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Soliton behavior in an optical waveguide in the Gerdjikov-Ivanov model

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Received August 17, 2024
Revised August 17, 2024
Accepted August 26, 2024

Behavior of a soliton moving in an optical waveguide along an inhomogeneous and time-dependent background is addressed in the framework of the Gerdjikov-Ivanov model. Equations describing the soliton motion were derived. Theory is illustrated by the soliton motion along a simple wave.

Keywords:soliton, optical waveguide, Gerdjikov-Ivanov model.

DOI: 10.61011/EOS.2024.08.60028.7000-24

1. Introduction

Light propagation in a nonlinear medium often acts like a liquid flow [1]. From a formal perspective, this is associated with a fact that the nonlinear Schrödinger equation (NSE) neglecting dispersion accurately reduces to the shallow-water equations in a defocusing nonlinearity case or to the „overturned" shallow-water equations in a focusing nonlinearity case. Consideration of dispersion leads to so-called dispersive fluid dynamics that describes various nonlinear optical effects in optical waveguides as well as polariton and atom condensates [2], including soliton and dispersive shock wave behavior [3–5]. Evolution of the dispersive shock wave theory in recent years has brought about the development of new methods allowing great advances to be made in the classical problem of soliton motion along an inhomogeneous and time-dependent large-scale wave.

Difficulty of this problem is in that the soliton motion cannot be separated with absolute accuracy from the background evolution because both of them are actually a single wave flow described by the same wave variables. Due to this, the soliton motion induces a counter flow in the background and this counter flow substantially affects the soliton behavior described approximately as particle-like excitation having a certain coordinate and momentum.

To solve this problem, various approximate methods of the perturbation theory were proposed [6–17] that are usually very cumbersome and not always provide a fairly simple description of the soliton behavior. It has been recently observed [18] that the theory could be significantly simplified supposing that a narrow soliton behavior is described by equations that are agreed in a particular way with zero-dispersion evolution of a large-scale background. Previous results were easily reproduced and new results were obtained on this course for soliton described by the Korteweg de Vries (KdV) equation [18] and NSE [19].

This study will elaborate a similar theory for the Gerdjikov-Ivanov equation [20,21] that will be written in standard dimensionless variables:

$$i\psi_t + \psi_{xx} + \frac{1}{2}|\psi|^4\psi + i\psi^2\psi_x^* = 0. \qquad (1)$$

This equation is often used to simulate propagation of ultrashort light pulses in waveguides [22,23], and in this context ψ is a light field envelope. The equation is written in a frame of reference that moves at a group carrier wave velocity, wherein x was used for a coordinate along the waveguide and t was used for time to emphasize the analogy with equations of fluid dynamics. The second term on the left-hand side of (1), as usual, describes the group velocity dispersion, the third term describes the fifth-order nonlinearity, and the fourth term describes medium response delay. These nonlinear terms are generally accounted for as small perturbations in NSE, but their isolation into a separate equation (1) gives an insight into effects that are qualitatively associated with them.

The next section will address the main relations of the Gerdjikov-Ivanov equation theory, including its soliton solution. Then, a soliton behavior equation will be derived and illustrated by a soliton motion along a simple wave.

2. General relations

First, equation (1) is rearranged to a so-called fluid dynamics form in which the behavior of light wave „photon liquid" may be conveniently described. For this, the following substitution is made

$$\psi = \sqrt{\rho}\exp\left(i\int^x u(x', t)dx'\right), \qquad (2)$$

and the real and imaginary parts are separated in the resulting equation. As a result, the following system is obtained

$$\rho_t + \left[\rho\left(2u + \frac{1}{2}\rho\right)\right]_x = 0,$$

$$u_t + 2uu_x - (\rho u)_x - \rho\rho_x + \left(\frac{\rho_x^2}{4\rho^2} - \frac{\rho_{xx}}{2\rho}\right)_x = 0. \quad (3)$$

These equations are similar to fluid dynamics equations where ρ serves as the density of medium and u serves as the flow velocity of medium. The first equation (3) is a modified continuity equation and the second one is the Euler equation with the „pressure" $P = -\rho(u + \rho/2)$ depending on the flow velocity, wherein the last term, that is often called the „quantum pressure", describes dispersive properties of a medium. For stability of a medium with respect to density fluctuations, the pressure shall be positive which implies the condition

$$u < -\rho/2 \quad (4)$$

of applicability of the light wave description in terms of fluid dynamics.

When small perturbations of a homogeneous state $\rho = \rho_0$, $u = u_0$ are considered, then linearization of equations (3) with respect to small deflections from this state $\rho = \rho_0 + \rho'$, $u = u_0 + u'$ gives a system

$$\rho'_t + 2u_0\rho'_x + 2\rho_0 u'_x + \rho_0\rho'_x = 0,$$

$$u'_t + 2u_0 u'_x - u_0\rho'_x - \rho_0 u'_x - \rho_0\rho'_x - \frac{1}{2\rho_0}\rho'_{xxx} = 0, \quad (5)$$

from which the linear wave dispersion law is obtained ρ', $u' \propto \exp[i(kx - \omega t)]$:

$$\omega = k\left(2u_0 \pm \sqrt{-\rho_0(2u_0 + \rho_0) + k^2}\right). \quad (6)$$

Radical expression is positive here provided that condition (4) is fulfilled.

Another important limiting case refers to long waves for which $|\rho_x/\rho| \ll 1$, $|u_x/u| \ll 1$, so the dispersion term may be neglected in the second equation (3) and zero-dispersion equations may be derived

$$\rho_t + \left[\rho\left(2u + \frac{1}{2}\rho\right)\right]_x = 0,$$

$$u_t + 2uu_x - (\rho u)_x - \rho\rho_x = 0. \quad (7)$$

In our theory, they describe evolution of the background along which the soliton travels.

Let's now get a soliton solution in a convenient form. For this, solution of equations (3) is sought in the form of the progressing wave $\rho = \rho(\xi)$, $u = u(\xi)$, $\xi = x - Vt$, where V is the wave propagation velocity. Then equations (3) become ordinary, are easily integrated once, and the first equation gives

$$u = \frac{A}{2\rho} + \frac{V}{2} - \frac{\rho}{4}, \quad (8)$$

where A is the integration constant. The second equation after integration and substitution of Eq.(8) into it may be integrated once more with the result rearranged to the following form

$$\rho_\xi^2 = -\frac{1}{4}\left[\rho^4 + 4V\rho^3 + 4(V + 3A + 4B) \times \rho^2 + 16C\rho + 4A^2\right], \quad (9)$$

where B and C are another two integration constants.

Three integration constants that occur in the solution may be expressed through the soliton velocity V and values of the background variables ρ_0, u_0 at infinity, i.e. far from the soliton. For this, note, above all, that equation (9) has solutions when ρ fluctuates between zeros $\rho_1 \le \rho_2 \le \rho_3 \le \rho_4$ of the polynomial on the right-hand side in the positivity intervals of this polynomial, i.e. in one of the intervals $\rho_1 \le \rho \le \rho_2$ or $\rho_3 \le \rho \le \rho_4$. As is well known, the soliton corresponds to a separatrix solution that occurs when $\rho_2 = \rho_3 = \rho_0$, so in this case equation (9) may be rewritten as

$$\rho_\xi^2 = \frac{1}{4}(\rho_0 - \rho)^2(\rho - \rho_1)(\rho_4 - \rho). \quad (10)$$

Let's assume for definiteness that the case of $\rho_1 \le \rho \le \rho_2 = \rho_0$ corresponding to a dark soliton is considered and $\rho = \rho_{min} = \rho_1$ with $\xi = 0$, so the elementary integration gives

$$\rho = \rho_0 - \frac{(\rho_4 - \rho_0)(\rho_0 - \rho_1)}{(\rho_4 - \rho_1)\operatorname{ch}^2\theta - (\rho_1 - \rho_0)}, \quad (11)$$

where

$$\theta = \frac{1}{4}\sqrt{(\rho_4 - \rho_0)(\rho_0 - \rho_1)}\xi,$$

$$V = -\frac{1}{4}(\rho_1 + \rho_4 + 2\rho_0). \quad (12)$$

To express ρ_1 and ρ_4 in terms of V, ρ_0, u_0, note that $A = \rho_0(\rho_0/2 + 2u_0 - V)$ follows from Eq. (8) and $\rho_1\rho_4\rho_0^2 = 4A^2$ follows from Eq. (9), so

$$\rho_1\rho_4 = (2V - 4u_0 - \rho_0)^2. \quad (13)$$

The second equation (12) and equation (13) imply the values of ρ_1, ρ_4:

$$\rho_1 = -2V - \rho_0 - 2\sqrt{2(\rho_0 + 2u_0)(V - u_0)},$$

$$\rho_4 = -2V - \rho_0 + 2\sqrt{2(\rho_0 + 2u_0)(V - u_0)}, \quad (14)$$

Taking into account (4), we find that these parameters are real and positive in line with the determination of ρ in Eq. (2), when the soliton velocity satisfies the condition

$$V < u_0, \quad (15)$$

i.e. it is negative and greater in magnitude than the background flow velocity u_0.

Equations (11) and (12) imply that the inverse soliton halfwidth κ defined by the limit relation $\rho_0 - \rho \propto \exp(-\kappa|\xi|)$ is equal to

$$\kappa = \frac{1}{2}\sqrt{(\rho_4 - \rho_0)(\rho_0 - \rho_1)}. \tag{16}$$

Substitution of expressions (14) here gives

$$(V + \rho_0)^2 - 2(\rho_0 + 2u_0)(V + \rho_0) + 2(\rho_0 + u_0)$$
$$\times (\rho_0 + 2u_0) + \kappa^2 = 0,$$

from which the desired expression for the soliton velocity is derived through the inverse soliton halfwidth:

$$V = 2u_0 \pm \sqrt{-\rho_0(\rho_0 + 2u_0) - \kappa^2}. \tag{17}$$

As reported by Stokes [24] for the Korteweg de Vries soliton, this velocity is expressed through the linear wave dispersion law (6) by

$$V = \frac{\omega(i\kappa)}{i\kappa}, \tag{18}$$

that has a clear physical meaning: linear waves $\propto e^{i(kx-\omega t)}$ as well as low-amplitude soliton tails $\propto e^{-\kappa|x-Vt|}$ are described by the same linearized equations (5), so their solutions go into one another when the substitution $k \to i\kappa$ is made, wherein the linear wave phase velocity goes into the soliton velocity (18).

Now we can pass to the derivation of soliton behavior equations.

Soliton behavior

It is supposed that the soliton travels along an inhomogeneous wave $\rho = \rho(x, t)$, $u = u(x, t)$, wherein the typical length at which these wave variables change substantially is much higher than 1. Since the soliton width is in the region of one, then it can be assumed with fair accuracy that the soliton's instantaneous velocity is given by expression (17) where ρ_0 and u_0 are replaced with the local values of $\rho(x, t)$, $u(x, t)$ in the soliton location point at time t. Therefore, if the dependence of κ on ρ and u is found, then equation (17) will be transformed into an equation that defines the soliton motion along a large-scale wave.

To find $\kappa = \kappa(\rho, u)$, let's turn to considerations similar to Stokes' arguments [24] in his justification of equation (18) and find first the dependence of the carrier wave wavenumber k on ρ and u for a high-frequency packet propagating along a large-scale wave. For this, note that, as long as the wavelength $\sim k^{-1}$ is assumed to be lower or approximately equal to 1, then the packet size may be taken to be much lower than the typical amount of background variations, so the packet position may be described with fair accuracy by introducing the coordinate $x = x(t)$. As is known from the optical-mechanical analogy [25,26], motion of the packet in the geometric-optical approximation obeys the Hamilton equations

$$\frac{dx}{dt} = \frac{\partial\omega}{\partial k}, \qquad \frac{dk}{dt} = -\frac{\partial\omega}{\partial x}, \tag{19}$$

where $\omega = \omega(k, x, t)$ serves as the Hamiltonian. In our case ω depends on x and t only through the wave variables $\rho(x, t)$, $u(x, t)$ according to equation (6). It is assumed that k is also a function of only ρ and u, and the evolution of ρ and u according to the zero-dispersion equations (7) keeps the Hamilton equations (19) valid. As was found in [27–30], these conditions of Hamiltonian structure preservation by the zero-dispersion flow give equations that define $k = k(\rho, u)$ at least in the range of large values of k. For the Gerdjikov–Ivanov equation, calculations similar to those conducted in [27–30] give these equations written as

$$\frac{\partial k^2}{\partial \rho} = 2(u + \rho),$$
$$\frac{\partial k^2}{\partial u} = 2\rho - 2\sqrt{k^2 - \rho(\rho + 2u)}. \tag{20}$$

These equations are easily solved and we obtain

$$k^2 = (q - u)^2 + \rho(\rho + 2u), \tag{21}$$

where q is the integration constant.

Now, following Stokes, it is supposed that solution of equations (20) similarly to solution of linearized equations (5) may be extended to the regions of negative values of k^2, so equation (21) transforms into

$$\kappa^2 = -(q - u)^2 - \rho(\rho + 2u) \tag{22}$$

for the inverse soliton halfwidth that defines $\kappa = \kappa(\rho, u)$. Substitution of this equation into (18) gives (suppose $q - u > 0$)

$$\frac{dx}{dt} = V = \frac{\omega(i\kappa)}{i\kappa} = q + u(x, t), \tag{23}$$

that defines the soliton path $x = x(t)$. q is defined by the initial soliton velocity at the origin of its path.

Note that the Gerdjikov–Ivanov equation is fully integrable and, therefore, modulation Whitham equations in a diagonal Riemannian form may be derived for it. Then equation (23) may be derived from the Whitham equation in the soliton limit (see this kind of theory in [14–17] for the KdV and NSE equations). Our method provides a simpler derivation procedure. Moreover, equation (23) may form a basis for derivation of the Hamilton equations for the soliton behavior similarly to [19] so that it is converted into the integral of these equations. If there is the term $U(x)\psi$ with potential $U(x)$ on the right-hand side of equation (1), then the Hamilton equations may be rearranged to the Newton equation taking the potential into account. However, such statement of the problem is likely not relevant for the soliton behavior in waveguides and, therefore, we restrict ourselves to the case where an external potential is not available.

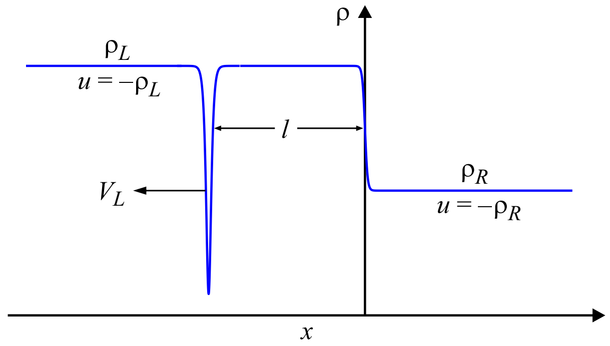

Figure 1. Initial distribution of density ρ corresponding to the soliton with the initial coordinate $-l$ and velocity $V_L < 0$ as well as to the discontinuity of density (28).

Example: soliton motion along a simple wave

By way of illustration, let's consider the soliton motion along a simple wave formed by the initial discontinuity in the distributions of density ρ and u. Zero-dispersion system (7) may be rearranged in a diagonal form

$$\frac{\partial r_+}{\partial t} + v_+ \frac{\partial r_+}{\partial x} = 0, \qquad \frac{\partial r_-}{\partial t} + v_+ \frac{\partial r_-}{\partial x} = 0 \tag{24}$$

for variables

$$r_\pm = u \pm \sqrt{-\rho(\rho + 2u)}, \tag{25}$$

wherein characteristic velocities are equal to

$$v_\pm = 2u \pm \sqrt{-\rho(\rho + 2u)}. \tag{26}$$

In a simple wave, one of the Riemannian invariants $r_\pm$ is constant. As can be seen from (25), $r_+ = 0$, if u and ρ are interrelated by

$$u = -\rho, \tag{27}$$

that is fulfilled all round the simple wave. In this case we have the initial discontinuity of density ρ

$$\rho(x, 0) = \begin{cases} \rho_L, & x \le 0, \\ \rho_R, & x > 0, \end{cases} \tag{28}$$

where $\rho_L > \rho_R$, and $u(x, 0)$ has an opposite discontinuity:

$$u(x, 0) = \begin{cases} -\rho_L, & x \le 0, \\ -\rho_R, & x > 0, \end{cases} \tag{29}$$

so everywhere $r_+ = 0$.

Let at the initial time $t = 0$ the soliton be located on the left of the discontinuity in point $x = -l$ (Figure 1) and has the initial velocity $V_L < 0$. After the time t, the discontinuity will turn into the depression wave in which everywhere $r_+ = 0$, i.e. the first equation (24) is satisfied automatically, and the second equation implies self-similar solution $v_- = -3\rho = x/t$, i.e.distribution of density is written as

$$\rho(x, t) = \begin{cases} \rho_L, & x \le -3\rho_L t, \\ -x/(3t), & -3\rho_L t < x < -3\rho_R t, \\ \rho_R, & x > -3\rho_R t. \end{cases} \tag{30}$$

Left edge of this distribution moves at the velocity of $-3\rho_L$, so the soliton falls into the depression wave region at time

$$t_1 = \frac{l}{|-3\rho_L - V_L|} = \frac{l}{3\rho_L + V_L} \tag{31}$$

at point

$$x_1 = -L + V_L t_1 = -\frac{3l\rho_L}{3\rho_L + V_L}. \tag{32}$$

As can be easily found from (17), we always have $|V_L| < 3\rho_L$, i.e. such time $t_1 > 0$ always comes. When $t > t_1$, the soliton moves along the depression wave (30) according to equation (23), i.e

$$\frac{dx}{dt} = q - \frac{x}{3t}, \tag{33}$$

where q, we find from condition that $(dx/dt)_{t=t_1} = V_L$, so $q = V_L + \rho_L$. Equation (33) is easily solved with the initial condition $x(t_1) = x_1$, so for the soliton path we get

$$x(t) = \frac{3}{2}(V_L + \rho_L)t - \frac{3}{2} l^{2/3} [(3\rho L + V_L)t]^{1/3}. \tag{34}$$

As can be seen, it differs considerably from the soliton motion law $x = -l + V_L t$ with the homogeneous distribution of density $\rho = \rho_L$ up until transition to the simple wave region.

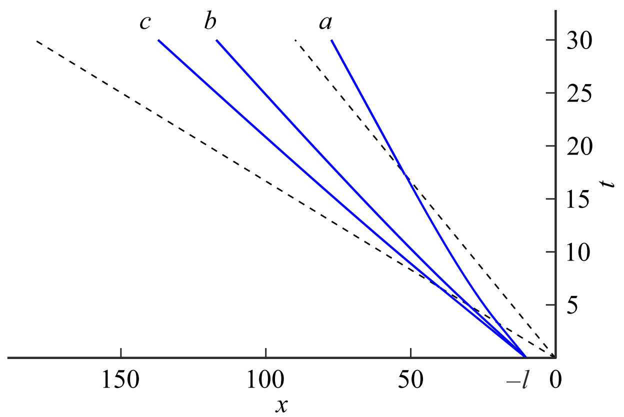

Figure 2. Paths of solitons with various initial velocities when moving across a depression wave formed by the initial discontinuity with $\rho_L = 2$, $\rho_R = 1$. The initial coordinate of the soliton is $-l = -10$. Dashed lines show the motion of the depression wave edges. Curve (a) corresponds to the initial velocity $V_L = -3$ that satisfies condition (36) where the soliton passes through the depression wave. Curve (b) corresponds to special case (50) when the soliton velocity asymptotically approaches the front edge velocity $-3\rho_R$. For curve (c), the soliton in the asymptotic state is carried by the depression wave at velocity (39).

The soliton achieves the opposite depression wave edge that moves at the velocity of $-3\rho_R$ at time

$$t_2 = \frac{l(3\rho_L + V_L)^{1/2}}{(V_L + \rho_L + 2\rho_R)^{3/2}}, \tag{35}$$

if its initial velocity satisfies

$$|V_L| < \rho_L + 2\rho_R. \tag{36}$$

After this time and after collision with the depression wave, it moves along the homogeneous background $\rho = \rho_R$ at the velocity

$$V_R = V_L + \rho_L - \rho_R, \tag{37}$$

i.e. its velocity decreases in magnitude. It is obvious that by virtue of (36), V_R is inherently negative as must be the case for the soliton moving along a homogeneous background. The soliton path for this case is illustrated in Figure 2 (curve (a)).

However, if

$$|V_L| > \rho_L + 2\rho_R, \tag{38}$$

then the soliton motion stabilizes inside the depression wave and its velocity at $t \to \infty$ tends to a constant value

$$V_{\text{asymp}} = \frac{3}{2}(V_L + \rho_L). \tag{39}$$

In this asymptotic state, the soliton is always at a point on the depression wave profile with the density

$$\rho_{\text{asymp}} = -\frac{1}{2}(V_L + \rho_L), \tag{40}$$

i.e. it is carried by the depression wave flow being at rest with respect to the wave (Figure 2, curve (c)).

In a special case $V_L = -\rho_L - 2\rho_R$, the soliton path is given as

$$x(t) = -3\rho_R t - 3(l/2)^{2/3} t^{1/3}, \tag{41}$$

i.e. the soliton velocity tends at $t \to \infty$ to the rear edge velocity $-3\rho_R$, but the distance between the soliton and rear edge grows with time as $t^{1/3}$ (curve (b) in Figure 2).

A similar behavior of soliton „trapping“ by the depression wave in case of NSE was discussed in [15,31] using various methods.

Conclusion

Methods used in this work relies on Stokes' elementary considerations that correlate the properties of solitons with the properties of high-frequency packets propagating along large-scale wave profiles. In the case of the Gerdjikov−Ivanov equation, equations (20) that occur in our theory admit an exact solution of (21), which yields an extremely simple soliton motion law (23). However, note that, generally speaking, equations of type (20) defining the function $k = k(\rho, u)$, that depends on two arguments, are not always consistent. At this point, it can be said that they are consistent in case of fully integrable equations, but examples of their consistency for non-integrable equations are not known. However, it can be expected that the proposed method remains also applicable to non-integrable equations when equations of type (20) admit approximate solution that is true in the range of high values of $k \gg 1$. Assuming that this approximate solution may be extended to the region of negative $k^2 = -\kappa^2 < 0$, a relation interconnecting the inverse soliton halfwidth with the local wave variables of the background is obtained and an approximate soliton motion equation is finally derived. Hopefully, such generalization of theory will help solve important soliton behavior problems, including in the field of nonlinear optics.

Acknowledgments

The study was supported by research project FFUU-2021-0003 of the Institute of Spectroscopy RAS (Section 2−3) and grant „Basis“ (Section 4).

Conflict of interest

The authors declare that they have no conflict of interest.

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Translated by E.Ilinskaya