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Generation of coherent atomic states arising by light scattering from a BEC

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Using the solution of the system of Maxwell-Schrödinger equations, the coherent properties of atomic states arising by light scattering from a Bose-Einstein condensate of a rarefied gas enclosed in a harmonic trap are investigated.

Keywords: Bose–Einstein condensate, coherent atomic states, superradiant light scattering.

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Introduction

Superradiant light scattering from a Bose–Einstein condensate (BEC) of a rarefied atomic gas has been observed for the first time by Ketterle et al. in MIT [1–3]. In experiment [4], a BEC of Rb atoms held in a magnetic trap was irradiated by a pair of counterpropagating laser beams. Following multiple scattering events, BEC atoms acquired translational motion momenta close to $j\hbar k_0$ ($j = \pm 2, \pm 4, \ldots$), where k_0 is the magnitude of the pump field wave vector. This resulted in the emergence of a series of moving atomic clouds. The authors of [4] paid special attention to the accuracy of measuring the photon recoil momenta imparted to atoms and to their deviations from values that are multiples of double the photon momentum.

The theory of superradiant scattering from a BEC was discussed in [5–25]. The present study is focused on the influence of a harmonic trap potential on the kinetics of atomic clouds and demonstrates how close the quantum states of atoms produced as a result of light scattering are to the quantum coherent states that were proposed by Schrödinger [26] and later examined in detail and applied successfully by Glauber [27] in the field of quantum optics. The coherent properties of atomic waves were used in [4] to measure the photon recoil momentum via the interference effect.

Formulation of the problem. Main equations

An atom from a BEC is modeled as a two-level Bose particle with ground $|a\rangle$ and excited $|b\rangle$ electron states that is engaged in translational motion along the direction of counter laser pumping (as in the experiment in [4]). In what follows, we limit ourselves to a one-dimensional model of the condensate, assuming that the dependence on coordinates orthogonal to the direction of laser excitation is uniform inside a harmonic trap.

The single-atom wave function is sought in the form

$$\Psi(x,t) = \sum_{j=0,\pm 2,\ldots} \Big\{ a_j(x,t) e^{ijk_0x} |a\rangle + e^{-i\omega_0 t} b_{j+1}(x,t) e^{i(j+1)k_0x} |b\rangle \Big\}, \quad (1)$$

where x is the coordinate of translational motion of an atom along the direction of laser pumping; ω_0 and $k_0 = \omega_0/c$ are the frequency and the wave vector of the laser field, respectively; and $a_j(x,t)$ and $b_j(x,t)$ are the sought-for amplitudes of wave functions of the ground and excited atomic states, respectively.

In the ideal gas model, the wave functions of all BEC atoms may be assumed to be identical and satisfying the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \left[\hat{H}^0 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \hat{\mathbf{d}} \cdot \mathbf{E}(x,t) + \frac{1}{2} m\Omega^2 x^2\right]\Psi(x,t). \quad (2)$$

Here, $\hat{H}^0$ is the Hamiltonian of the electron state of a free atom, m is the atom mass, $\hat{\mathbf{d}}$ is the atomic dipole moment operator, $\mathbf{E}(x,t)$ is the vector of the total electric field (the exciting laser field and the „secondary“ field produced by the induced condensate polarization) strength, and $m\Omega^2x^2/2$ is the potential energy of an atom in a one-dimensional harmonic trap with natural frequency Ω.

The exciting laser field is written as

$$E_0(x,t) = E_0^+(t)\exp\left(-i\omega_0\left(1-\frac{x}{c}\right)\right) + E_0^-(t)\exp\left(-i\omega_0\left(1+\frac{x}{c}\right)\right) + \text{c.c.} \quad (3)$$

Following [4], we assume that the field polarization vector is perpendicular to the direction of laser excitation.

The condensate polarization is defined as the quantum-mechanical average of operator

$$\hat{P}(x,x') = N\hat{d}\delta(x-x'), \quad (4)$$

where N in our one-dimensional model is the number of condensate atoms per unit area of the irradiated face of the trap. The quantum-mechanical average of the condensate polarization is then given by

$$P(x,t) = \int_{-\infty}^{\infty} \Psi^*(x',t)N\hat{d}\delta(x-x')\Psi(x',t)dx' = Nd_{ab}\sum_{j=0,\pm2,\dots}\sum_{m=0,\pm2} b_{j+1}(x,t)a_m^*(x,t) \times \exp[-i\omega_0 t + ik_0x(j+1-m)] + \text{c.c.}, \quad (5)$$

where d_{ab} is the dipole moment of electron transition $a \leftrightarrow b$. By virtue of Maxwell equations, the „secondary" electromagnetic field induced by this polarization obeys the non-homogeneous wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E'(x,t) = \frac{4\pi}{c^2}\frac{\partial^2 P(x,t)}{\partial t^2} \quad (6)$$

and may be written as [28]

$$E'(x,t) = -\frac{2\pi}{c}\int_{-\infty}^{\infty} dx' \frac{\partial}{\partial t}P\left(x', t-\frac{|x-x'|}{c}\right). \quad (7)$$

Using the expression for polarization (5) in the approximation of slow amplitude variation, we obtain the following relation for the total field with the use of (3) and (7):

$$E(x,t) = E^+(x,t)\exp\left[-i\omega_0\left(t-\frac{x}{c}\right)\right] + E^-(x,t)\exp\left[-i\omega_0\left(t+\frac{x}{c}\right)\right] + \text{c.c.}, \quad (8)$$

where

$$E^+(x,t) = E_0^+(t) + i2\pi k_0 d_{ab}N\int_{-\infty}^{x} dx' \times \sum_{j=0,\pm2,\dots} b_{j+1}(x',t')\bar{a}_j(x',t'), \quad (8a)$$

$$E^-(x,t) = E_0^-(t) + i2\pi k_0 d_{ab}N\int_{x}^{\infty} dx' \times \sum_{j=0,\pm2,\dots} b_{j-1}(x',t')\bar{a}_j(x',t'). \quad (8b)$$

In what follows, we use a system of units with „width" $L = 2(\hbar\ln 2/m\Omega)^{1/2}$ of the ground state of a harmonic oscillator set to be the unit of length, „superradiant time" $\tau_R = \hbar/(\pi d_{ab}^2 k_0 N)$ being the unit of time, and $\hbar\tau_R^{-1}$ taken to be the unit of energy.

With retardation neglected, the Maxwell–Schrödinger system of equations (1), (2), (8) may be written as

$$\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x}\right)a_j = -i\varepsilon_j a_j - iux^2a_j + b_{j+1}\bar{E}^+ + b_{j-1}\bar{E}^-, \quad (9a)$$

$$\left(\frac{\partial}{\partial t} + v_{j+1}\frac{\partial}{\partial x}\right)b_{j+1} = = i\left(\Delta - \varepsilon_{j+1} - ux^2 + i\frac{\gamma}{2}\right)b_{j+1} - a_jE^+ - a_{j+2}E^-, \quad (9b)$$

$$E^+(x,t) = E_0^+(t) + 2\int_{-\infty}^{x} dx' \sum_{j=0,\pm2,\dots} b_{j+1}(x',t)\bar{a}_j(x',t), \quad (9c)$$

$$E^-(x,t) = E_0^-(t) + 2\int_{x}^{\infty} dx' \sum_{j=0,\pm2,\dots} b_{j-1}(x',t)\bar{a}_j(x',t). \quad (9d)$$

Here, $\varepsilon_j = \hbar j^2k_0^2\tau_R/(2m)$ and $\nu_j = \hbar jk_0\tau_R/(mL)$ are the kinetic energy (in units of frequency) and the velocity of an atom with mass m and momentum $j\hbar k_0$, respectively; $\Delta = (\omega_0 - \omega_{ab})\tau_R$ is the detuning of pump frequency ω_0 from resonance frequency ω_{ab} of the optical transition; the electric field strength amplitudes are expressed in units of $\hbar/(d_{ab}\tau_R)$; $\gamma = \Gamma\tau_R$, where Γ is the radiation constant of an excited electron state; and $u = 0.5m\tau_R(\Omega L)^2/\hbar$.

Scattered atomic waves

The values of problem parameters of the same order of magnitude as in the experiment in [4] were used to solve system (9): cyclic frequency of laser radiation $\omega_0 = 2.4\cdot10^{15}\,\text{s}^{-1}$, radiation constant of the electron transition $\Gamma = 0.38\cdot10^8\,\text{s}^{-1}$, wavelength $\lambda = 780\,\text{nm}$ and dipole moment $d_{ab} = 2.53\cdot10^{-29}\,\text{C}\cdot\text{m}$ of this transition, $N = 1.4\cdot10^{10}\,\text{cm}^{-2}$, natural frequency of the harmonic trap $\Omega/2\pi = 20\,\text{Hz}$, and Rb atom mass $m = 1.44\cdot10^{-22}\,\text{g}$. Under these conditions, the superradiant time is estimated as $\tau_R \sim 5\,\text{ns}$, and the parameters in Eqs. (9) then assume the following approximate values in our system

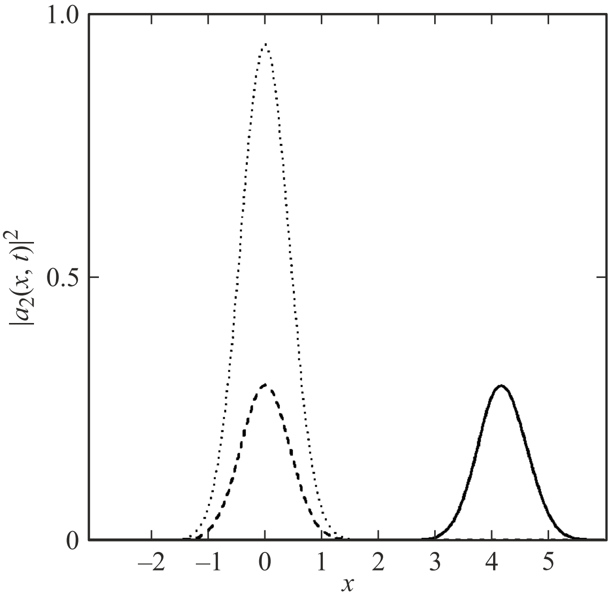

Figure 1. Dashed and solid curves represent the $|a_2(x,t)|^2 \cdot 10$ population distribution at time points $t = t_p$ and $t = 350t_p$, respectively. The dotted curve corresponds to initial BEC population distribution $|a_0(x, t = 0)|^2$.

of units: $\varepsilon_j = 1.2 \cdot 10^{-4} j^2$, $v_j = 7.5 \cdot 10^{-6} j$, $\gamma = 0.19$, $u = 8.62 \cdot 10^{-7}$. The condensate excitation was modeled by two counterpropagating laser pulses with duration $t_p \sim 4\,\mu$s (in our units, $t_p \sim 800$). System (9) was solved with account for the formation of 11 atomic states: a_0, $b_{\pm1}$, $a_{\pm2}$, $b_{\pm3}$, $a_{\pm4}$, $b_{\pm5}$. Pump amplitudes $E_0^+ = E_0^-$ were chosen (depending on the detuning) so that the fraction of atoms in the ground state of the condensate remained at 0.9 within the excitation time. The only non-zero initial condition in Eqs. (9) was set for amplitude a_0 as the wave function of the ground state of a harmonic oscillator.

We limit ourselves here to presenting the results obtained with detuning $\Delta_{ab}/2\pi = -50$ MHz (in our units, ~ -1.6) for the most populated scattered atomic states $a_{\pm2}$. According to the experimental data from [4] and our calculations, the populations of atomic states $a_{\pm4}, a_{\pm6}, \dots,$ corresponding to 2-fold, 3-fold, and higher multiplicities of scattering events turn out to be negligibly small. Specifically, the populations of states $a_{\pm4}$ at time t_p when the pumping is switched off turn out to be less than 1% of the population values of states $a_{\pm2}$.

The spatial distributions of the a_2 cloud population at two points in time are shown in Fig. 1.

The a_2 atomic cloud shifts with time due to the fact that photon recoil momenta are imparted to atoms. The distribution for the a_{-2} cloud is displaced symmetrically in the opposite direction. The shape of these atomic clouds remains roughly unchanged and is close to the shape of the ground state of a harmonic oscillator. The velocity of displacement of the a_2 cloud maximum corresponds roughly to the estimate of photon recoil, $2\hbar k_0$, which was taken into account in the expression (1) for the wave function. The cloud velocity value may be refined by calculating the Fourier transform of amplitudes $a_{\pm j}(x,t)\exp(\pm i j k_0 x)$:

$$\tilde{a}_{\pm j}(k,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \exp(-ikx) \times a_{\pm j}(x,t)\exp(\pm i j k_0 x)dx. \tag{10}$$

The obtained results for cloud a_2 are presented in Fig. 2.

As we have found out, the positive shift of the distribution maximum relative to the $2\hbar k_0$ value, which is observed at time point $t = t_p$, is attributable to the influence of secondary field E' induced by the polarization of the medium on the condensate. The slowdown in the velocity of cloud a_2 movement by the time $t = 350t_p$ is due to the influence of the trap potential.

Estimation of coherence of scattered atomic states

Let us compare scattered atomic states with coherent states of a harmonic oscillator [27].

It is worth reminding that quantum coherent states of an oscillator are the eigenfunctions of lowering operator

$$\hat{a} = \sqrt{\frac{m\Omega}{2\hbar}}\left(x + \frac{\hbar}{m\Omega}\frac{\partial}{\partial x}\right),$$

$$\hat{a}\psi_a(x) = a\psi_a(x), \tag{11}$$

where m is the oscillator mass, Ω is the natural frequency of a harmonic trap, and $a = \alpha + i\beta$ — is a complex eigenvalue. It is easy to verify by direct substitution that the solution of Eq. (11) may be expressed through the wave function of the ground state of a harmonic oscillator with a shifted

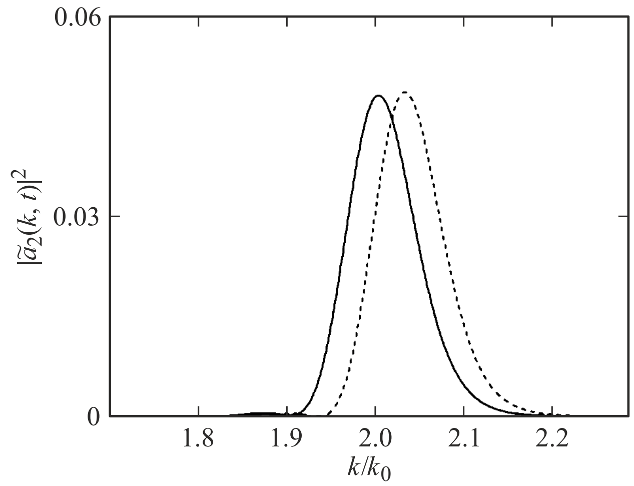

Figure 2. Shape of the momentum distribution of cloud a_2 at two points in time: $t = t_p$ — dashed curve, $t = 350t_p$ — solid curve.

argument value:

$$\psi_a(x) = \left(\frac{m\Omega}{\hbar\pi}\right)^{1/4} e^{-\beta^2} e^{-[(\frac{m\Omega}{2\hbar})^{1/2}x-a]^2}$$

$$= \left(\frac{m\Omega}{\hbar\pi}\right)^{1/4} e^{-[(\frac{m\Omega}{2\hbar})^{1/2}x-a]^2} e^{2i\beta[(\frac{m\Omega}{2\hbar})^{1/2}x-a]}. \quad (12)$$

Applying the Fourier transform to function (12), we obtain the wave function of the coherent state of a harmonic oscillator in the momentum representation:

$$\Phi_a(p) = (\pi\hbar m\Omega)^{-1/4} e^{-\beta^2} e^{-[\frac{p}{\sqrt{2\hbar m\Omega}}+ia]^2} e^{-a^2}$$

$$= (\pi\hbar m\Omega)^{-1/4} e^{-[\frac{p}{\sqrt{2\hbar m\Omega}}-\beta]^2} e^{-2i\alpha[\frac{p}{\sqrt{2\hbar m\Omega}}]}. \quad (13)$$

If an atom in the ground state of a harmonic trap acquires additional momentum p_0, its state may be characterized by wave function

$$\Phi(p-p_)) = (\pi\hbar m\Omega)^{-1/4} e^{-[\frac{p-p_0}{\sqrt{2\hbar m\Omega}}]^2}. \quad (14)$$

It is evident that this state matches the coherent state at

$$ia = -\frac{p_0}{\sqrt{2\hbar m\Omega}}, \quad (15)$$

i.e., $\alpha = 0$, $\beta = \frac{p_0}{\sqrt{2\hbar m\Omega}}$ and $e^{-a^2} = e^{\beta^2}$).

To characterize the time evolution of a coherent state, one performs the $a \to ae^{-i\Omega t}$ substitution [27]. In the present case,

$$a = \frac{ip_0}{\sqrt{2\hbar m\Omega}} \to \frac{ip_0 e^{-i\Omega t}}{\sqrt{2\hbar m\Omega}}$$

$$= \frac{p_0 \sin\Omega t}{\sqrt{2\hbar m\Omega}} + i\,\frac{p_0\cos\Omega t}{\sqrt{2\hbar m\Omega}}. \quad (16)$$

Therefore, parameters a and b at time point t assume the values of

$$\alpha(t) = \frac{p_0 \sin\Omega t}{\sqrt{2\hbar m\Omega}},$$

$$\beta(t) = \frac{p_0\cos\Omega t}{\sqrt{2\hbar m\Omega}}, \quad (17)$$

and wave functions (12) and (13) take the form

$$\psi(x,t) = \left(\frac{m\Omega}{\hbar\pi}\right)^{1/4} e^{-\frac{m\Omega}{2\hbar}[x-\frac{p_0}{m\Omega}\sin\Omega t]^2}$$

$$\times e^{i\frac{p_0 x}{\hbar}\cos\Omega t} e^{-i\frac{p_0^2}{\hbar m\Omega}\sin\Omega t\cos\Omega t}, \quad (18)$$

$$\varphi(p,t) = (\pi\hbar m\Omega)^{-1/4}$$

$$\times e^{\frac{(p-p_0\cos\Omega t)^2}{2\hbar m\Omega}} e^{-i\frac{p_0 p\sin\Omega t}{\hbar m\Omega}}. \quad (19)$$

The squared moduli of these functions are

$$|\psi(x,t)|^2 = \left(\frac{m\Omega}{\hbar\pi}\right)^{1/2}$$

$$\times e^{-(\frac{m\Omega}{\hbar})[x-\frac{p_0}{\Omega m}\sin(\Omega t)]^2}, \quad (20)$$

$$|\varphi(p,t)|^2 = (\pi\hbar m\Omega)^{-1/2} e^{-\frac{[p-p_0\cos(\Omega t)]^2}{\hbar m\Omega}}, \quad (21)$$

indicating that distributions (20) and (21) undergo harmonic oscillations along the abscissa axes with frequency Ω without altering their shape.

Let us determine the extent to which states a_j obtained as a result of solving the Maxwell–Schrödinger system of equations resemble coherent ones. This possibility arises due to the fact that these atomic states, which emerge as a result of light scattering from a BEC, are, as we have seen, close to the ground state of a harmonic oscillator with recoil momentum imparted to it. However, the momentum is transferred not instantly, but within a finite time of interaction of the BEC with the laser field. This is the reason why the a_j states found here differ from coherent states (18), (19).

To assess the degree of coherence of the obtained states, we compare their coherence lengths with coherence length $L_{\mathrm{cog}} = \lambda^2/\Delta\lambda$ of state (19). This quantity is used in optics to determine the maximum path-length difference allowing for interference at a given wavelength uncertainty $\Delta\lambda$ [29]. Expression $L_{\mathrm{cog}} = \frac{2\pi}{\Delta k}$ is derived from obvious relation $\Delta k = \frac{2\pi}{\lambda^2}\Delta\lambda$. Taking the root of the momentum distribution variance as an estimate of uncertainty Δk, we find $\Delta k = \sqrt{\frac{m\Omega}{2\hbar}}$ for coherent state (19). Note that the uncertainty of the wave number and the coherence length do not depend on time in this case. Inserting the chosen system parameters, we obtain $\Delta k/k_0 = 0.0364$ and $L_{\mathrm{cog}}/\lambda = 27.5$.

Calculations of these characteristics based on the obtained solutions of the Maxwell–Schrödinger equations revealed that the uncertainty of the atomic momentum increases with observation time, while the coherence length of the scattered atomic wave decreases in the process. However, they still remain close to the corresponding values for the coherent state. Specifically, at time point $t = t_p$, $\Delta k/k_0 = 0.0417$ and $L_{\mathrm{cog}}/\lambda = 24.0$ were obtained for state a_2; at $t = 350t_p - \Delta k/k_0 = 0.0422$ and $L_{\mathrm{cog}}/\lambda = 23.7$.

Conclusion

The formation of moving atomic clouds in the process of scattering of light from a Bose–Einstein condensate of a rarefied gas confined in a harmonic trap was characterized by solving the system of Maxwell–Schrödinger equations. Our findings revealed how close the atomic states produced this way are to the coherent quantum states of a harmonic oscillator with a momentum shifted by the value of the photon recoil. The obtained coherence estimates suggest that these atomic states hold promise for application in atomic interferometry.

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