Method of calculation of relativistic charged particles density

© L.A. Bakaleinikov, V.I. Kuznetsov, E.Yu. Flegontova, D.P. Barsukov, I.K. Morozov

Ioffe Institute, St. Petersburg, Russia
E-mail: morozov22505@gmail.com

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The Q, G method for studying non-stationary processes in collisionless plasma with relativistic particles has been developed. Analytical expressions for the particles density and current in a slightly perturbed electric field are obtained. Based on the obtained expressions, a theory of stability of a relativistic Boursian diode stationary states in the absence of electron reflection from potential barriers is constructed.

Keywords: plasma diode, electron and positron beams, solutions stability.

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Plasma produced by various astrophysical objects [1] and laboratory setups [2] often contains charged particles accelerated to relativistic velocities. Pulsars are one of the most complex objects with relativistic plasma. They have been discovered approximately 50 years ago, but the nature of generation of their RF radiation still remains unclear [3]. Fluxes of charged particles in the indicated objects normally leave the emitter surface with a known velocity distribution function (VDF) and propagate without collisions in a nonstationary electric field. The Q, G-method for calculating the density of nonrelativistic particles was proposed in [4]. It relies on an analytical expression that is an integral over the regions of initial velocities at the emitter. Compared to the stationary case [5], this expression contains two additional functions: G and Q, which are associated with the field change over time. Exact expressions were obtained for them in [4]. In the present study, a method of this kind is developed for relativistic particles. It is demonstrated that the method provides an opportunity to investigate the stability of stationary states (constructed in [5]) of a Bursian diode with relativistic electrons.

A planar diode with distance d and potential difference U between the electrodes is considered. It is assumed that a relativistic electron flux is supplied from an electrode with a known VDF $f_0(v_0, t_0)$ and moves in a nonstationary electric field without collisions.

The equation of motion of a relativistic electron with momentum $p = \gamma m v$ ($\gamma = \left[1 - v^2/c^2\right]^{-1/2}$ is the relativistic factor, c is the speed of light, and m is the electron mass) in electric field $E(z, t)$ takes the form

$$\frac{dp}{dt} = -eE(z, t) = e\frac{\partial}{\partial z}\varphi(z, t). \quad (1)$$

Here, e is the electron charge and $\varphi(z, t)$ is the potential. Multiplying (1) by velocity v and using the expressions for kinetic energy $W_{kin} = (\gamma - 1)mc^2$ and its derivative $\frac{dW_{kin}}{dt} = v\frac{dp}{dt}$, we obtain the law of conservation of energy with an additional term on the right-hand side — $eG(z, t; v_0, t_0)$:

$$(\gamma - 1)mc^2 - e\varphi - (\gamma_0 - 1)mc^2$$
$$= -e\int_{t_0}^{t} dt' \frac{\partial}{\partial t'}\varphi(z, t')\Big|_{z=z(t'; v_0, t_0)} \equiv eG(z, t; v_0, t_0). \quad (2)$$

Here, γ_0 is the relativistic factor at the emitter. Quantity $e\,G(z, t; v_0, t_0)$ is the amount of energy acquired by an electron leaving the emitter with velocity v_0 at time t_0, propagating in a nonstationary field, and arriving at point z at time t. Equation (2) relates the velocity of the particle at point (t, z) to its velocity at the moment of emission.

The calculation of particle density follows the one performed in [4]. We divide the VDF at the emitter tentatively into groups („beams") of electrons leaving the emitter with velocities from a narrow interval $(v_0,\ v_0 + \Delta v_0)$ within a short period of time $(t_0,\ t_0 + \Delta t_0)$. It is assumed that density $\Delta n(z, t; v_0, t_0)$ for each beam is related to its average velocity $v(z, t; v_0, t_0)$ by a formula similar to formula (5) from [4]:

$$\Delta n(z, t; v_0, t_0) = \frac{f_0(v_0, t_0)v_0 dv_0}{\left|D(z, t; v_0, t_0)\right|},$$
$$D(z, t; v_0, t_0) = -\frac{\partial}{\partial t_0} z(t, v_0, t_0). \quad (3)$$

This relation may be obtained by applying the law of conservation of energy for a particle and the continuity equation for a beam.

To find the trajectory of an electron emitted with velocity v_0 at time t_0, we integrate equation of motion (1) once. This yields

$$p(t; v_0, t_0) = m\gamma_0 v_0$$
$$- e\int_{t_0}^{t} dt' E[z(t'; v_0, t_0), t'] \equiv F(t; v_0, t_0). \quad (4)$$

Velocity v determined from this relation is

$$v(t;v_0,t_0)=\frac{d}{dt}z(t;v_0,t_0)=\frac{F(t;v_0,t_0)}{[m^2+F^2(t;v_0,t_0)/c^2]^{1/2}}. \tag{5}$$

Solving differential equation (5), we find trajectory $z(t;v_0,t_0)$:

$$z(t;v_0,t_0)=\int_{t_0}^{t}dt'\frac{F(t';v_0,t_0)}{[m^2+F^2(t';v_0,t_0)/c^2]^{1/2}}$$

$$=\int_{0}^{t-t_0}dt'\frac{F(t'+t_0;v_0,t_0)}{[m^2+F^2(t+t_0';v_0,t_0)/c^2]^{1/2}}. \tag{6}$$

Functions $D(z,t;v_0,t_0)$ and $Q(z,t;v_0,t_0)$ are then written as

$$D(z,t;v_0,t_0)=v(t;v_0,t_0)+e\,m^2$$

$$\times\int_0^{t-t_0}dt'\,\frac{\int_0^{t'}dt''\frac{d}{dt_0}E[z(t''+t_0;v_0,t_0),t''+t_0]}{[m^2+F^2(t'+t_0)/c^2]^{3/2}},$$

$$Q(z,t;v_0,t_0)=v-D=-e\,m^2$$

$$\times\int_0^{t-t_0}dt'\,\frac{\int_0^{t'}dt''\frac{d}{dt_0}E[z(t''+t_0;v_0,t_0),t''+t_0]}{[m^2+F^2(t'+t_0)/c^2]^{3/2}}. \tag{7}$$

To calculate the density, we sum the contributions of all beams that may reach point z at time t:

$$n(z,t)=\sum_{i=0,1}\int_{\Omega_i(z,t)}\frac{f_0(v_0)\,v_0dv_0}{|v(z,t;v_0,t_0)-Q(z,t;v_0,t_0)|},$$

$$v(z,t;v_0,t_0)$$

$$=c\,\frac{\left(\left\{\gamma_0+e/(mc^2)[\varphi(z,t)+G(z,t;v_0,t_0)]\right\}^2-1\right)^{1/2}}{\gamma_0+e/(mc^2)\,[\varphi(z,t)+G(z,t;v_0,t_0)]}. \tag{8}$$

Here, $i=0$ and 1 correspond to particles arriving at point z with positive and negative velocities. The relation between velocity v_0 and time of emission t_0 of particles from the boundary may be found by solving equation of motion (6). In addition, at given z and t, regions $\Omega_i(z,t)$ are determined from the shape of curve $v_0=v_0(t_0;t,z)$.

Let us consider the motion of particles without reflections from potential barriers in a weakly perturbed field; i.e., it is assumed that

$$\varphi(z,t)=\varphi_0(z)+\tilde\varphi(z)\exp(-i\,\omega\,t),\quad |\tilde\varphi(z)|\ll|\varphi_0(z)|. \tag{9}$$

As in the nonrelativistic case, we assume that $G(z,t)=\tilde G(z)\exp(-i\,\omega\,t)$, $Q(z,t)=\tilde Q(z)\,\exp(-i\omega t)$. Let us find $\tilde Q(z)$. First, we find the derivative of field E with respect to t_0 using (9):

$$\frac{d}{dt_0}E[z(t+t_0;v_0,t_0),t+t_0]$$

$$=\left[i\,\omega\frac{d\tilde\varphi(z)}{dz}-\varphi_0''\,\tilde Q(z)\right]\exp[-i\omega\,(t+t_0)]. \tag{10}$$

Inserting this expression into (7), we obtain an integral equation for $\tilde Q(z)$. Substituting integration variable t in the integrand with variable z $(dt=dz/v(z))$ and switching from $\tilde Q(z)$ to new function $W(z)=\tilde Q(z)\exp[-i\,\omega\,\sigma(z)]$, we obtain

$$W(z)=\int_0^z\frac{dx}{v(x)\gamma^3(x)}\int_0^x\frac{dy}{v(y)}$$

$$\times\left\{(vv'\gamma^3)'W(y)-i\omega\frac{e}{m}\frac{d\tilde\varphi(y)}{dy}\exp\left[-i\omega\sigma(y)\right]\right\}. \tag{11}$$

Here, $\sigma(z)=\int_0^z\frac{dx}{v(x)}$ is the time of flight to point z. This Volterra integral equation may be solved by the method proposed in [4]. Thus, we find $W(z)$ and $\tilde Q(z)$

$$\tilde Q(z)=-i\omega\frac{e}{m}v(z)\exp\left[i\omega\sigma(z)\right]$$

$$\times\int_0^z\frac{dx}{v^3(x)\gamma^3(x)}\int_0^x dy\tilde\varphi'(y)\exp\left[-i\omega\sigma(y)\right]. \tag{12}$$

It is notable that expression (12) for $\tilde Q(z)$ differs from the nonrelativistic case only in the presence of factor γ^3 in the denominator of the outer integral.

The formula for $\tilde G(z)$ is the same as in the nonrelativistic case [4]:

$$\tilde G(z;v_0,t_0)=-\tilde\varphi(z)$$

$$+\int_0^z dx\tilde\varphi'(x)\exp\left\{i\omega\left[\sigma(z)-\sigma(x)\right]\right\}. \tag{13}$$

The developed method allows one to investigate the stability of stationary potential distributions in a Bursian diode that were found in [5]. It is convenient to switch to dimensionless quantities and use the energy of electrons entering from the left boundary and the Debye length at the left boundary as the units of energy and length [5]: $W_0=(\gamma_0-1)m_0c^2$, $\lambda_D=\left[(2\tilde\varepsilon_0W_0)/(e^2n_0)\right]^{1/2}$. The unit of velocity is $v_0=c\sqrt{\gamma_0^2-1}/\gamma_0$. The dimensionless coordinate, potential, electric field strength, velocity, and time are $\xi=z/\lambda_D$, $\eta=e\varphi/(2W_0)$, $\varepsilon=eE\lambda_D/(2W_0)$, $u=v/v_0$, and $\tau=t/(\lambda_D/v_0)$. The dimensionless gap length and potential difference between the electrodes are written as $\delta=d/\lambda_D$ and $V=eU/(2W_0)$. It is convenient to represent stationary solutions by points on the $\left\{\varepsilon_0,\delta\right\}$ plane, where

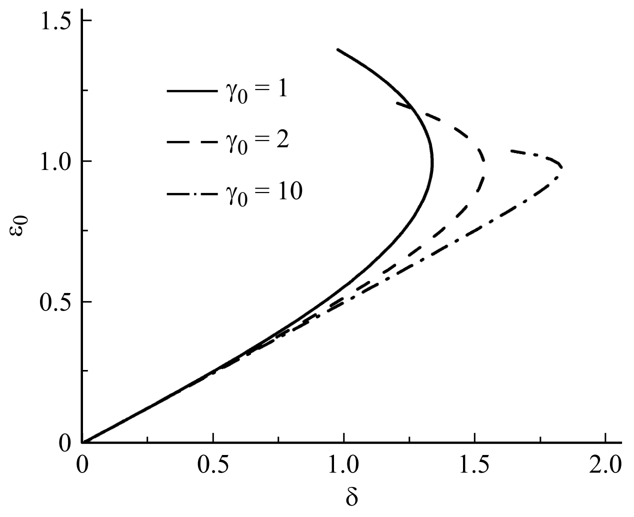

Figure 1. Branches of stationary solutions for a Bursian diode for different values of relativistic factor γ_0. The potential difference between the electrodes is $V = 0$.

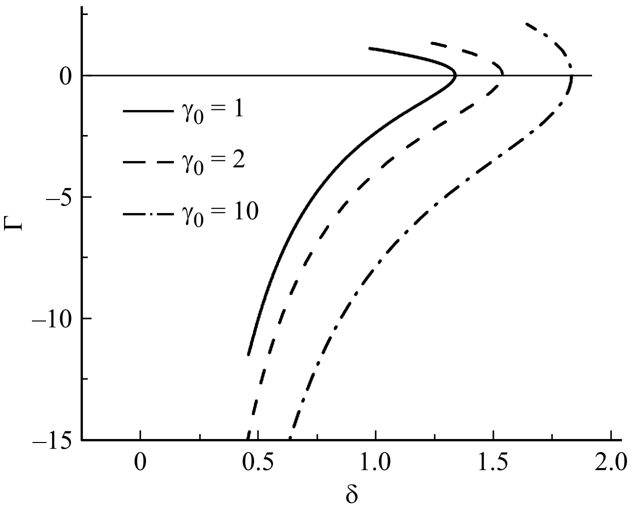

Figure 2. Dependences of the Γ increment on δ for the stationary solutions in a Bursian diode corresponding to the branches in Fig. 1.

ε_0 is the dimensionless electric field strength at the emitter. At fixed V, these points form continuous curves (branches of stationary solutions). In the regime without electron reflection examined in the present study, such branches corresponding to $V = 0$ are shown in Fig. 1 for a series of γ_0 values. The lower and upper parts of these branches are often called normal and overlap branches, respectively, in literature. Note that overlap branches are interrupted on the left at the points of origin of solutions with electron reflection from a virtual cathode.

To examine the stability of stationary solutions, we insert electron density (8) into the Poisson equation in which the potential distribution has the form of expression (9) and perform linearization in the amplitude of small perturbation $\tilde{\varphi}$. An integro-differential equation for $\tilde{\varphi}$ containing $\tilde{G}$ and $\tilde{Q}$ is obtained as a result. We use expressions (12) and (13) for these functions. The resulting equation may be integrated once in z to obtain an integral equation for $\tilde{\varphi}'(z)$. If the electron VDF at the emitter is a δ-function, the equation for amplitude in dimensionless variables takes the form

$$\tilde{\eta}'(\xi) + \frac{2\gamma_0^2}{\gamma_0 + 1} \int\limits_0^{\xi} \frac{dx}{u^3(x)\gamma^3(x)} \int\limits_0^{x} dy\, \tilde{\eta}'(y)$$

$$\times \exp\{i\Omega[q(\xi) - q(y)]\} = -\frac{i}{\Omega}\tilde{J}. \tag{14}$$

Here, $q = \int_0^{\xi} \frac{dx}{u(x)}$ is the dimensionless time of flight of an electron from the emitter to point ξ, $\Omega = \omega/(\lambda_D/v_0)$ is the dimensionless frequency, and $\tilde{J}$ is the dimensionless amplitude of the total current perturbation. The derivative is taken with respect to coordinate ξ.

To solve Eq. (14), one needs to calculate the characteristics of stationary solutions: u and γ. Using the expression for the momentum of a relativistic electron, Eq. (1), and the Poisson equation, we obtain the following for these quantities:

$$u(q)\gamma(q) = \frac{\gamma_0^2}{\gamma_0 + 1} q^2 - \frac{2\gamma_0^2}{\gamma_0 + 1} q + \gamma_0 = f(q),$$

$$u(q) = \frac{f(q)}{\left\{1 + [(\gamma_0^2 - 1)/\gamma_0^2] f^2(q)\right\}^{1/2}},$$

$$\xi(q) = \int\limits_0^{q} \frac{dt\, f(t)}{\left\{1 + [(\gamma_0^2 - 1)/\gamma_0^2]\, f^2(t)\right\}^{1/2}}. \tag{15}$$

Using the law of conservation of energy (2), we obtain the dependence of potential η on q

$$\eta(q) = -\frac{1}{2} + \frac{(\gamma_0 + 1) f^2(q)}{2\gamma_0^2 \left\{\left[((\gamma_0^2 - 1)/\gamma_0^2) f^2(q) + 1\right]^{1/2} + 1\right\}}. \tag{16}$$

It can be seen from formulae (15) and (16) that the stationary solutions in a relativistic diode are governed by three external parameters: δ, V, and relativistic factor γ_0.

Let us solve Eq. (14) numerically to study the stability of the obtained potential distributions. First, we divide it by $\tilde{\eta}'(0) = -\frac{i}{\Omega}\tilde{J}$ and introduce $\Psi(\xi;\Omega) = \tilde{\eta}'(\xi)/\tilde{\eta}'(0)\exp(-i\Omega\tau(\xi))$. Changing the order of integration in the double integral in (14) and multiplying the entire equation by $\exp(-i\Omega\tau(\xi))$, we obtain

$$\Psi(\xi;\Omega) + \frac{2\gamma_0^2}{\gamma_0 + 1} \int\limits_0^{\xi} K(\xi, y)\Psi(y;\Omega)dy = \exp(-i\Omega\tau(\xi)),$$

$$K(\xi, y) = \int\limits_y^{\xi} \frac{dx}{(u(x)\gamma(x))^3}. \tag{17}$$

Let us divide the entire $[0, \delta]$ interval into N intervals with length h and boundaries ξ_i, $i = 0, 1, \ldots, N$ in a way that $0 = \xi_0 < \xi_1 < \cdots < \xi_N = \delta$ and substitute the integrals with sums. The result is a system of linear equations for determining the value of unknown function $\Psi(\xi_i; \Omega)$ at nodes ξ_i.

The discrete form of the relation specifying the connection between Ω and δ (dispersion relation) is

$$\sum_{j=1}^{N-1} \Psi(\xi_j; \Omega) \exp\big(i\Omega\tau(\xi_j)\big) + \frac{1}{2}\left[\Psi\big(\xi_N; \Omega\big) \exp(i\Omega\tau(\xi_N)\big) + 1\right] = 0.$$

We find $\Omega(\delta)$ from this relation and determine the stability of branches. The dispersion branches of stability are shown in Fig. 2. In the case of a nonrelativistic diode ($\gamma_0 = 1$), the $\Gamma(\delta)$ dependences match those plotted in [6]. It is evident from Fig. 2 that, as in the nonrelativistic case, normal branches are aperiodically stable, while overlap branches are unstable with respect to small perturbations.

Thus, an analytical method for calculating the density of relativistic charged particles leaving an emitter with a known VDF and propagating in a nonstationary electric field without collisions was developed. The obtained expression for particle density was used to examine the stability of solutions for a diode with relativistic particles in the case of their motion without reflection from potential barriers. Analytical expressions for functions G and Q in the case of a small electric field perturbation were obtained. The stability of stationary states of a Bursian diode with relativistic electrons [5] was investigated for illustrative purposes. The developed method may be used to study the stability of diodes with relativistic fluxes of charged particles.

Conflict of interest

The authors declare that they have no conflict of interest.

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