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Multiphoton scattering by resonant atoms as a problem of the theory of integrable quantum systems

© V.I. Rupasov, V.I. Yudson [1,2,3]

[1] International Laboratory of Condensed Matter Physics, HSE University, Moscow, Russia
[2] Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, Russia
[3] Russian Quantum Center, Skolkovo, Moscow, Russia

e-mail: rupasov.valery@gmail.com; vyudson@hse.ru

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The problem of scattering of a multiphoton electromagnetic field of arbitrary statistics by resonant atoms has been described by methods of the theory of integrable quantum systems. Using a full set of exact eigenstates of the integrable model „quantum field + two-level atoms", we have derived an exact expression for the multiparticle wave function of scattered photons. The general formalism has been illustrated in the particular case where the incident field is in a stationary coherent state. The results for the spectrum and the second order correlation function of the scattered field for this case coincide with the known ones, but they have been derived without using the Lindblad's approach, the quantum regression theorem, and all that. The developed formalism is applicable to an arbitrary (including non-classical - not coherent) state of incident photons.

Keywords: integrable systems, quantum optics, resonant fluorescence.

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1. Introduction

Nonlinear resonance media optics has become one of the objects of application of the classical integrable system theory [1,2]. However, the quantum nature of field is of critical importance in such phenomena as spontaneous emission and resonance fluorescence, and their consistent description is, strictly speaking, possible only in the framework of the quantum field theory. Both of the above-mentioned phenomena can be formally described as the quantum Cauchy problem with various initial conditions: „incident photons + atoms in the ground state" (resonance fluorescence) or „field vacuum + atoms in the excited state" (spontaneous emission). The problem of incident quantum field scattering was first formulated and solved (for a weak field — in the single-photon approximation) by Weisskopf [3] (see also [4]). Weisskopf's theory was made applicable to a strong field case using various approaches and approximations in many studies that formed the resonance fluorescence theory [5–9].

Nevertheless, description of resonance fluorescence as a dynamic problem in the framework of the theory of integrable quantum systems, i.e. direct construction of an exact solution of the quantum Cauchy problem for the multiparticle Schrödinger equation of the „photons + atoms" system, is of obvious interest. A similar program was implemented earlier [10] to describe spontaneous emission of excited compactly arranged atoms (Dicke configuration [11]) into photonic vacuum. Using a similar approach for a problem of multiphoton scattering, an exact expression is derived for the multiparticle state of scattered photons in case of an arbitrary state of the incoming field (fundamentals of the approach were formulated in early work [12] that was not brought to regular journal publication). The derived expression for the out-state contains full information about quantum statistics and scattered field correlation functions. In particular case when the incoming field is in a stationary coherent state, the scattered photon spectrum corresponds to a well-known Mollow triplet [7]. In the approach described below, this result is obtained through a regular quantum-mechanical calculation without involvement of sophisticated approaches with the Lindblad equation, regression theorem, etc. [13].

Section 2 contains a description of the model and formal statement of the scattering problem. In Section 3, a full set of system Hamiltonian eigenstates is used to derive the out-state of the multiphoton scattering problem. The developed general formalism is illustrated in Sections 4 and 5 as a particular case where the incoming field is in the stationary coherent state; scattered field correlation functions are calculated.

2. Description of the model and statement of the scattering problem

Hamiltonian of the „quantum field + two-level atom" system in the dipole resonance approximation can be written as: $H_0 + H_{int}$, where

$$H_0 = \sum_{j,m} \int \frac{d\omega}{2\pi}(\omega - \omega_{12})c^{\dagger}_{jm}(\omega)c_{jm}(\omega) + \omega_{12}\left(s^3 + \frac{1}{2}\right), \quad (1)$$

$$H_{int} = \sqrt{\gamma} \int \frac{d\omega}{2\pi}[c^{\dagger}(\omega)s^{-} + c(\omega)s^{+}] \quad (2).$$

Here, $\gamma = 4\omega_{12}^3 d^2/3$, ω_{12} and d are the radiation-induced relaxation rate, atomic transition frequency and dipole moment (speed of light and Planck's constant are taken equal to 1). Spin operators $\mathbf{s} = (s^1, s^2, s^3)$ with commutation relations $[s^j, s^k] = ie_{jkl}s^l$ describe a two-level atom ($s = 1/2$); $s^{\pm} = s^1 \pm is^2$. Electromagnetic field operators are expanded as series in spherical harmonics [4]:

$$a_{\mathbf{k}} = \sum_{j,m} \int \frac{d\omega}{2\pi}\Phi_{jm\omega}(\mathbf{k}/|\mathbf{k}|)c_{jm}(\omega). \quad (3)$$

Due to the dipole approximation, equation (2) may be limited to consideration of interaction between the atom and only electrodipole photons (with the angular momentum $j = 1$). Wherein $c(\omega) \equiv c_{j=1,m=0}(\omega)$, where the momentum projection $m = 0$ is established by the incident field polarization (here the linear polarization is chosen for definiteness).

Let's introduce a notation for the field operators (the formal Fourier variable x may be assumed as a „radial coordinate“: incoming photons have $x < 0$, outgoing photons have$x > 0$):

$$\varepsilon(x) = \int \frac{d\omega}{2\pi}c(\omega)e^{i(\omega-\omega_{12})x}, \quad (4)$$

corresponding to the commutation relations $[\varepsilon(x), \varepsilon^{\dagger}(x')] = \delta(x - x')$, and rewrite the Hamiltonian of the „atom + electrodipole photons“ system as

$$H = \int dx\{-i\varepsilon^{\dagger}(x)\partial_x\varepsilon(x) + \sqrt{\gamma}\delta(x)[\varepsilon^{\dagger}(x)s^{-} + \varepsilon(x)s^{+}]\} + \omega_{12}\left(s^3 + \frac{1}{2}\right). \quad (5)$$

Thus, the problem of resonance interaction between the atom and quantum electromagnetic field is described effectively by a one-dimensional model. This model was studied before [10] in view of the spontaneous Dicke superradiance. The Bethe ansatz was used to make a full set of eigenstates of Hamiltonian (5), which forms the basis for description of the time evolution of an arbitrary initial system state. Unlike the initial atomic excitation decay problem addressed in [10], the initial system state ($t \to -\infty$) in the resonance fluorescence problem contains photons approaching the atom. Physical characteristics of the scattered field are defined by the final system state ($t \to +\infty$) that is found by solving the Schrödinger equation:

$$i\partial_t |\Psi(t)\rangle = H|\Psi(t)\rangle,$$

$$|\Psi(t \to -\infty)\rangle = e^{-iH_0t}|In\rangle, \quad (6)$$

$$|\Psi(t \to +\infty)\rangle = e^{-iH_0t}|Out\rangle. \quad (7)$$

Let's consider a configuration where a photon beam with the typical wave vector $\mathbf{k}$ oriented along the z axis falls on an atom located at the origin. It is assumed that the beam has the cross-section S and the length L along the z axis. The Fock state of the incident field containing N free photons is given by the following expression

$$|In, N\rangle = \frac{1}{\sqrt{N}}[a^{\dagger}_{\mathbf{k}}]^N|0\rangle, \quad (8)$$

where $|0\rangle$ is the vacuum state of the field. An arbitrary state of a beam of identical independent photons is written in the Fock basis as:

$$|In\rangle = \sum_N A_N|In, N\rangle. \quad (9)$$

A case when the incident field is in the coherent state is of particular interest

$$|In, \alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2} + \alpha a^{\dagger}_{\mathbf{k}}\right)|0\rangle, \quad (10)$$

which corresponds to the Poisson coefficient statistics in expression (9): $|A_N|^2 = \exp(-\bar{N})\bar{N}^N/N!$, where $\bar{N} = |\alpha|^2$ is the mean photon number. As will be shown below, coherent states form a convenient basis for the investigation of field scattering with arbitrary statistics.

Scattered light properties are defined by averaging of field operators over the out-state. The stationary fluorescence case in the external field corresponds to the limit transition

$$\bar{N} \to \infty;\ L \to \infty;\ \frac{\bar{N}}{L} \to \text{Const}. \quad (11)$$

Our immediate problem is to find the out-state of the scattered field.

3. Multiparticle scattering problem

1. In expansion (3) for $a_{\mathbf{k}}$, the electrodipole photon operator is extracted:

$$a_{\mathbf{k}} = \tilde{a}_{\mathbf{k}} + \sqrt{\frac{\sigma_1}{SL}}c(\omega), \quad (12)$$

where $\tilde{a}_{\mathbf{k}}$ includes the photon operators with the angular momenta $j > 1$ that don't interact with the atom; σ_1 is the „target area“ of photons with $j = 1$ [14]. Taking into account (12), the Fock state (8) is written as:

$$|In, N\rangle = \frac{1}{\sqrt{N!}}\sum_{n=0}^{N} C_N^n\left(\frac{\sigma_1}{SL}\right)^{n/2}(\tilde{a}^{\dagger}_{\mathbf{k}})^{N-n}[c^{\dagger}(\omega)]^n|0\rangle. \quad (13)$$

Thus, the initial scattering problem reduces to the problem of scattering of the $|\text{in}, n\rangle$ state containing only „dipole“

photons

$$
\begin{aligned}
|\mathrm{in}, n\rangle =& \frac{1}{\sqrt{n!}}[c^\dagger(\omega)]^n|0\rangle \\
=& \frac{1}{\sqrt{n!}}\int d^n x \prod_{j=1}^{n} e^{i(\omega-\omega_{12})x_j}\varepsilon^\dagger(x_j)|0\rangle,
\end{aligned} \tag{14}
$$

whose time evolution is defined by Hamiltonian (5).
2. Accurate solution of problem (5), (14) results from complete integrability of model (5). The full set of eigenstates of Hamiltonian (5) is written as [10]:

$$
\begin{aligned}
|\lambda\rangle \equiv& |\lambda_1, \ldots, \lambda_n\rangle = \int d^n y \mathscr{F}_B(\lambda, \mathbf{y}) \prod_{j=1}^{n} e^{i\lambda_j y_j} \\
&\times \left[\varepsilon^\dagger(y_j) + \frac{\sqrt{\gamma}}{\lambda_j} s^+ \delta(y_j)\right] |vac\rangle,
\end{aligned} \tag{15}
$$

where the $|vac\rangle$ state corresponds to an unexcited atom and the absence of photons, and the Bethe factor

$$
\mathscr{F}_B(\lambda, \mathbf{y}) = \prod_{j<l,l}^{n} \left[1 + \frac{i\gamma}{\lambda_j - \lambda_l}\mathrm{sign}(y_j - y_l)\right] \tag{16}
$$

accounts for multiparticle correlations. Allowable values of the complex quantities $\{\lambda_j\}$ are defined by the wave function boundedness condition (15) [10]. Energy of state (15) is equal to $E(\lambda) = \sum_j \lambda_j$.

Evolution of an arbitrary initial state $|\Psi_{in}\rangle$ is defined using an exact integral representation [15]:

$$
\begin{aligned}
|\Psi(t)\rangle =& e^{-iHt}|\Psi_{in}\rangle = \int_\Gamma d^n\lambda e^{-it\sum_{j=1}^{n}\lambda_j}|\lambda\rangle(\lambda|\Psi_{in}\rangle \\
\to& e^{-iH_0 t}|\Psi_{out}\rangle
\end{aligned} \tag{17}
$$

where integration follows a special contour Γ in a n-dimensional complex space of variables $\{\lambda\}$, and the auxiliary state $|\lambda)$ differs from (15) in that the Bethe factor is absent. By omitting the details specified in [Y–85], exact expressions for the sought out-state are written:

$$
|\Psi_{out}\rangle = \int d^n x \Phi_{out}(\mathbf{x}) \prod_{j=1}^{n} \varepsilon^\dagger(x_j)|0\rangle, \tag{18}
$$

$$
\Phi_{out}(\mathbf{x}) = \int d^n y \mathcal{S}(\mathbf{x}, \mathbf{y}) \Phi_{in}(\mathbf{y}), \tag{19}
$$

where $\mathcal{S}$ — the matrix that connects the wave functions of outgoing and incoming photons is written as:

$$
\begin{aligned}
\mathcal{S}(\mathbf{x}, \mathbf{y}) =& \int \frac{d^n\lambda}{(2\pi)^n} \prod_{j<l}^{n} \frac{\lambda_j - \lambda_l - i\gamma\mathrm{sign}(x_j - x_l)}{\lambda_j - \lambda_l + i\gamma} \\
&\times \prod_{j=l}^{n} \frac{\lambda_j - i\gamma/2}{\lambda_j + i\gamma/2} e^{i\lambda_j(x_j - y_j)} = \sum \theta(x_1 > \ldots > x_n) \\
&\times \prod_{j=l}^{n} \left[\delta(x_j - y_j) - \theta(y_j - x_j)e^{\gamma(x_j - y_j)/2}\right].
\end{aligned} \tag{20}
$$

Summation in (20) is performed over permutations of the x_j coordinates.

Expression (20) for the $\mathcal{S}$-matrix corresponds to a single-atom resonance fluorescence case. In case of photon scattering on the ensemble M of closely spaced atoms, the following replacement shall be made in expression (20)

$$
\frac{\lambda_j - i\gamma/2}{\lambda_j + i\gamma/2} \to \frac{\lambda_j - iM\gamma/2}{\lambda_j + iM\gamma/2}. \tag{21}
$$

This makes the mathematical structure of the theory much more sophisticated — bunched („string“) photon states will occur in the final state of the scattered field. This work is restricted to a relatively simple case of single-atom resonance fluorescence.

3. Expression (20) is a sum of 2^n terms (without permutations) describing all possible scattering processes of n photons, wherein $\delta(x-y)$ type multipliers correspond to unscattered (free passing) photons. Unscattered dipole photon operators added to the $\tilde{a}^\dagger_{\mathbf{k}}$ operators of photons with $j > 1$ (3) recover the $a^\dagger_{\mathbf{k}}$ operator in the out-state. Eventually, the final state of the $|Out, N\rangle$ field is given by the following expression

$$
|Out, N\rangle = \sum_{n=0}^{N} \sqrt{C_N^n} \left(\frac{\sigma_1}{SL}\right)^{n/2} |scat, n\rangle \otimes |In, N-n\rangle. \tag{22}
$$

Here, the $|In, N-n\rangle$ state (8) corresponds to $N-n$ photons in the passing (unscattered) beam, and $|scat, n\rangle$ describes the state of n scattered photons:

$$
|scat, n\rangle = \int d^n x \Phi_{scat}(\mathbf{x}) \prod_{j=1}^{n} \varepsilon^\dagger(x_j)|0\rangle, \tag{23}
$$

$$
\Phi_{scat}(\mathbf{x}) = \int d^n y T(\mathbf{x}, \mathbf{y}) \Phi_{in}(\mathbf{y}), \tag{24}
$$

$$
\begin{aligned}
T(\mathbf{x}, \mathbf{y}) =& (-\gamma)^n L^{-n/2}\sqrt{n!}\theta(x_1 > \ldots > x_n) \\
&\times \prod_{j=l}^{n} \theta(y_j - x_j) e^{\gamma(x_j - y_j)/2},
\end{aligned} \tag{25}
$$

where the T-matrix (25) is a component of sum (20) that doesn't contain any unscattered dipole photons. It shall be emphasized that expressions (22)−(25) correctly account for all possible scattering channels. Note, however, certain conditionality in using symbol $\otimes$ in (22). The point is that the Hilbert space of the $\varepsilon^\dagger(x)$ operators that form the $|scat, n\rangle$ state is a subspace of the Hilbert space of the $a^\dagger_{\mathbf{k}}$ operators that generate the $|In, N-n\rangle$ state. Therefore, when calculating averages with respect to state (22), nonzero value, following from (4) and (3), of the commutator

$$
[\varepsilon(x), a^\dagger_{\mathbf{k}}] = \left(\frac{\sigma_1}{SL}\right)^{1/2} e^{i(\omega-\omega_{12})x}. \tag{26}
$$

shall be taken into account.

4. For initial state (14), wave function (24) of the scattered photons is written as:

$$\Phi_{scat}(\mathbf{x}) = (-1)^n \frac{2^n n!}{L^{n/2}} \theta(x_1 > \ldots > x_n) \prod_{j=1}^{n} \psi(x_{j-1}, x_j), \tag{27}$$

$$\psi(x_{j-1}, x_j) = \frac{\gamma}{2} \int_{x_j}^{x_{j-1}} dy e^{i(\Delta > +i\gamma/2)(y - x_j)}$$

$$\times \theta(-L/2 < y < L/2), \tag{28}$$

(here, $x_{j=0} \equiv L/2$). For simplicity, we restrict ourselves hereinafter to the case of exact resonance where the incident radiation frequency offset $\Delta = \omega - \omega_{12} = 0$. Wherein

$$\psi(x_{j-1}, x_j) = \theta(x_{j-1} - x_j) \left[1 - e^{-\gamma(x_{j-1} - x_j)/2}\right];$$

$$j = 1, \ldots, n;\ x_n > -L/2, \tag{29}$$

$$\psi(x_{n-1}, x_n) = \theta(x_{n-1} - x_n) \left[e^{\gamma(x_n + L/2)/2} - e^{-\gamma(x_{n-1} - x_n)/2}\right];$$

$$x_n < -L/2. \tag{30}$$

Equation (30) reflects the fact that the last photon of the scattered field can be emitted even after expiration of the excitation light pulse lifetime.

A set of expressions (22)−(30) is the main result of the work that makes it possible to describe completely the out-state of the scattered field for an arbitrary state of the incoming photon field. The next section demonstrates how the developed formalism works using the simplest, but physically interesting, situation in which the incoming field is in the coherent state. Correlation functions of the scattered field will be derived by direct calculation of averages over the corresponding operators with respect to the out-state (without reference to the Lindblad formalism, regression theorem, etc.).

4. Field scattering in the coherent state

1. Taking into account expressions (22), the out-state is found for the case where the incident field is in the coherent state (10):

$$|Out, \alpha\rangle = \sum_{n=0}^{N} \frac{\alpha^n}{\sqrt{n!}} \left(\frac{\sigma_1}{SL}\right)^{n/2} |scat, n\rangle \otimes |In, \alpha\rangle\,. \tag{31}$$

Expression (31) describes both scattered and unscattered fields, including the correlation between them. In the case when characteristics of only the scattered field are of interest, then, using (31), an effective description may be developed solely in terms of the scattered field. For this, note that, when averaging the arbitrary functional $\mathrm{O}(\varepsilon^\dagger, \varepsilon)$ of the scattered field operators over (31), the following expressions occur naturally:

$$e^{-\bar{N}} \langle 0| e^{\sqrt{N} a_{\bar{k}}} \prod_{j=1}^{n} \varepsilon(x'_j)\, \mathrm{O}(\varepsilon^\dagger, \varepsilon) \prod_{l=1}^{n} \varepsilon^\dagger(x_l) e^{sqrtbarNa^\dagger_{\bar{k}}} |0\rangle \tag{32}$$

(α was selected for real definiteness, $\alpha = \sqrt{N}$). By changing the exponents of the operators and using relation (26), expression (32) is written as

$$\langle 0| \prod_{j=1}^{n} [\varepsilon(x'_j) + \sqrt{\rho}\theta(-L/2 < x'_j < L/2)] \mathrm{O}(\varepsilon^\dagger, \varepsilon)$$

$$\times \prod_{l=1}^{n} [\varepsilon^\dagger(x_l) + \sqrt{\rho}\theta(-L/2 < x_l < L/2)]|0\rangle, \tag{33}$$

where $\rho = \sigma_1 \bar{N}/(SL)$ is the dipole photon flux density in the incident beam. When passing from (32) to (33), it was assumed in line with a general physical statement of the scattering problem (final beam aperture) that the observed O is measured in spatial points where the transmitted wave field is absent. Consequently, operators included in $\mathrm{O}(\varepsilon^\dagger, \varepsilon)$ commute with the $a^\dagger_{\bar{k}}$ and $a_{\bar{k}}$ operators and, therefore, are not shifted. Thus, the average of the operator $\mathrm{O}(\varepsilon^\dagger, \varepsilon)$ is written as

$$\langle Out, \alpha| \mathrm{O}(\varepsilon^\dagger, \varepsilon)\, |Out, \alpha\rangle = \langle \widetilde{Out}, \alpha| \mathrm{O}(\varepsilon^\dagger, \varepsilon) |\widetilde{Out}, \alpha\rangle, \tag{34}$$

where the effective out-state $|\widetilde{Out}, \alpha\rangle$ is written as:

$$|\widetilde{Out}, \alpha\rangle = |0\rangle + \sum_{n=1}^{\infty} (-1)^n (4\rho)^{n/2} \int d^n x$$

$$\times \prod_{j=1}^{n} \psi(x_{j-1}, x_j) [\varepsilon^\dagger(x_j)$$

$$+ \sqrt{\rho}\theta(-L/2 < x_j < L/2)]|0\rangle. \tag{35}$$

Derivation of expression (35) directly generalizes to the case of an arbitrary envelope shape $E(x)$ of the incident pulse, which leads to replacement $\sqrt{\rho}\theta(-L/2 < x_j < L/2) \to E(x)$ in (35). $E(x)$ will also occur in the expression under the integral sign in (28). The developed formalism is thereby also applicable to the non-stationary fluorescence excitation.

2. Expanding the product in square brackets in expression (35), write (35) as a sum of terms with different photon numbers:

$$|\widetilde{Out}, \alpha\rangle = \sum_{n=0}^{\infty} (-1)^n (4\rho)^{n/2} \int d^n x v(x_n)$$

$$\times \prod_{j=1}^{n} u(x_{j-1}, x_j) \varepsilon^\dagger(x_j) |0\rangle. \tag{36}$$

Here, the functions $u(x, y)$ and $v(x)$ satisfying the following equations are introduced

$$u(x, y) = \psi(x, y) - 2\rho \int_{-L/2}^{L/2} \psi(x, z) u(z, y) dz, \tag{37}$$

$$v(x) = 1 - 2 \int_{-L/2}^{L/2} u(x, y) dy. \tag{38}$$

In the case of fluorescence in a constant-amplitude field, $\psi(x,y)$ and $u(x,y)$ depend only on the difference of positions and expression (36) takes a simple form:

$$\begin{aligned}|\widetilde{Out},\alpha\rangle =& v(L)|0\rangle + \sum_{n=1}^{\infty}(-1)^n(4\rho)^{n/2}\\ &\times\int d^n x u(L/2-x_1)u(x_1-x_2)\ \dots\ u(x_{n-2}-x_{n-1})\\ &\times\Big[u(x_{n-1}-x_n)v(x_n+L/2)+u(x_{n-1}+L/2)\\ &\times\theta(-L/2-x_n)e^{\gamma(x_n+L/2)/2}\Big]\prod_{j=1}^{n}\varepsilon^{\dagger}(x_j)|0\rangle.\end{aligned} \tag{39}$$

$u(x)$ and $v(x)$ are nonzero only when $x>0$. Their Fourier components determined from (37) and (38) are given by

$$u(\omega) = -\frac{\gamma/2}{(\omega-\omega_+)(\omega-\omega_-)}, \tag{40}$$

$$v(\omega) = \frac{i(\omega+i\gamma/2)}{(\omega-\omega_+)(\omega-\omega_-)}, \tag{41}$$

where

$$\omega_{\pm} = -i\gamma/4 \pm \sqrt{\gamma\rho-(\gamma/4)^2}. \tag{42}$$

A set of results obtained in (34) – (42) contains all information about the scattered radiation and makes it possible to calculate its physical properties.

5. Calculation of physical quantities

Application of the derived expressions for calculation of some correlation functions of scattered radiation is illustrated below.

1. Let's make sure first that the derivation of state didn't break the unitarity and (34) is normalized correctly, i.e. $\langle\widetilde{Out},\alpha|\widetilde{Out},\alpha\rangle = 1$. From (34) follows

$$\begin{aligned}\langle\widetilde{Out},\alpha|\widetilde{Out},\alpha\rangle =& v^2(L) + \sum_{n=1}^{\infty}(4\rho)^n\\ &\times\int d^n x u^2(L/2-x_1)u^2(x_1-x_2)\ \dots\ u^2(x_{n-2}-x_{n-1})\\ &\times\Big[u^2(x_{n-1}-x_n)v^2(x_n+L/2)+u^2(x_{n-1}+L/2)\\ &\times\theta(-L/2-x_n)e^{\gamma(x_n+L/2)}\Big].\end{aligned} \tag{43}$$

Introducing notations $U(x)=u^2(x)$ and $V(x)=v^2(x)$ with the Fourier components

$$U(\omega) = -\frac{i\gamma^2/2}{(\omega+i\gamma/2)(\omega-2\omega_+)(\omega-2\omega_-)}, \tag{44}$$

$$V(\omega) = i\frac{(\omega+i\gamma/2)(\omega+i\gamma)-2\rho\gamma}{(\omega+i\gamma/2)(\omega-2\omega_+)(\omega-2\omega_-)}, \tag{45}$$

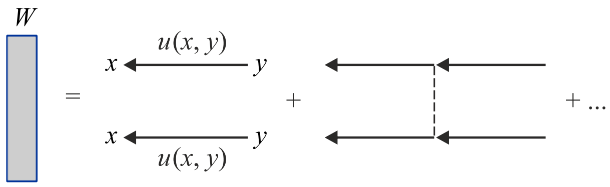

Figure 1. The dashed line shows $\delta(x_j-x'_j)$ corresponding to pairing of $\varepsilon^{\dagger}(x_j)$ and $\varepsilon(x'_j)$.

and passing to the Fourier representation in (43), we have

$$\begin{aligned}\langle\widetilde{Out},\alpha|\widetilde{Out},\alpha\rangle =& V(L) + \int\frac{d\omega}{2\pi}e^{-i\omega L}W(\omega)\\ &\times\left[V(\omega)+1/\gamma\right].\end{aligned} \tag{46}$$

Here

$$\begin{aligned}W(\omega) &= \sum_{n=1}^{\infty}[4\rho U(\omega)]^n = \frac{4\rho U(\omega)}{1-4\rho U(\omega)}\\ &= -2i\,\frac{\rho\gamma}{(\omega+i0)(\omega-\Omega_+)(\omega--\Omega_-)},\end{aligned} \tag{47}$$

where

$$\Omega_{\pm} = -3i\gamma/4 \pm \sqrt{\Omega_R^2-(\gamma/4)^2}. \tag{48}$$

The following quantity is introduced in this expression

$$\Omega_R = \sqrt{4\rho\gamma}, \tag{49}$$

that coincides with the Rabi frequency. In the given limit $L\to\infty$ (whereas $V(L)\to 0$), the nonvanishing contribution to the integral over ω is given only by the pole $\omega=-i0$ and we get

$$\langle\widetilde{Out},\alpha|\widetilde{Out},\alpha\rangle = \int\frac{d\omega}{2\pi}e^{-i\omega L}\frac{i}{\omega+i0} = 1. \tag{50}$$

Summation occurring during calculation of the norm and leading to $W(\omega)$ is illustrated in Figure 1 where the dashed line shows $\delta(x_j-x'_j)$ corresponding to pairing of $\varepsilon^{\dagger}(x_j)$ and $\varepsilon(x'_j)$.

2. The correlator calculation scheme will be described next as an example

$$D(x,y) = \langle\widetilde{Out},\alpha|\varepsilon^{\dagger}(x)\varepsilon(y)|\widetilde{Out},\alpha\rangle, \tag{51}$$

the Fourier component of which with respect to $(x-y)$ defines the spectral density of scattered light. Using (39), we find the following expression for (51) at $x>y$:

$$D(x,y) = D_1(x,y) + D_2(x,y), \tag{52}$$

where

$$\begin{aligned}D_1(x,y) =& \sum_{k,l,m=0}^{\infty}(4\rho)^{k+l+m+3}\\ &\times\int dx'dx''dy'dy''\langle L/2|\hat{U}^k|x'\rangle\Gamma_1(x',x,x'')\\ &\times\langle x''|\hat{U}^l|y'\rangle\Gamma_1(y',y,y'')\langle y''|\hat{U}^m(\hat{V}+\hat{\gamma}^{-1})|-L/2\rangle;\end{aligned} \tag{53}$$

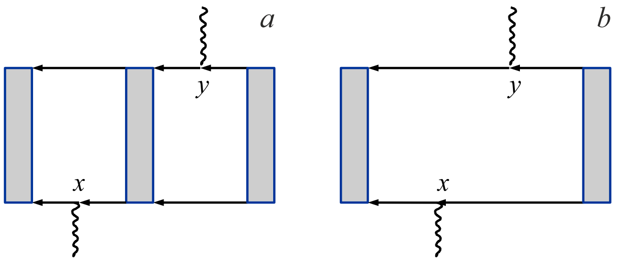

Figure 2. The wavy lines correspond to external coordinates x and y.

$$D_2(x,y)=\sum_{k,l=0}^{\infty}(4\rho)^{k+l+2}\int dx'dy'\langle L/2|\hat{U}^k|x'\rangle$$

$$\times\Gamma_2(x',x,y,y')\langle y'|\hat{U}^l(\hat{V}+\hat{\gamma}^{-1})|-L/2\rangle. \quad (54)$$

Here, the following notations are used:

$$\langle x_1|\hat{f}\hat{g}|x_2\rangle\equiv\int dx\, f(x_1,x)g(x,x_2),$$

$$\Gamma_1(x',x,x'')=u(x'-x)u(x-x'')u(x'-x''), \quad (55)$$

$$\Gamma_2(x',x,y,y')=u(x'-x)u(x-y')u(x'-y)u(y-y'). \quad (56)$$

Figure 2, a and 2, b shows typical fragments of the general term in sums (53) and (54), respectively; wavy lines correspond to external coordinates x and y. Resulting diagrams are identical to those in the atom density matrix method [16] that uses technique [17].

It is more convenient to calculate quantities (53) and (54) by passing to the Fourier representation in the expressions under the integral sign followed by summation over photon numbers and, finally, by inverse Fourier transform. In the given stationary excitation limit ($L\to\infty$), we get

$$D(x-y)=\frac{\Omega_R^2\gamma/4}{\Omega_R^2+\gamma^2/2}\left\{\frac{\gamma^2}{\Omega_R^2+\gamma^2/2}+e^{-\gamma|x-y|/2}\right.$$

$$+\frac{\Omega_R^2-\gamma^2/2}{\Omega_R^2+\gamma^2/2}\cos\left[\Omega(x-y)\right]e^{-3\gamma|x-y|/4}$$

$$+\frac{\gamma}{\Omega_R}\frac{(5/4)\Omega_R^2-\gamma^2/2}{\Omega_R^2+\gamma^2/2}\sin\left(\Omega|x-y|\right)$$

$$\left.\times e^{-3\gamma|x-y|/4}\right\}, \quad (57)$$

where $\Omega\equiv\sqrt{\Omega_R^2-(\gamma/4)^2}$. Fourier transform of this correlator with respect to $(x-y)$ gives the scattered radiation spectrum where, besides the central peak, there are broadened side peaks at frequencies $\pm\Omega_R$ (forming together the famous Mollow triplet at $\Omega_R>\gamma/4$) [7].

3. Resulting exact expression (39) for the scattered light state allows complete description of statistical properties of radiation. Thus, for example, it is easy to see that (Figure 3) the n-th order correlation function

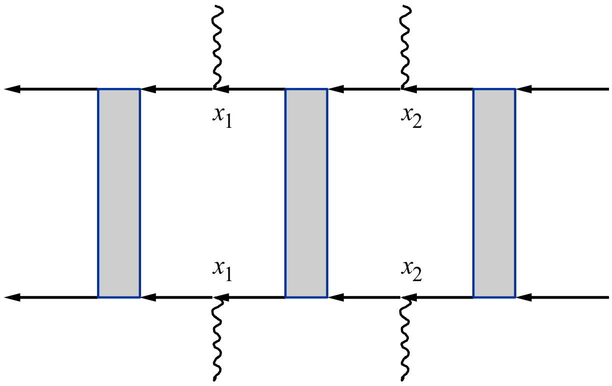

Figure 3. Factorization of the n-th order correlation function

$$\rho^{(n)}(x_1,\ \ldots,x_n)$$

$$=\langle\widetilde{Out},\alpha|\varepsilon^{\dagger}(x_1)\ \ldots\ \varepsilon^{\dagger}(x_n)\varepsilon(x_n)\ \ldots\ \varepsilon(x_1)|\widetilde{Out},\alpha\rangle \quad (58)$$

is expressed through the second-order correlation function $\rho^{(2)}(x_1,x_2)$. Specifically, in the region $x_1>\ \ldots\ >x_n$, we have

$$\rho^{(n)}(x_1,\ \ldots,x_n)=\rho^{(2)}(x_1,x_2)\rho^{(2)}(x_2,x_3)\ \ldots$$

$$\times\rho^{(2)}(x_{n-1},x_n). \quad (59)$$

Calculation of $\rho^{(2)}(x_1,x_2)$ is identical to the derivation shown above for expression (51) and gives

$$\rho^{(2)}(x_1,x_2)=\left(\frac{\Omega_R^2\gamma/2}{\Omega_R^2+\gamma^2/2}\right)^2\left\{1-\left[\cos\left(\Omega|x_1-x_2|\right)\right.\right.$$

$$\left.\left.+\frac{3\gamma}{4\Omega}\sin\left(\Omega|x_1-x_2|\right)\right]e^{-3\gamma|x_1-x_2|/4}\right\}. \quad (60)$$

Note that expression (60) (and, therefore, (59)) describes the so-called photon antibunching phenomenon, i.e. correlators (60), (58) vanish when positions coincide. In the approach developed here, this obviously results from the „Fermi“ structure of the wave function of scattered photon state (39) that vanishes when photon positions coincide.

Expressions (57) and (60) derived for the particular case of stationary excitation by the coherent field discussed in this section agree with the known results and illustrate the application of the field-theoretic approach described in the previous sections.

6. Conclusion

We have presented an exact field-theoretic approach to description of multiphoton state scattering in an electromagnetic field with arbitrary statistics on the resonance atom. The approach is based on direct regular application of the

Hamilton formalism without using the density matrix and description reduction related thereto. Such usually unfeasible approach can be implemented due to the exceptional circumstance where the given „atom +field" system is an integrable one, which makes it possible to find a full set of multiphoton eigenstates. Hence, the resonance fluorescence description may be represented as an exact solution of the quantum Cauchy problem that corresponds to the dynamic transformation of the given initial in-state (incoming field) into the final out-state (scattered and unscattered field). The constructed exact out-state makes it possible to calculate the scattered field characteristics for an arbitrary quantum state of the excited field. The general formalism developed in Sections 2−3 is illustrated in Sections 4 − 5 as a particular case where the incoming field is in the stationary coherent state. Spectrum and second-order correlation function of the scattered field coincide with the known results. Derivation proposed here is a regular quantum-mechanical calculation without involvement of sophisticated approaches with the Lindblad equation, regression theorem, etc. [13]. It is expected that the present approach will be in high demand for modern quantum optics applications where the incident field is in nonclassical (incoherent) state.

This paper is dedicated to the memory of V.M. Agranovich (1929−2024) - the founder and first head of the theoretical department at the Institute of Spectroscopy.

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6.1. Conflict of interest

The authors declare that they have no conflict of interest.

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