The differential flow near by polar cap surface of neutron star in the case of inclined magnetic field

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Received April 27, 2024
Revised July 30, 2024
Accepted October 30, 2024

The flow in a liquid layer at the neutron star surface due to magnetospheric electric current in the case of homogeneous inclined to star surface magnetic field is considered.

Keywords: radiopulsar, neutron star, magnetic hydrodynamics.

Introduction

One of the main neutron star deceleration mechanisms is short circuiting of electric current, that flows in the magnetosphere along magnetic field lines, in the star's surface layers [1,2]. Pulsar deceleration by electric current flowing through the magnetosphere and distribution of the braking torque on the star surface layers were discussed, for example, in [3]. Flow induced by electric current in the flat layer of conducting liquid was addressed, for example, in [4]. This work, following studies [5,6], discusses the flow induced by this current in the liquid layer on the neutron star surface.

1. Model

As in [6], we assume that the neutron star surface is covered with liquid layer „ocean" with the depth $L \sim 10{-}100\,\mathrm{m}$ [7]. Since $L \ll r_{ns}$, where $r_{ns} \approx 10\,\mathrm{km}$ — is the neutron star radius, then we neglect the surface curvature and assume that the ocean is an infinite flat layer with the depth L (Figure 1). We address the liquid flow in the neutron star's frame of reference and assume that the liquid flow doesn't depend on the time t in this frame of reference. In this work, as in [6], we limit ourselves to the simplest case, when the pressure p depends only on the liquid density ρ. Liquid viscosity and conductivity are also assumed as isotropic for simplicity. In this case, magnetohydrodynamics equations may be written as

$$\rho\left(2\left[\boldsymbol{\Omega}\times\mathbf{v}\right]+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=-\nabla p+\frac{1}{c}\left[\mathbf{j}\times\mathbf{B}\right]+\mathscr{F}_{vis}+\rho\mathbf{g}, \tag{1}$$

$$-\nabla\Phi+\frac{1}{c}\left[\mathbf{v}\times\mathbf{B}\right]=R\mathbf{j},\ \ \operatorname{div}\mathbf{B}=0,$$

$$\operatorname{rot}\mathbf{B}=\frac{4\pi}{c}\,\mathbf{j},\ \ \operatorname{div}(\rho\mathbf{v})=0, \tag{2}$$

where $\mathbf{v}$ is the liquid flow velocity in the frame of reference that rotates together with the star, $\mathbf{B}$ is the magnetic field induction, $\mathbf{j}$ is the current density, Φ is the electrostatic potential, R is the liquid resistance, $p=p(\rho)$, $\mathscr{F}_{vis}$ is the viscosity force, $\mathbf{g}=-g\mathbf{e}_z$ is the gravity field intensity, and we assume that g is independent of coordinates, $\boldsymbol{\Omega}$ is the star's angular rotation rate, $\Omega=2\pi/P$, P is the star's rotation period. We assume that to the zero approximation the ocean has no flow $\mathbf{v}=0$ and electric currents $\mathbf{j}=0$.

Equations (1) and (2) then are written as

$$\nabla p_{(0)}=\rho_{(0)}\cdot g\mathbf{e}_z,\ \ \operatorname{div}\mathbf{B}_{(0)}=0,\ \ \operatorname{rot}\mathbf{B}_{(0)}=0,\ \ \nabla\Phi_{(0)}=0, \tag{3}$$

where $p_{(0)}=p(\rho_{(0)})$ and (0) mark the zero approximation quantities. In this work, we limit ourselves only to a special case of uniform magnetic field $\mathbf{B}_{(0)}=B_{(0)}(\cos\beta\mathbf{e}_z+\sin\beta\mathbf{e}_x)$, where $B_{(0)}$ and β are constant quantities (Figure 1). Assume also $\Phi_{(0)}=0$ and that $p_{(0)}$, $\rho_{(0)}$ and $R_{(0)}$ depend only on z. We now consider small disturbance induced by the current flow through the ocean. We assume $p=p_{(0)}+\delta p$ and $\rho=\rho_{(0)}+\delta\rho$ and limit ourselves only to a linear case in $\mathbf{v}$, $\mathbf{j}$, δp and $\delta\rho$. Then

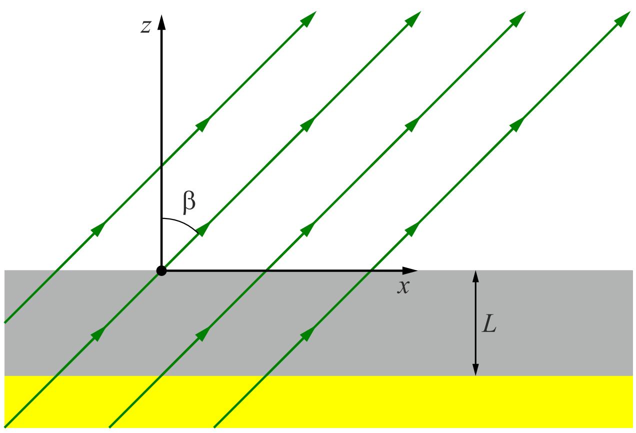

Figure 1. Schematic diagram of the liquid layer on the star surface. The liquid layer is shown grey, hard crust is shown yellow, magnetic field lines are shown by green arrows.

equations (1) and (2) may be written as:

$$2\rho_{(0)}[\boldsymbol{\Omega}\times\mathbf{v}]=\frac{B_{(0)}}{c}[\mathbf{j}\times\mathbf{e}_B]+\mathscr{F}_{vis}-c_s^2\nabla\delta\rho-\delta\rho g\mathbf{e}_z, \tag{4}$$

$$-\nabla\Phi+\frac{B_{(0)}}{c}[\mathbf{v}\times\mathbf{e}_B]=R_{(0)}\mathbf{j},\ \operatorname{div}\mathbf{j}=0 \text{ and } \operatorname{div}(\rho_{(0)}\mathbf{v})=0, \tag{5}$$

where $\mathbf{e}_B=\mathbf{B}_{(0)}/B_{(0)}$ and $c_s^2=\frac{dp}{d\rho}(\rho_{(0)})$. As boundary conditions on the ocean surface $z=0$, we require $\frac{\partial v_x}{\partial z}=0$, $\frac{\partial v_y}{\partial z}=0$ and $v_z=0$ to be met, the first two conditions correspond to the tangential component continuity of the stress tensor at the boundary of the liquid layer and magnetosphere, the latter means that there is no liquid flow from the ocean to the magnetosphere [5]. We also assume that j_z is set at $z=0$, i.e. the current flowing into the ocean from the magnetosphere [5]. On the ocean bottom at $z=-L$, we assume $\mathbf{v}=0$ [5]. In addition, for simplicity we assume that the hard crust has infinite conductivity and, thus, $\Phi=0$ at $z=-L$. We suppose that $L\sim10^2$ m, $\rho\sim10^6\,\mathrm{g\,cm^{-3}}$ [7], the shear-viscosity coefficient $\eta_{(0)}\sim10^4\,\mathrm{g\,m^{-1}\,s^{-1}}$ [8,9], $R_{(0)}\sim10^{-19}$ CGS [10]. The Ekman number $E=\eta_{(0)}/(\Omega L^2\rho_{(0)})\sim10^{-11}$ and the Hartman number $\mathrm{Ha}=(B_{(0)}L)/(c\sqrt{\eta_{(0)}R_{(0)}})\sim10^{11}$. Since $\mathrm{Ha}^2\gg E^{-1}\gg1$, then the Coriolis forces can be neglected and everywhere, except the surface layers, the viscous forces $\mathscr{F}_{vis}$ can be also neglected. The nonlinear term $\rho(\mathbf{v}\cdot\nabla)\mathbf{v}$ is also neglected because the Reynolds numberRe $=\rho_{(0)}vL/\eta_{(0)}$ in our case will be very low $\mathrm{Re}\sim10^{-2}$–10^{-4}. Equation (1) also neglects the small term $\rho[\dot{\boldsymbol{\Omega}}\times\mathbf{x}]$ and doesn't assume centrifugal forces due to their smallness compared with the gravity $\rho\mathbf{g}$ [5]. An approximate solution of equations (4) and (5) outside the boundary layers may be written as

$$v_x=-\frac{1}{\cos^2\beta}\frac{c^2}{B_{(0)}^2}\left(R_f\frac{\partial\delta\hat{p}_0}{\partial\tilde{x}}+\mathrm{tg}\beta\tilde{R}_f\frac{\partial^2\delta\hat{p}_0}{\partial y^2}\right)+\frac{1}{\cos^2\beta}\frac{c}{B_{(0)}}\frac{\partial\hat{j}_B}{\partial y}\left(\tilde{R}_{(0)}+\frac{\sin^2\beta}{\rho_{(0)}(z)}\tilde{K}(z)\right), \tag{6}$$

$$v_y=\frac{1}{\cos^2\beta}\frac{c^2}{B_{(0)}^2}\left(-R_f\frac{\partial\delta\hat{p}_0}{\partial y}+\mathrm{tg}\beta\tilde{R}_f\frac{\partial^2\delta\hat{p}_0}{\partial\tilde{x}\,\partial y}\right)+\frac{c}{B_{(0)}}\left(\mathrm{tg}\beta R_{(0)}\hat{j}_B-\frac{\tilde{R}_{(0)}}{\cos^2\beta}\frac{\partial\hat{j}_B}{\partial\tilde{x}}\right), \tag{7}$$

$$v_z=\mathrm{tg}\beta\frac{c}{B_{(0)}}\frac{1}{\rho_{(0)}(z)}\frac{\partial\hat{j}_B}{\partial y}\tilde{K}(z), \tag{8}$$

where the following notations are introduced

$$\tilde{R}_{(0)}(z)=\int_{-L}^{z}R_{(0)}(z')dz',\ R_f(z)=R_{(0)}(z)f(z),$$

$$\tilde{R}_f(z)=\int_{-L}^{z}R_f(z')dz'$$

and

$$\tilde{K}(z)=\int_{-L}^{z}\rho_{(0)}(z')R_{(0)}(z')(f(z')K_0-1)dz',$$

constant K_0 is defined as

$$K_0=\int_{-L}^{0}\rho_{(0)}(z')R_{(0)}(z')dz'\Big/\int_{-L}^{0}\rho_{(0)}(z')R_{(0)}(z')f(z')dz'.$$

$f(z)$ is defined as

$$f(z)=\exp\left(\int_{z}^{0}g/c_s^2(z')dz'\right).$$

$\delta\hat{p}_0$ and $\hat{j}_B$ depend only on $\tilde{x}=x-\mathrm{tg}\beta z$ and y. $\delta\hat{p}_0$ is equal to

$$\delta\hat{p}_0(\tilde{x},y)=\sin\beta\frac{B_{(0)}}{c}\frac{K_0}{4\pi}\int_{-\infty}^{+\infty}\ln\big((\tilde{x}-\tilde{x}')^2+(y-y')^2\big)\frac{\partial\hat{j}_B}{\partial y}(\tilde{x}',y')d\tilde{x}'dy'. \tag{9}$$

Correction to pressure δp is equal to $\delta p=\delta\hat{p}_0(\tilde{x},y)\cdot f(z)$ and correction to density $\delta\rho$ is equal to $\delta\rho=\delta p/c_s^2(z)$, respectively. $\hat{j}(\tilde{x},y)$ is defined as $\hat{j}_B(x,y)=j_z(x,y,0)/\cos\beta$ and equal to the density of current flowing in the magnetosphere along the magnetic field lines.

Contribution of the surface layers and consideration of the Coriolis force only give the corrections $\sim1/(\mathrm{Ha}\cdot\cos(\beta))$ and $\sim E^{-1}/\mathrm{Ha}^2$ to expressions (6)–(8). Moreover, the liquid flow velocity $\mathbf{v}$ in the upper boundary layer remains almost unchanged and expressions (6) and (8) give correct velocities on the ocean surface with an accuracy to corrections $\sim1/(\mathrm{Ha}\cdot\cos(\beta))$. To illustrate the arising flow, let's consider the simplest model case, when the profile of current flowing from the magnetosphere into the liquid layer is axisymmetric, i.e. $\hat{j}_B(\tilde{x},y)=\hat{j}_B(r)$, where $r=\sqrt{\tilde{x}^2+y^2}$, and assume that $c_s^2=\mathrm{const}(z)$ and $R_{(0)}=R_s\exp\big(-\gamma(z+L)\big)$. The current profile is taken the same as in [5], i.e. assume that $\hat{j}_B(r)=I_s\int_0^{+\infty}\mathscr{J}_B(k)J_0(kR)dk$, where

$$\mathscr{J}_B(k)=\big(a^2b^2/(b^2-a^2)\big)\big((1/b)J_1(kb)-(1/a)J_1(ka)\big)\exp(-\varepsilon k^2).$$

This current profile for $a=0.9b$ and $\varepsilon=10^{-4}$ is shown in Figure 2, the left curve. The right curve in Figure 2 shows

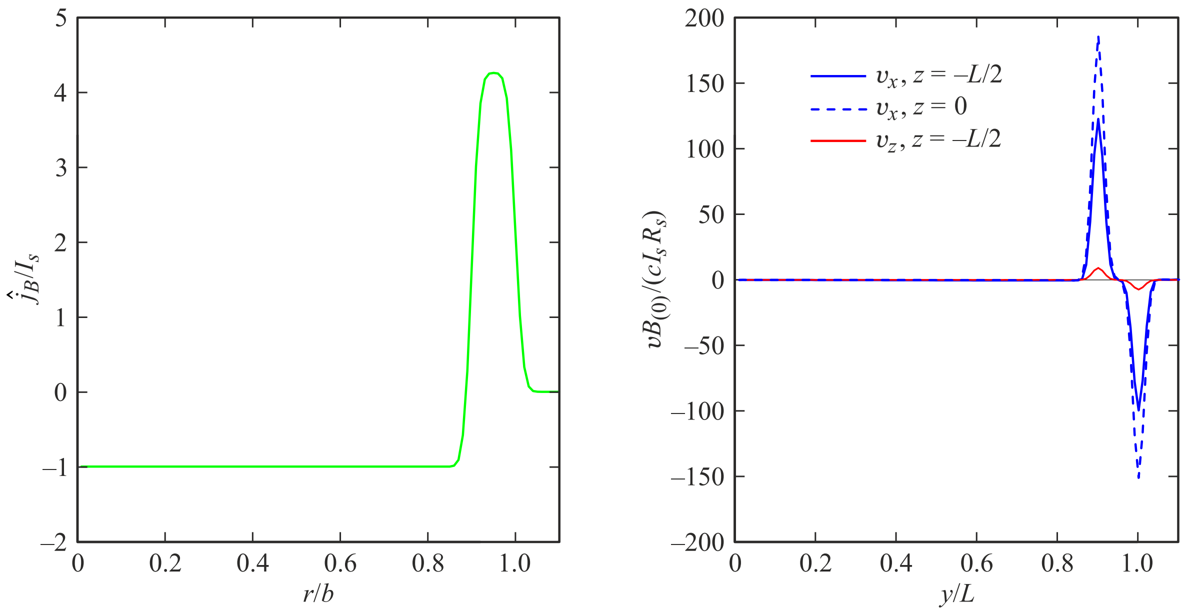

Figure 2. The left curve shows the current profile for current flowing from the magnetosphere. The right curve shows the dependence of the velocity components v_x and v_z on y at $\tilde{x} = 0$ for $z = 0$ and $z = -L/2$. Due to the boundary conditions $v_z = 0$ at $z = 0$.

the velocity components v_x and v_z of the flow that occurs at such current in the plane $\tilde{x} = 0$ for $z = 0$ and $z = -L/2$ for the case of $\beta = 45^\circ$, $b = L$, $gL/c_s^2 = 1$ and $\gamma L = 1$. It can be seen that, as in the case of the vertical magnetic field $\beta = 0$ [5], almost all current is accumulated near the pulsar tube boundary, where a current gradient exists.

Conclusion

Expression for liquid flow induced in the ocean on the neutron star surface by the current $\hat{j}_B$flowing in the magnetosphere in the case of inclined uniform magnetic field $\mathbf{B}_{(0)}$ was addressed. As in [5,6], the resulting flow velocity is extremely low. With $B_{(0)} \sim 10^{12}$ Gs and $P \sim 1$ s, assuming that $\hat{j}_B \sim \Omega B_{(0)}/(2\pi c)$, we have $v \sim 10^{-10}$–10^{-8} cm s^{-1}. This reflects the fact that current isn't almost short circuited in the liquid layer [5], which agrees with the results of [3] and, in particular, supports the conclusions of [2] regarding deceleration of the J0901-4046 pulsar by current loss due to currents flowing through the vacuum gap, including also the case when a small-scale magnetic field exists on the neutron star surface. Unlike the case of $\beta = 0$ addressed in [5,6], consideration of the field inclination leads to the occurrence of a vertical liquid flow components compared in magnitude with horizontal components. Such flow can probably induce a very slowly growing instability that is a little similar to that addressed in [11]. Note also that, when the magnetic field is close to the horizontal field, $\cos\beta \lesssim 0.1$, the flow velocity may grow by a factor of $\sim 10^2$–10^3 and a situation with Re > 1 is possible in the area with sharp current gradients. In this case, the obtained solution is not applicable any more and instability may occur [11].

Acknowledgments

The authors are grateful to A.I. Tsygan and M.V. Vorontsov for their support, comments and useful discussions, A.I. Chugunov, A.V. Biryukov, I.F. Malov, V.A. Urpin and E.A. Mikhailov for their comments and useful discussions, as well as an anonymous referee for comments regarding the case of $\cos\beta \approx 0$.

Conflict of interest

The authors declare no conflict of interest.

References

[1] V.S. Beskin. Phys.-Uspekhi, **61** (4), 353 (2018). DOI: 10.3367/UFNe.2017.10.038216
[2] D.N. Sob'yanin. Phys. Rev. D, **107**, L081301 (2023). DOI: 10.1103/PhysRevD.107.L081301
[3] D.N. Sob'yanin. Phys. Rev. D, **109**, L061302 (2024). DOI: 10.1103/PhysRevD.109.L061302
[4] E.A. Mikhailov, A.Y. Chudnovsky. J. Appl. Industrial Mathem., **14**, 722 (2020). DOI: 10.1134/S1990478920040109
[5] A.I. Tsygan, D.A. Shalybkov, D.P. Barsukov, O.A. Goglichidze. Nauchno-Tekhicheskie Vedomosti SPbPU. Fiziko-matematicheskie nauki, **194** (2), 120 (2014) (in Russian).
[6] M.V. Vorontsov, D.P. Barsukov. J. Phys. Conf. Ser., **1400**, 022022 (2019). DOI: 10.1088/1742-6596/1400/2/022022
[7] P. Haensel, A.Y. Potekhin, D.G. Yakovlev. *Neutron stars I. Equation of State and Structure* (Springer, NY., 2007), p. 13. ISBN 0387335439
[8] A.I. Chugunov, D.G. Yakovlev, Astronomy Reports, **49** (9), 724 (2005). DOI: 10.1134/1.2045323

[9] D.D. Ofengeim, D.G. Yakovlev. Europhys. Lett., **112** (5), 59001 (2015). DOI: 10.1209/0295-5075/112/59001
[10] A.Y. Potekhin. A&A, **351**, 787 (2015). DOI: 10.48550/arXiv.astro-ph/9909100
[11] E.A. Kuznetsov, E.A. Mikhailov. eprint arXiv:2402.16989 (2024). DOI: 10.48550/arXiv.2402.16989

Translated by E.Ilinskaya