and 0.016 (*b*).

parameter μ decreases, a step-by-step transition from the periodic regime to the five-frequency quasi-periodic one is observed in the quasi-periodic subsystem in accordance with the Landau–Hopf scenario [15]. Note that the pattern remains unchanged even if the chosen parameter values are varied slightly.

In what follows, we identify the regime of system (1) based on the spectrum of Lyapunov exponents in accordance with the table. Figure 2 shows the plots of eight largest exponents as functions of coupling parameter k of subsystems. The chosen value of parameter $\mu = 0.25$ corresponds to the three-frequency regime in the van der Pol oscillator ensemble [15]. The region of four-frequency regime (4-torus) $4T$, which has $\Lambda_{1,2,3,4} = 0$, is seen on the right in Fig. 2. Thus, the interaction with the chaotic subsystem led to an increase in the number of incommensurate frequencies (the torus dimension), and chaos was suppressed. As coupling strength k decreases, the four-frequency torus undergoes doubling bifurcations at points $D4T_1$ and $D4T_2$. These are evidenced by vanishing of the Λ_5 exponent, which remains negative in the vicinity of bifurcation points [21]. The torus then collapses, giving rise to chaos C with $\Lambda_1 > 0$. In the present case, chaos has a non-trivial feature: four zero exponents ($\Lambda_{2,3,4,5} = 0$). Thus, three other exponents are added to zero exponent $\Lambda_2 = 0$ that is „mandatory" for a continuous-time system. The examples of chaos discussed earlier had only one [22–25] and two [26] such exponents. Note that rigorous proofs are still lacking; therefore, a more correct term, which has already been used in [24], will be a Lyapunov exponent „indistinguishable from zero in numerics." The required accuracy of calculations of exponents was set for this purpose in the process of plotting the dependences in Fig. 2.

Let us now illustrate the pattern of regimes obtained at various values of „internal" coupling parameter μ of the quasi-periodic subsystem and two values of k (Fig. 3). In the case of small k (Fig. 3, a), classical chaos with $\Lambda_1 > 0$, $\Lambda_2 = 0$, and the remaining exponents being negative is observed at strong coupling μ. The type of chaotic regime changes at point L_1: an additional zero Lyapunov exponent emerges, so that $\Lambda_1 > 0$, $\Lambda_{2,3} = 0$, and the remaining exponents are negative. At points L_2, L_3, L_4, and L_5, chaotic regimes with two, three, four, and five additional zero Lyapunov exponents arise successively. These are certain special bifurcations, which we have designated L (derived from Lyapunov exponents).

Let us now raise parameter k of coupling of the quasi-periodic and chaotic subsystems (Fig. 3, b). Chaos is suppressed in this case. The Neimark−Sacker (NS) bifurcation of birth of a two-frequency torus with $\Lambda_{1,2} = 0$ from limit cycle P with $\Lambda_1 = 0$ is then found in the right part of the figure. As μ decreases, a successive cascade of quasi-periodic Hopf bifurcations $QH_{1,2,3,4}$ of soft birth of 3-torus, 4-torus, etc., is observed. The criterion for a bifurcation of this type is the equality of two negative exponents through to its threshold [21]. For example, exponents $\Lambda_3 = \Lambda_4$ and $\Lambda_4 = \Lambda_5$ match on approach to points QH_1 and QH_2, respectively. This pattern may be associated with the Landau−Hopf scenario. This suggests that the Landau−Hopf scenario observed in the ensemble of van der Pol generators is stable and is not disrupted in interaction with chaos if their coupling is relatively strong. Moreover, one more Hopf bifurcation of birth of a stable 6-torus is added. This is what distinguishes the examined pattern from the Ruelle−Takens scenario [27].

Thus, in the studied system, the interaction of subsystems with chaotic dynamics and the Landau−Hopf scenario gives rise to several significant features: the emergence of a higher-dimension torus, its doubling, and a cascade of points with a gradually increasing number of zero Lyapunov exponents in the chaotic regime. It was also demonstrated that the Landau−Hopf scenario is stable with respect to interaction with a chaotic subsystem if their coupling is relatively strong.

Funding

This study was carried out under the state assignment of the Kotel'nikov Institute of Radio Engineering and Electronics of the Russian Academy of Sciences.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] A. Pikovsky, M. Rosenblum, J. Kurths, *Synchronization: a universal concept in nonlinear science* (Cambridge University Press, 2001).
[2] A.G. Balanov, N.B. Janson, D.E. Postnov, O. Sosnovtseva, *Synchronization: from simple to complex* (Springer, 2009).
[3] Yu.A. Kuznetsov, *Elements of applied bifurcation theory* (Springer, 2023).
[4] Yu.A. Kuznetsov, H.G.E. Meijer, *Numerical bifurcation analysis of maps: from theory to software* (Cambridge University Press., 2019). DOI: 10.1017/9781108585804
[5] X. Chen, S. Qian, F. Yu, Z. Zhang, H. Shen, Y. Huang, S. Cai, Z. Deng, Y. Li, S. Du, Complexity, **2020**, 1 (2020). DOI: 10.1155/2020/8274685
[6] L. Přibylová, J. Ševčík, V. Eclerová, P. Klimeš, M. Brázdil, H.G.E. Meijer, Network Neurosci., **8** (1), 293 (2024). DOI: 10.1162/netn_a_00351
[7] M. Bucolo, A. Buscarino, L. Fortuna, S. Gagliano, Front. Phys., **10**, 862376 (2022). DOI: 10.3389/fphy.2022.862376
[8] M. Kopp, J. Telecommun. Electron. Comput. Eng., **16** (1), 13 (2024). DOI: 10.54554/jtec.2024.16.01.003
[9] A.V. Kurbako, V.I. Ponomarenko, M.D. Prokhorov, Tech. Phys. Lett., **48** (10), 38 (2022). DOI: 10.21883/TPL.2022.10.54796.19328.
[10] A.P. Kuznetsov, Yu.V. Sedova, N.V. Stankevich, Tech. Phys. Lett., **48** (12), 56 (2022). DOI: 10.21883/TPL.2022.12.54949.19296.
[11] I.A. Korneev, A.V. Slepnev, V.V. Semenov, T.E. Vadivasova, Izv. Vyssh. Uchebn. Zaved. Prikl. Nelineinaya Din., **28** (3), 324 (2020) (in Russian). DOI: 10.18500/0869-6632-2020-28-3-324-340
[12] B. Singhal, I.Z. Kiss, J.S. Li, SIAM J. Appl. Dyn. Syst., **22** (3), 2180 (2023). DOI: 10.1137/22M152120
[13] P. Mircheski, J. Zhu, H. Nakao, Chaos, **33** (10), 103111 (2023). DOI: 10.1063/5.0161119
[14] A.S. Dmitriev, E.V. Efremova, N.A. Maksimov, A.I. Panas, Generatsiya khaosa (Tekhnosfera, 2012) (in Russian).
[15] A.P. Kuznetsov, S.P. Kuznetsov, I.R. Sataev, L.V. Turukina, Phys. Lett. A, **377**, (45-48), 3291 (2013). DOI: 10.1016/j.physleta.2013.10.013
[16] N.V. Stankevich, A.P. Kuznetsov, E.P. Seleznev, Tech. Phys., **62** (6), 971 (2017). DOI: 10.1134/S106378421706024X.
[17] N.V. Stankevich, E.S. Popova, A.P. Kuznetsov, E.P. Seleznev, Tech. Phys. Lett., **45** (12), 1233 (2019). DOI: 10.1134/S1063785019120265.
[18] A.P. Kuznetsov, E.S. Seliverstova, D.I. Trubetskov, L.V. Tyuryukina, Izv. Vyssh. Uchebn. Zaved. Prikl. Nelineinaya Din., **22** (4), 3 (2014) (in Russian). DOI: 10.18500/0869-6632-2014-22-4-3-42
[19] L.D. Landau, Dokl. Akad. Nauk USSR, **44**, 311 (1944).
[20] Yu.P. Emel'yanova, A.P. Kuznetsov, Tech. Phys., **56** (4), 435 (2011). DOI: 10.1134/S106378421104013X.
[21] R. Vitolo, H. Broer, C. Simó, Regul. Chaot. Dyn., **16** (1-2), 154 (2011). DOI: 10.1134/S1560354711010060
[22] H. Broer, C. Simó, R. Vitolo, Nonlinearity, **15** (4), 1205 (2002). DOI: 10.1088/0951-7715/15/4/312
[23] N.V. Stankevich, N.A. Shchegoleva, I.R. Sataev, A.P. Kuznetsov, J. Comput. Nonlinear Dyn., **15** (11), 111001 (2020). DOI: 10.1115/1.4048025
[24] E.A. Grines, A. Kazakov, I.R. Sataev, Chaos, **32** (9), 093105 (2022). DOI: 10.1063/5.0098163

[25] I. Garashchuk, A. Kazakov, D. Sinelshchikov, Chaos Solit. Fract., **182**, 114785 (2024).
DOI: 10.1016/j.chaos.2024.114785
[26] A.P. Kuznetsov, Y.V. Sedova, N.V. Stankevich, Chaos Solit. Fract., **169**, 113278 (2023).
DOI: 10.1016/j.chaos.2023.113278
[27] D. Ruelle, F. Takens, Commun. Math. Phys., **20**, 167 (1971).
DOI: 10.1007/BF01646553

Translated by D.Safin