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# Fundamental and regular transport solutions of Maxwell's equations and their properties at superluminal speeds: shock electromagnetic waves

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Transport solutions of Maxwell's equations under the action of moving emitters of electromagnetic waves moving with a constant velocity in a fixed direction are considered. Using the Fourier transform of generalized functions, fundamental and generalized solutions are constructed for velocities exceeding the velocity of electromagnetic wave propagation in a medium and coinciding with it, which is called the *light velocity*. Their regular integral representations are given in analytical form. Construction of solutions for arbitrary moving sources is based on the property of convolution of fundamental solutions of differential equations with the right-hand side. It is shown that shock electromagnetic waves arise at such velocities. Using the method of generalized functions, conditions are obtained for jumps in electromagnetic field strengths at shock wave fronts. It is shown that shock electromagnetic waves are transverse, and the vectors of electric and magnetic strength are orthogonal to each other and lie in the tangent bundle to the shock wave front.

**Keywords**: Maxwell's equations, light speed , motion speed, Mach number, Green's tensor, generalized solutions, shock electromagnetic waves, conditions on the fronts.

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## Introduction

Maxwell's Equations (ME) linking the vectors of electric and magnetic intensity with electric currents and charges constitute the basis of the modern electrodynamics and allow determining the electromagnetic (EM) field at known charges and currents, and vice versa. Many scientists have been solving various tasks for them since the second half of the 19th century. The bibliography in this area is extensive and there is a lot of educational literature on electromagnetism [1–7].

Movable sources mounted on platforms of different vehicles are the most common existing sources of EM waves emission. It is evident that the travel velocity significantly impacts the processes of EM wave propagation in the media with different electrical conductivity and permeability, as well as the shape of the source itself and the nature of its operation. The studies in this area are not that abundant, and these are related to certain type of the emission source [8–14].

Previously, we constructed fundamental and generalized transport solutions of the ME system under the action of movable EM wave sources moving in a fixed direction at a constant velocity, which is lower than the velocity of propagation of EM waves in the medium called *light* [15]. Formulas were obtained for calculating EM fields for moving emitters of various types and arbitrary shapes, useful for radio engineering applications.

Here we construct fundamental and generalized solutions to the problem of motion at velocities exceeding the light in the considered electromagnetic medium and coinciding with it. Regular integral representations are constructed in an analytical form.

At light and superlight velocities the system of transport ME becomes strictly hyperbolic, its solutions describe *shock* EM waves, at the fronts of which the electric and magnetic field intensity vectors are discontinuous. The methods of the theory of generalized functions were used to obtain the conditions at the shock wave fronts which confirm the known properties of the transversity of EM waves and the orthogonality of the electric and magnetic field strength vectors at their fronts and phase surfaces.

## 1. Transport ME. Mach number

Let us consider the ME system:

$$\begin{gathered}\operatorname{rot}\mathbf{E}+\mu\mu_0\frac{\partial\mathbf{H}}{\partial t}=\mathbf{j}^m(x_1,x_2,x_3,t),\\ \operatorname{rot}\mathbf{H}-\varepsilon\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}=\mathbf{j}^e(x_1,x_2,x_3,t),\\ \operatorname{div}\mathbf{H}=0,\quad \operatorname{div}\mathbf{D}=\rho^e.\end{gathered}\tag{1}$$

where $\mathbf{j}^m\,[\mathrm{V/m^2}]$ is the magnetic current density vector, $\mathbf{j}^e\,[\mathrm{A/m^2}]$ is the electric current density vector, $\mathbf{E}\,[\mathrm{V/m}]$ is the electric field strength vector, $\mathbf{H}(x,t)\,[\mathrm{A/m}]$ is the magnetic field strength vector, $\rho^e$ [C/m$^3$] is the volumetric density of electric charge.

Material ratios:

$$\mathbf{B} = \mu\mu_0\mathbf{H}, \quad \mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}, \tag{2}$$

where $\mu$ is the magnetic permeability of the medium, $\varepsilon$ is the electrical conductivity of the medium, $\mathbf{B}(x_1, x_2, x_3, t)$ is the magnetic field induction vector, $\mathbf{D}(x_1, x_2, x_3, t)$ is the electric field induction vector.

Magnetic currents $\mathbf{j}^m(x_1, x_2, x_3, t)$ are introduced in the equations (1). $\mathbf{j}^m = 0$ in the ME. Next, let us remove this constraint.

Let us consider mobile transport sources of EM waves that move at a constant velocity $V$ in a certain direction ($\mathbf{e}_z$). They can be described by currents of the form $\mathbf{j}^m(x_1, x_2, z)$, $z = x_3 + Vt$. In the moving coordinate system $(x_1, x_2, z)$:

$$\frac{\partial}{\partial t} = V\frac{\partial}{\partial z}$$

and the vector ME will have the following form:

$$\frac{\partial E_z}{\partial x_2} - \frac{\partial E_2}{\partial z} + V\mu\mu_0\frac{\partial}{\partial z}H_1 = j_1^m(x_1, x_2, z),$$

$$\frac{\partial E_1}{\partial z} - \frac{\partial E_z}{\partial x_1} + V\mu\mu_0\frac{\partial}{\partial z}H_2 = j_2^m(x_1, x_2, z),$$

$$\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} + V\mu\mu_0\frac{\partial}{\partial z}H_z = j_z^m(x_1, x_2, z),$$

$$\frac{\partial H_z}{\partial x_2} - \frac{\partial H_2}{\partial z} - V\varepsilon\varepsilon_0\frac{\partial}{\partial z}E_1 = j_1^e(x_1, x_2, z),$$

$$\frac{\partial H_1}{\partial z} - \frac{\partial H_z}{\partial x_1} - V\varepsilon\varepsilon_0\frac{\partial}{\partial z}E_2 = j_2^e(x_1, x_2, z),$$

$$\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} - V\varepsilon\varepsilon_0\frac{\partial}{\partial z}E_z = j_z^e(x_1, x_2, z). \tag{3}$$

Two scalar equations (1) do not change their form. Let us call this system *transport* ME. Let us write it in matrix form [15]:

$$\mathbf{M}(\partial_1, \partial_2, \partial_z)\mathbf{u} = \mathbf{J}, eqno(4)$$

where $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, z$; $\mathbf{M}(\partial_1, \partial_2, \partial_z)$ — Maxwell's transport differential operator, which has the following form:

$$\mathbf{M} = \begin{pmatrix} 0 & -\partial_z & \partial_2 & V\mu\mu_0\partial_z & 0 & 0 \\ \partial_z & 0 & -\partial_1 & 0 & V\mu\mu_0\partial_z & 0 \\ -\partial_2 & \partial_1 & 0 & 0 & 0 & V\mu\mu_0\partial_z \\ -V\varepsilon\varepsilon_0\partial_z & 0 & 0 & 0 & -\partial_z & \partial_2 \\ 0 & -V\varepsilon\varepsilon_0\partial_z & 0 & \partial_z & 0 & -\partial_1 \\ 0 & 0 & -V\varepsilon\varepsilon_0\partial_z & -\partial_2 & \partial_1 & 0 \end{pmatrix};$$

$$\mathbf{u} = \begin{pmatrix} \mathbf{E}(x_1, x_2, z) \\ \mathbf{H}(x_1, x_2, z) \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \mathbf{j}^m(x_1, x_2, z) \\ \mathbf{j}^e(x_1, x_2, z) \end{pmatrix}.$$

Next, let us use the following notations: $c = \frac{1}{\sqrt{\mu\mu_0\varepsilon\varepsilon_0}}$ — the propagation velocity of EM waves in the considered medium. Let us call it *light.*

Let us call the ratio $M = \frac{V}{c}$ as the *Mach number* like the ratio of the velocity of motion of the source of disturbance in the medium with respect to the velocity of wave propagation in the medium is called in the continuum mechanics.

There are three possible cases of motion that change the type of equations (4) and the type of its solutions: *sublight* $M < 1$, *light* $M = 1$ and *superlight* $M > 1$. We built and studied ME transport solutions at sublight velocities earlier in Ref. [15]. We have elliptic type equations in this case. Here we consider two other cases that lead to systems of strictly hyperbolic and parabolic type, respectively, of the velocity of motion, which significantly affects the type of solution and its properties.

## 2. The Green tensor of transport ME at superlight velocities

**Definition**. *The Green ME tensor is a matrix of fundamental solutions of equations* (4) *at*

$$\mathbf{J} = \delta(x_1)\delta(x_2)\delta(z)\{\delta_{ij}\}_{6\times 6},$$

*satisfying the emission conditions, which describe waves propagating from a movable wave source and attenuating in the infinity.*

The Green tensor satisfies the equation

$$\mathbf{M}(\partial_1, \partial_2, \partial_z)\mathbf{U}(x_1, x_2, z) = \delta(x_1)\delta(x_2)\delta(z)\{\delta_{ij}\}_{6\times 6}, \tag{5}$$

and where $\delta_{ij}$ — Kronecker symbol, $\delta(\ldots)$ — Dirac delta function. We use Fourier transform in the space of slow-growth generalized functions to build it [{]16,17}.

The relation with the original coordinates have the following form in the transformation space $(k_1, k_2, k_3)$

$$(x_1, x_2, z) \leftrightarrow (k_1, k_2, k_3).$$

The Fourier transform has the following form for regular functions:

$$\mathrm{F}[f(x_1, x_2, z)] = \bar{f}(k_1, k_2, k_3) = \int_{R^3} f(x_1, x_2, z) \times e^{i(x_1k_1 + x_2k_2 + zk_3)}dx_1dx_2dz,$$

Inverse Fourier transform:

$$\mathrm{F}^{-1}[\bar{f}(k_1, k_2, k_3)] = f(x_1, x_2, z) = \frac{1}{(2\pi)^3}\int_{R^3} \bar{f}(k_1, k_2, k_3)e^{-i(x_1k_1 + x_2k_2 + zk_3)}dk_1dk_2dk_3. \tag{6}$$

Using the property of Fourier transform of the derivative: $\partial_j \Leftrightarrow -ik_j$, and delta functions: $F[\delta(x_1)\delta(x_2)\delta(z)] = 1$, we obtain a system of linear algebraic equations for determining the components of the Green tensor transform:

$$\mathbf{M}(-ik_1, -ik_2, -ik_z)\bar{\mathbf{U}}(k_1, k_2, k_3) = \{\delta_{ij}\}_{6\times 6}. \quad (7)$$

Here $\mathbf{M}(-ik_1, -ik_2, -ik_z)$ — the Fourier transform of the Maxwell transport differential operator:

$$\mathbf{M}(-ik_1, -ik_2, -ik_z) = \begin{pmatrix} 0 & ik_3 & -ik_2 & -ik_3V\mu\mu_0 & 0 & 0 \\ -ik_3 & 0 & ik_1 & 0 & -ik_3V\mu\mu_0 & 0 \\ ik_2 & -ik_1 & 0 & 0 & 0 & -ik_3V\mu\mu_0 \\ ik_3V\varepsilon\varepsilon_0 & 0 & 0 & 0 & ik_3 & -ik_2 \\ 0 & ik_3V\varepsilon\varepsilon_0 & 0 & -ik_3 & 0 & ik_1 \\ 0 & 0 & ik_3V\varepsilon\varepsilon_0 & ik_2 & -ik_1 & 0 \end{pmatrix}. \quad (8)$$

The solution of equations (7) has the form of an inverse matrix:

$$\bar{\mathbf{U}}(k_1, k_2, k_3) = \big(\mathbf{M}(-ik_1, -ik_2, -ik_z)\big)^{-1}, \quad (9)$$

the columns of which are components of the Green tensor at superlight velocities presented below:

$$\{\bar{U}_{m1}\} = \begin{bmatrix} 0 \\ \frac{-ik_3}{k_1^2+k_2^2-k_3^2m^2} \\ \frac{ik_2}{k_1^2+k_2^2-k_3^2m^2} \\ \frac{ik_1^2-ik_3^2M^2}{V\varepsilon\varepsilon_0 k_3(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{ik_1k_2}{V\varepsilon\varepsilon_0 k_3(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{ik_1}{V\varepsilon\varepsilon_0(k_1^2+k_2^2-k_3^2m^2)} \end{bmatrix}, \{\bar{U}_{m2}\} = \begin{bmatrix} \frac{ik_3}{k_1^2+k_2^2-k_3^2m^2} \\ 0 \\ \frac{-ik_1}{k_1^2+k_2^2-k_3^2m^2} \\ \frac{ik_1k_2}{V\varepsilon\varepsilon_0 k_3(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{ik_2^2-ik_3^2M^2}{V\varepsilon\varepsilon_0 k_3(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{ik_2}{V\varepsilon\varepsilon_0(k_1^2+k_2^2-k_3^2m^2)} \end{bmatrix},$$

$$\{\bar{U}_{m3}\} = \begin{bmatrix} \frac{-ik_2}{k_1^2+k_2^2-k_3^2m^2} \\ \frac{ik_1}{k_1^2+k_2^2-k_3^2m^2} \\ 0 \\ \frac{ik_1}{V\varepsilon\varepsilon_0(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{ik_2}{V\varepsilon\varepsilon_0(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{-ik_3m^2}{V\varepsilon\varepsilon_0(k_1^2+k_2^2-k_3^2m^2)} \end{bmatrix}, \{\bar{U}_{m4}\} = \begin{bmatrix} -\frac{ik_1^2-ik_3^2M^2}{V\mu\mu_0 k_3(k_1^2+k_2^2-k_3^2m^2)} \\ -\frac{ik_2k_1}{V\mu\mu_0 k_3(k_1^2+k_2^2-k_3^2m^2)} \\ -\frac{ik_1}{V\mu\mu_0(k_1^2+k_2^2-k_3^2m^2)} \\ 0 \\ \frac{-ik_3}{k_1^2+k_2^2-k_3^2m^2} \\ \frac{ik_2}{k_1^2+k_2^2-k_3^2m^2} \end{bmatrix},$$

$$\{\bar{U}_{m5}\} = \begin{bmatrix} \frac{-ik_2k_1}{V\mu\mu_0 k_3(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{-ik_2^2-ik_3^2M^2}{V\mu\mu_0 k_3(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{-ik_2}{V\mu\mu_0(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{ik_3}{k_1^2+k_2^2-k_3^2m^2} \\ 0 \\ -\frac{ik_1}{k_1^2+k_2^2-k_3^2m^2} \end{bmatrix}, \{\bar{U}_{m6}\} = \begin{bmatrix} \frac{ik_1}{V\mu\mu_0(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{ik_2}{V\mu\mu_0(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{-ik_3m^2}{V\mu\mu_0(k_1^2+k_2^2-k_3^2m^2)} \\ \frac{-ik_2}{k_1^2+k_2^2-k_3^2m^2} \\ \frac{ik_1}{k_1^2+k_2^2-k_3^2m^2} \\ 0 \end{bmatrix}.$$

We obtain the following in denominators using similar terms at $M > 1$, since $1 - M^2 < 0$

$$k_1^2 + k_2^2 + k_3^2 - M^2k_3^2 = k_1^2 + k_2^2 - m^2k_3^2, \; m = \sqrt{M^2 - 1}.$$

It should be noted that the components of the Green tensor are expressed in terms of the following basic functions and their originals:

$$\bar{f}_0(k_1, k_2, k_3) = \frac{1}{k_1^2 + k_2^2 - m^2k_3^2} \Leftrightarrow f_0(x_1, x_2, z), \quad (10)$$

$$\bar{f}_1(k_1, k_2, k_3) = -\frac{1}{ik_3}\bar{f}_0(k_1, k_2, k_3) \Leftrightarrow f_0(x_1, x_2, z)$$
$$= \partial_z f_1(x_1, x_2, z). \quad (11)$$

By using them, the properties of the Fourier transform of derivatives the original $\mathbf{U}(x_1, x_2, z)$ is represented through these basic functions:

$$\{\bar{U}_{m1}\} = \begin{bmatrix} 0 \\ -ik_3\bar{f}_0(k_1, k_2, k_3) \\ ik_2\bar{f}_0(k_1, k_2, k_3) \\ \frac{ik_1^2-ik_3^2M^2}{\varepsilon\varepsilon_0 V}\bar{f}_1(k_1, k_2, k_3) \\ \frac{k_1k_2}{\varepsilon\varepsilon_0 V}\bar{f}_1(k_1, k_2, k_3) \\ \frac{ik_1}{\varepsilon\varepsilon_0 V}\bar{f}_0(k_1, k_2, k_3) \end{bmatrix} \Rightarrow$$

$$\Rightarrow \{U_{m1}\} = \begin{bmatrix} 0 \\ \partial_z f_0(x_1, x_2, z) \\ -\partial_2 f_0(x_1, x_2, z) \\ \frac{1}{\varepsilon\varepsilon_0 V}(\partial_1^2 - M^2\partial_z^2)f_1(x_1, x_2, z) \\ \frac{1}{\varepsilon\varepsilon_0 V}\partial_1\partial_2 f_1(x_1, x_2, z) \\ -\frac{1}{\varepsilon\varepsilon_0 V}\partial_1 f_0(x_1, x_2, z) \end{bmatrix},$$

$$\{\bar{U}_{m2}\} = \begin{bmatrix} ik_3\bar{f}_0(k_1,k_2,k_3) \\ 0 \\ -ik_1\bar{f}_0(k_1,k_2,k_3) \\ \frac{k_1k_2}{\varepsilon\varepsilon_0 V}\bar{f}_1(k_1,k_2,k_3) \\ \frac{ik_2^2-ik_3^2M^2}{\varepsilon\varepsilon_0 V}\bar{f}_1(k_1,k_2,k_3) \\ \frac{ik_2}{\varepsilon\varepsilon_0 V}\bar{f}_0(k_1,k_2,k_3) \end{bmatrix} \Rightarrow \{U_{m2}\} = \begin{bmatrix} -\partial_z f_0(x_1,x_2,z) \\ 0 \\ \partial_1 f_0(x_1,x_2,z) \\ -\frac{1}{\varepsilon\varepsilon_0 V}\partial_1\partial_2 f_1(x_1,x_2,z) \\ -\frac{1}{\varepsilon\varepsilon_0 V}(\partial_2^2-M^2\partial_z^2)f_1(x_1,x_2,z) \\ -\frac{1}{\varepsilon\varepsilon_0 V}\partial_2 f_0(x_1,x_2,z) \end{bmatrix},$$

$$\{\bar{U}_{m3}\} = \begin{bmatrix} -ik_2\bar{f}_0(k_1,k_2,k_3) \\ ik_1\bar{f}_0(k_1,k_2,k_3) \\ 0 \\ -\frac{ik_1}{\varepsilon\varepsilon_0 V}\bar{f}_0(k_1,k_2,k_3) \\ -\frac{ik_2}{\varepsilon\varepsilon_0 V}\bar{f}_0(k_1,k_2,k_3) \\ \frac{ik_3m^2}{\varepsilon\varepsilon_0 V}\bar{f}_0(k_1,k_2,k_3) \end{bmatrix} \Rightarrow \{U_{m3}\} = \begin{bmatrix} \partial_2 f_0(x_1,x_2,z) \\ -\partial_1 f_0(x_1,x_2,z) \\ 0 \\ \frac{1}{\varepsilon\varepsilon_0 V}\partial_1 f_0(x_1,x_2,z) \\ \frac{1}{\varepsilon\varepsilon_0 V}\partial_2 f_0(x_1,x_2,z) \\ \frac{m^2}{\varepsilon\varepsilon_0 V}\partial_z f_0(x_1,x_2,z) \end{bmatrix},$$

$$\{\bar{U}_{m4}\} = \begin{bmatrix} \frac{ik_1^2-ik_3^2M^2}{\mu\mu_0 V}\bar{f}_1(k_1,k_2,k_3) \\ \frac{k_1k_2}{\mu\mu_0 V}\bar{f}_1(k_1,k_2,k_3) \\ \frac{ik_1}{\mu\mu_0 V}\bar{f}_0(k_1,k_2,k_3) \\ 0 \\ -ik_3\bar{f}_0(k_1,k_2,k_3) \\ ik_2\bar{f}_0(k_1,k_2,k_3) \end{bmatrix} \Rightarrow \{U_{m4}\} = \begin{bmatrix} \frac{1}{\mu\mu_0 V}(\partial_1^2-M^2\partial_z^2)f_1(x_1,x_2,z) \\ \frac{1}{\mu\mu_0 V}\partial_1\partial_2 f_1(x_1,x_2,z) \\ -\frac{1}{\mu\mu_0 V}\partial_1 f_0(x_1,x_2,z) \\ 0 \\ \partial_z f_0(x_1,x_2,z) \\ -\partial_2 f_0(x_1,x_2,z) \end{bmatrix},$$

$$\{\bar{U}_{m5}\} = \begin{bmatrix} \frac{k_1k_2}{\mu\mu_0 V}\bar{f}_1(k_1,k_2,k_3) \\ \frac{-ik_3^2M^2+ik_2^2}{\mu\mu_0 V}\bar{f}_1(k_1,k_2,k_3) \\ \frac{ik_2}{\mu\mu_0 V}\bar{f}_0(k_1,k_2,k_3) \\ ik_3\bar{f}_0(k_1,k_2,k_3) \\ 0 \\ -ik_1\bar{f}_0(k_1,k_2,k_3) \end{bmatrix} \Rightarrow \{U_{m5}\} = \begin{bmatrix} \frac{1}{\mu\mu_0 V}\partial_1\partial_2 f_1(x_1,x_2,z) \\ \frac{-1}{\mu\mu_0 V}(M^2\partial_z^2-\partial_2^2)f_1(x_1,x_2,z) \\ \frac{-1}{\mu\mu_0 V}\partial_2 f_0(x_1,x_2,z) \\ -\partial_z f_0(x_1,x_2,z) \\ 0 \\ \partial_1 f_0(x_1,x_2,z) \end{bmatrix},$$

$$\{\bar{U}_{m6}\} = \begin{bmatrix} \frac{-ik_1}{\mu\mu_0 V}\bar{f}_0(k_1,k_2,k_3) \\ \frac{-ik_2}{\mu\mu_0 V}\bar{f}_0(k_1,k_2,k_3) \\ \frac{-ik_3m^2}{\mu\mu_0 V}\bar{f}_0(k_1,k_2,k_3) \\ -ik_2\bar{f}_0(k_1,k_2,k_3) \\ ik_1\bar{f}_0(k_1,k_2,k_3) \\ 0 \end{bmatrix} \Rightarrow \{U_{m6}\} = \begin{bmatrix} \frac{1}{\mu\mu_0 V}\partial_1 f_0(x_1,x_2,z) \\ \frac{1}{\mu\mu_0 V}\partial_2 f_0(x_1,x_2,z) \\ \frac{m^2}{\mu\mu_0 V}\partial_3 f_0(x_1,x_2,z) \\ \partial_2 f_0(x_1,x_2,z) \\ -\partial_1 f_0(x_1,x_2,z) \\ 0 \end{bmatrix}. \quad (12)$$

Hence, the Green tensor components are defined through the original basic functions. Let us build them.

## 3. Building original basic functions at $M > 1$

Let us consider the Fourier transform of the basis function:

$$\bar{f}_0(k_1, k_2, k_3) = \frac{1}{k_1^2 + k_2^2 - m^2 k_3^2}, \quad (13)$$

which is the Fourier transform of the fundamental solution of the equation:

$$\frac{\partial^2 f_0}{\partial x_1^2} + \frac{\partial^2 f_0}{\partial x_2^2} - m^2 \frac{\partial^2 f_0}{\partial x_3^2} = \delta(x_1)\delta(x_2)\delta(x_3). \quad (14)$$

This is a hyperbolic wave equation. We use the fundamental solution of the wave equation in 2$D$space to build its solution [16,17]:

$$\left(\frac{\partial^2 \Psi}{\partial x_1^2} + \frac{\partial^2 \Psi}{\partial x_2^2}\right) - a^{-2} \frac{\partial^2 \Psi}{\partial t^2} = \delta(x_1)\delta(x_2)\delta(t), \quad (15)$$

$$\Psi(x_1, x_2, t) = -\frac{aH(at - r)}{2\pi\sqrt{a^2t^2 - r^2}}, \quad r = \sqrt{x_1^2 + x_2^2}, \quad (16)$$

which satisfies the radiation conditions

$$\Psi(x_1, x_2, t) = 0 \quad \text{for} \quad t < 0 \text{ and } \quad r > at. \quad (17)$$

We obtain the original of the first basic function comparing with (14):

$$f_0(x_1, x_2, z) = -\frac{H(z - mr)}{2\pi\sqrt{z^2 - m^2r^2}}. \quad (18)$$

Here $H(z)$ — Heaviside function.

Next, let us find $f_1(x_1, x_2, z)$ using convolution $(*)$ with the Heaviside function. By virtue of (11) and the properties of $H'(z) = \delta(z)$:

$$f_1(x_1, x_2, z) = f_0(x_1, x_2, z)_z * H(z)$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{H(\xi - mr)}{\sqrt{\xi^2 - m^2r^2}} H(z - \xi)d\xi$$
$$= -\frac{H(z)}{2\pi}\int_{mr}^{z} \frac{1}{\sqrt{\xi^2 - m^2r^2}} d\xi$$
$$= -\frac{H(z - mr)}{2\pi} \ln\left(\frac{z + \sqrt{z^2 - m^2r^2}}{mr}\right). \quad (19)$$

The basic functions are built. It should be noted that their carrier is the interior of the cone: $z > mr$, outside of which $\mathbf{U}(x_1, x_2, z) = 0$. That is, the surface of the Mach cone $z = mr$ is the front of the shock EM wave on which the components of the Green tensor are singular, since $f_0(x_1, x_2, z) \to \infty$ at $r \to \frac{z}{m}$.

So, all components of the Green tensor are built. Let us build solutions to these equations for an arbitrary right-hand side using the property of the Green tensor.

## 4. Building transport solutions of ME at $M > 1$

The solution, up to the solution of a homogeneous system of equations, has the form of a tensor-functional convolution of the right side of the equations (4) with the Green tensor:

$$\mathbf{u}(x, z) = \mathbf{U}(x, z) * \mathbf{J}(x, z), \quad (20)$$

$$\begin{pmatrix} \mathbf{E}(x, z) \\ \mathbf{H}(x, z) \end{pmatrix} = \mathbf{U}(x, z) * \begin{pmatrix} \mathbf{j}^m(x, z) \\ \mathbf{j}^e(x, z) \end{pmatrix},$$

or component-by-component

$$u_i(x, z) = \sum_{k=1}^{6} U_{ik}(x, z) * j_k(x, z), \quad i = 1, \dots, 6. \quad (21)$$

The formula (20) contains convolutions of basic functions and their derivatives with current components of the following form:

$$u_1(x, z) = f_k * g(x, z), \quad k = 0, 1,$$
$$u_2(x, z) = \partial_j f_k * g(x, z), \quad j = 1, 2, z,$$
$$u_3(x, z) = \partial_j \partial_m f_k * g(x, z), \quad m = 1, 2, z.$$

Here, using $g(x_1, x_2, z)$ , we conditionally designate the components of the currents $j_k(x_1, x_2, z)$, where $k = 1, \dots, 6$.

Since

$$\partial_z H(z - mr) = \delta(z - mr),$$
$$\partial_j H(z - mr) = -mr_{,j}\delta(z - mr),$$

where $\delta(z - mr)$ — a simple layer on a light cone which is a singular generalized function, so the derivatives of the basis functions are also singular:

$$\partial_z f_0 = \frac{H(z - mr)}{2\pi\sqrt{z^2 - m^2r^2}} = -\frac{zH(z - mr)}{2\pi\left(\sqrt{z^2 - m^2r^2}\right)^3}$$
$$- \frac{1}{2\pi\sqrt{z^2 - m^2r^2}}\delta(z - mr),$$

$$\partial_j f_0 = \frac{H(z - mr)}{2\pi\sqrt{z^2 - m^2r^2}} = \frac{m^2x_jH(z - mr)}{2\pi\left(\sqrt{z^2 - m^2r^2}\right)^3}$$
$$+ \frac{mr_{,j}}{2\pi\sqrt{z^2 - m^2r^2}}\delta(z - mr).$$

Here and further $r_{,j} = \partial r/\partial x_j = \frac{x_j}{r}$.

As you can see, here the density of the simple layer on the cone is equal to infinity, which does not allow direct differentiation of the basic functions. Therefore, the convolution differentiation property should be used when calculating convolutions [16,17]:

$$u_2(x,z) = \partial_j\big(f_k * g(x,z)\big) = \big(f_k * \partial_j g(x,z)\big) = \partial_j f_k * g(x,z),$$

$$u_3(x,z) = \partial_j\partial_m f_k * g(x,z) = \partial_j\partial_m\big(f_k * g(x,z)\big). \quad (22)$$

That is, the solutions of ME (21), for which $\mathbf{j}^m(x,z) = (0,0,0)$, have the form

$$E_x = \frac{1}{\epsilon\epsilon_0 V}\big\{(\partial_1^2 - M^2\partial_3^2)(f_1 * j_1^e) + \partial_1\partial_2(f_1 * j_2^e) - \partial_1(f_0 * j_z^e)\big\},$$

$$E_y = -\frac{1}{\epsilon\epsilon_0 V}\big\{\partial_1\partial_2(f_1 * j_1^e) + (\partial_1^2 - M^2\partial_3^2)(f_1 * j_2^e) + \partial_2(f_0 * j_z^e)\big\},$$

$$E_z = \frac{1}{\epsilon\epsilon_0 V}\big\{\partial_1(f_0 * j_1^e) + \partial_2(f_0 * j_2^e) - (1-M^2)\partial_3(f_0 * j_z^e)\big\}, \quad (23)$$

$$H_x = \partial_3(f_0 * j_2^e) - \partial_2(f_0 * j_z^e),$$
$$H_y = -\partial_3(f_0 * j_1^e) + \partial_1(f_0 * j_z^e),$$
$$H_z = \partial_2(f_0 * j_1^e) - \partial_1(f_0 * j_2^e). \quad (24)$$

If $\mathbf{j}^e(x,z)$ — regular functions, then the solution can be represented in integral form using the integral representation of convolutions (22):

$$u_1(x,z) = \big(f_k * g(x,z)\big) = H(z)\int\limits_{r\le \frac{z}{m}}\Bigg(\int\limits_{mr(x,y)}^{z} f_k(x-y,\xi)g(y,z-\xi)\,d\xi\Bigg)dy_1\,dy_2,$$

$$u_2(x,z) = \partial_j\big(f_k * g(x,z)\big) = H(z)\partial_j\int\limits_{r\le z/m}\Bigg(\int\limits_{mr(x,y)}^{z} f_k(x-y,\xi)g(y,z-\xi)d\xi\Bigg)dy_1dy_2,$$

$$u_3(x,z) = \partial_i\partial_j\big(f_k * g(x,z)\big) = H(z)\partial_i\partial_j\int\limits_{r\le z/m}\Bigg(\int\limits_{mr(x,y)}^{z} f_k(x-y,\xi)g(y,z-\xi)d\xi\Bigg)dy_1dy_2,$$

where

$$x = (x_1,x_2), \quad y = (y_1,y_2), \quad r(x,y) = \|x-y\|.$$

Here the external integral over the domain $y\in\mathbb{R}^2$: $r(x,y)\le\frac{z}{m}$ is a circle of radius $z/m$ centered at point $x$. The introduction of the derivative under the sign of the integral depends on the differentiability properties of the components of the electric current density $\mathbf{j}^e(x,z)$.

If the currents are differentiable, then convolutions (22) should be calculated using formulas

$$u_2(x,z) = \partial_j f_k * g(x,z) = f_k * \partial_j g(x,z),$$

$$u_3(x,z) = \partial_j\partial_m f_k * g(x,z) = f_k * \partial_j\partial_m g(x,z). \quad (25)$$

Then not $g(x,z)$, but their derivatives are used in the integral representation of these convolutions. If the components are singular generalized functions, then the convolutions in solution (21) should be taken according to the definition of convolutions in the space of generalized functions [16,17].

It should be noted also that the formulas (23), (24), in addition to currents distributed in the $3D$-space, allow building solutions of transport systems for EM wave emitters, whose carriers are concentrated at points, on filaments or surfaces of arbitrary shapes, which can be modeled by singular generalized functions of simple and multidimensional layer type and surfaces of different dimensions, as shown by us for sublight velocities in Ref. [15].

## 5. Shock EM waves as generalized solutions of ME. Conditions on fronts

The transportation system is strictly hyperbolic at superlight velocities. Therefore, it can have non-differentiable solutions that are discontinuous on characteristic surfaces in addition to smooth differentiable solutions.

Let's consider such solutions that describe shock EM waves, on the fronts of which solutions and their derivatives experience jumps. Let us use the Method of Generalized Functions to determine the conditions at the shock wave fronts [18–20]. To do this let us consider a system of ME in the space of generalized vector functions, the components of which belong to the class of generalized functions $D'(R^3)$ [16,17]:

$$\mathbf{M}(\partial_1,\partial_2,\partial_z)\begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix} = \mathbf{M}(\partial_1,\partial_2,\partial_z)\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} + \mathbf{M}(n_1,n_2,n_z)\left[\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right]_F \delta_F(x_2,z) = \hat{\mathbf{j}}(x,z) + \mathbf{M}(n_1,n_2,n_z)\left[\begin{pmatrix}\mathbf{E}(x,z)\\ \mathbf{H}(x,z)\end{pmatrix}\right]_F \delta_F(x_2,z), \quad (26)$$

where

$$M(n_1,n_2,n_z) = \begin{pmatrix} 0 & -n_z & n_2 & V\mu\mu_0 n_z & 0 & 0\\ n_z & 0 & -n_1 & 0 & V\mu\mu_0 n_z & 0\\ -n_2 & n_1 & 0 & 0 & 0 & V\mu\mu_0 n_z\\ -V\varepsilon\varepsilon_0 n_z & 0 & 0 & 0 & -n_z & n_2\\ 0 & -V\varepsilon\varepsilon_0 n_z & 0 & n_z & 0 & -n_1\\ 0 & 0 & -V\varepsilon\varepsilon_0 n_z & -n_2 & n_1 & 0 \end{pmatrix}.$$

Here, the cap on top denotes a generalized vector function. Using the property of differentiating discontinuous regular functions in $D'(R^3)$, we obtain the right part (26) which has a simple layer on the surface $F$ which is the front of the shock EM wave:

$$\mathbf{M}(n_1, n_2, n_z)\left[\begin{pmatrix}\mathbf{E}(x,z)\\ \mathbf{H}(x,z)\end{pmatrix}\right]_F \delta_F(x_2, z),$$

the density of which is determined by a surge of the electric and magnetic field strength vectors on $F$. For $\hat{\mathbf{E}}(x,z), \hat{\mathbf{H}}(x,z)$ to be a generalized solution of ME (4), its density should be zero:

$$\mathbf{M}(n_1, n_2, n_z)\left[\begin{pmatrix}\mathbf{E}(x,z)\\ \mathbf{H}(x,z)\end{pmatrix}\right]_F = \mathbf{0}. \tag{27}$$

This leads to conditions for surges of $\mathbf{E}(x,z), \mathbf{H}(x,z)$ components at the front of the shock EM wave:

$$-n_z[E_2] + n_2[E_3] + V\mu\mu_0 n_z[H_1] = 0,$$
$$-V\epsilon\epsilon_0 n_z[E_1] - n_z[H_2] + n_2[H_z] = 0,$$
$$n_z[E_1] - n_1[E_3] + V\mu\mu_0 n_z[H_2] = 0,$$
$$-V\epsilon\epsilon_0 n_z[E_2] + n_z[H_1] - n_1[H_z] = 0,$$
$$-n_2[E_1] + n_1[E_2] + V\mu\mu_0 n_z[H_3] = 0,$$
$$-V\epsilon\epsilon_0 n_z[E_3] - n_2[H_1] + n_1[H_2] = 0. \tag{28}$$

It is convenient to represent the equations (28) in vector form:

$$V\mu\mu_0[\mathbf{H}(x,z)] = [[\mathbf{E}]_F, \mathbf{n}(x,z)],$$
$$V\epsilon\epsilon_0[\mathbf{E}(x,z)]_F = [[\mathbf{H}]_F, \mathbf{n}(x,z)], \tag{29}$$

where the vector products of the surge of the intensity vector at the wave front to the normal to the front are on the right in the equations . It follows that the electric field and magnetic field strength jumps are orthogonal to each other and orthogonal to the normal to the wave front. If the medium is undisturbed in front of the wave front, then it follows from (29) that:

$$V\mu\mu_0\,\mathbf{H}(x,z)|_F = [\mathbf{E}_F, \mathbf{n}(x,z)],$$
$$V\epsilon\epsilon_0\,\mathbf{E}(x,z)|_F = [\mathbf{H}_F, \mathbf{n}(x,z)]. \tag{30}$$

Here $\mathbf{E}(x,z)|_F = \mathbf{E}_F$, $\mathbf{H}_F = \mathbf{H}(x,z)|_F$ — the value of the stresses at the front of the shock EM wave.

As follows from these relations, EM shock waves are transverse and the vectors of electric and magnetic intensity at the front of the shock wave are orthogonal to each other and lie in a tangent bundle to it.

This fact is well known for phase surfaces of electric and magnetic intensity of EM waves [1–7]. We showed this in Ref. [20] for ME discontinuous solutions that describe shock EM waves. Here this property of EM waves is proved for supersonic transport solutions of ME equations.

## 6. The Green tensor at the light velocity of the radiation source

For $V = c$, $M = 1$. In formulas (9) $m = 0$. The Fourier transform of the basis function

$$f(k_1, k_2, k_3) = \frac{1}{k_1^2 + k_2^2} \tag{31}$$

is the Fourier transform of the fundamental solution of the Laplace equation:

$$\frac{\partial^2 f_0}{\partial x_1^2} + \frac{\partial^2 f_0}{\partial x_2^2} = \delta(x)\delta(z). \tag{32}$$

Its solution has the form

$$f_0(x_1, x_2, z) = -\frac{\ln r}{2\pi}\delta(z), \tag{33}$$

$$f_1(x_1, x_2, z) = f_0(x_1, x_2, z)_z * H(z) = -\frac{1}{2\pi}H(z)\ln r. \tag{34}$$

It should be noted that a half-space is the support of these functions: $z > 0$ — outside of which $\mathbf{U}(x_1, x_2, z) = 0$. That is, the plane $z = 0$ is the front of an electromagnetic shock wave on which the components of the Green tensor are discontinuous. The solution of the ME will have a similar form (20), (23) and (24), only the functions should be taken as basic functions (33), (34). The conditions at the front of the emitter shock wave have the form (29), where $V = \frac{1}{\sqrt{\mu\mu_0\varepsilon\varepsilon_0}}$.

## Conclusion

ME transport solutions are built in Ref. [15] and in this paper in the entire velocity range, from sublight to superlight velocities, which allow calculating EM fields from emitters of arbitrary shapes, which can be modeled using both regular and singular functions, both in the absence of magnetic currents and with their presence. The question arises when and where to use superlight transport solutions of ME.

It is known that charged particles that move in a liquid medium at a velocity higher than the velocity of light in this medium cause a cone-shaped glow, which is called *Vavilov–Cherenkov radiation*, or simply *Cherenkov radiation* [8,21–24]. Cherenkov's experiments clearly demonstrate the presence of such shock waves [21]. The Cherenkov cone is the front of the shock wave, which is the envelope of the Mach cones at its front.

This phenomenon was mathematically described using harmonic waves the phase velocity of which exceeded the light velocity in the considered EM medium (see [8]). The solutions built here make it possible to describe this effect for any superlight emitters, and the presence of Cherenkov radiation already suggests that the solutions obtained in this paper can be used to study EM fields in a variety of media, and not only in liquid, but also in bodies and tissues under laser and other types of irradiation.

Cherenkov radiation is used in the nuclear industry [24], where the research presented here can be very useful for use.

It should also be noted that the obtained solutions can be used to solve diffraction boundary value problems in EM media limited by cylindrical surfaces and shells. Such a class of subsonic and supersonic transport boundary value problems for an isotropic elastic medium was considered and published by us earlier in Ref. [25–27]. We assume that a similar class of transport boundary value problems in cylindrical domains should be considered for EM media. We already carry out these studies within the framework of the specified grant project.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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