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The distribution function of the electrical strength of a dielectric layer with randomly located air inclusions

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Received February 24, 2024
Revised May 6, 2024
Accepted May 28, 2024

A layer of a solid-state polymer dielectric with inclusions randomly located in it with low density, when they have random sizes, is under consideration. The electrical strength of the inclusions is less than the electrical strength of the material. If layer thickness is sufficiently small due to random distribution of inclusions, the electrical strength is heterogeneous along the surface of the sample and takes random values in its various parts. Based on the previously constructed macroscopic statistical model of the electrical breakdown development in the described physical situation, the presence of an additional maximum of the distribution density of the specified random variable is established.

Keywords: dielectric, electrical breakdown voltage, inclusion density, unimodality, electrical strength

DOI: 10.61011/TP.2024.10.59350.48-24

Introduction

Statistical data on the electrical strength of multilayer polymer films with defects in the form of air bubbles, which form in the process of film synthesis, are available in literature [1]. Air inclusions are distributed randomly in the film matrix (on average, uniformly along its plane) and have random sizes. The shape of such inclusions is hemispherical, which, again, is attributable to the specifics of the process of synthesis by application of each successive thin ($\propto 1-2$ mm) layer in the form of a polymer solution. Owing to the presence of randomly distributed air inclusions with their electrical strength being lower than the one of the polymer film material, the electrical breakdown voltage varies from one region of the film surface to the other (i.e., has a nonzero statistical dispersion) if the density of such inclusions is low. The electrical strength of the film may then be considered as a random variable, and the corresponding statistical data are presented in the form of histograms. The histograms presented in [1] indicate the presence of a peculiar effect in the form of non-unimodality of the distribution of the electrical strength. At first glance, this effect seems rather strange, since the emergence of several maxima in the probability distribution should be induced by a certain physical mechanism. It was noted in [2] that, at the level of general concepts of probability theory, this effect is anything but unusual, since the distribution density of random electrical strength may be represented as a probability distribution of the sums of a small number of independent, random, and identically distributed quantities: radii $\tilde{r}$ of air inclusions. Here an elsewhere, quantities marked with a tilde are assumed to be random. These sums are represented as terms in a sequence of independent trials, the state space of which is the set of their random values. However, the analysis of a specific theoretical model with a special type of probability distribution of random variables $\tilde{r}$ in [3–5] revealed that, owing to the peculiarities of available statistical data on the electrical strength of multilayer films, such a conclusion requires additional justification (specifically, physical justification) obtained as a result of examination of various physically admissible unimodal probability distributions of random inclusion sizes.

The analysis carried out in the present study demonstrates that the effect of non-unimodality of the distribution of electrical strength is, generally speaking, not necessarily associated with the multi-layer nature of a dielectric film if, owing to the application of a special technology for layer deposition, a maximum of one inclusion may appear in each layer. A similar effect may occur in any sufficiently thin dielectric film. It is assumed in the analyzed model that all inclusions have a spherical shape due to the isotropy of pressure in the air inclusion bubble. For the same physical reason, the probability distribution of random bubble radii $\tilde{r}$ is such that very small values are unlikely. At the same time, the proposed theoretical model assumes that the probability distribution for the $\tilde{r}$ values is not localized around a certain average bubble size; instead, it is spread within a certain $[0, r_*]$ interval with a dispersion comparable to this average value. This assumption enables the calculation of the distribution density of electrical strength in the asymptotic limit (where it is almost uniform). We have demonstrated that a peak additional to the one found at zero inclusions emerges in this case in the distribution of electrical strength.

It is also significant that, owing to the low density of inclusions, electrical breakdown in our model is assumed to be generated, in most cases, by a single electron avalanche originating at the terminal that supplies electrical voltage to the dielectric layer. This holds true if the average distance between inclusions is much greater than the typical size of terminals. Naturally, electron avalanches may form as a result of fluctuations of electrical strength of the material at any depth within it, but we are only interested in the scenario with an electrical breakdown all the way through the film.

1. Phenomenological representation of electrical breakdown of a dielectric layer

Our study relies on a simple statistical model that was constructed based on the most general qualitative physical macroscopic concepts regarding the phenomenology of electrical breakdown of a dielectric layer and foregoes microscopic-level analysis involving complex derivations within physical kinetics. The model builds on a simple physical notion that the deviation of the electrical strength of a film from its value in a defect-free material is, on average, proportional to the fraction of the film thickness that contains air inclusions in its cross section. It is this concept that was used in [1,3–5].

Electrical breakdown is caused by impact ionization, which is accompanied by the breaking of bonds between dielectric atoms under the direct influence of an electric field. Electrical strength u_{st} of solid dielectrics against electrical breakdown is the ratio of breakdown voltage U of the layer to its thickness d in the direction of the applied voltage. The values of electrical strength of polymer materials fall within the $(200-400)\cdot 10^3$ V/cm range (polyethylene, polystyrene, etc.) [6]. Regarding the properties of dielectrics in strong electric fields, see [7,8]. If air inclusions of random sizes are distributed randomly in a dielectric material, the breakdown voltage is, generally speaking, a random variable $(\tilde{U})$ if the site of its application to the opposite planes of the layer is fixed. Therefore, the electrical strength measured at this site is also a random variable. This is attributable to the fact that the electrical strength of air is significantly lower than the electrical strength of the dielectric material. Randomness of the specified type is insignificant if the statistical distribution of $\tilde{U}$ has a small dispersion due to the smallness of ratio $\tilde{r}/d$. In the contrary case when the dispersion of this distribution differs noticeably from zero, the histograms of random variable $\tilde{U}$ should represent the statistical spread of breakdown voltages and, consequently, electrical strength.

The number of air inclusions through which the breakdown electron avalanche propagates is random. With a low density of inclusions, this number is not very high. Therefore, it may be assumed that, as was rightly noted in [2], the statistical distribution of electrical strength should have several maxima if the probability distribution of random bubble sizes is localized near their fixed average value. The number of such maxima is equal in order of magnitude to the average number of air inclusions through which the electrical breakdown propagates. An analysis of the mathematical model proposed below reveals that if this distribution is not localized, the distribution of electrical strength is indeed non-unimodal at an arbitrarily low density of inclusions: a peak additional to the one corresponding to the case of zero inclusions is always present.

One consequence in the proposed theory is that such an effect is not necessarily associated with the multilayer nature of the polymer material examined in experiments mentioned in the cited review. The effect of non-unimodality of the distribution of electrical strength may manifest itself in dielectric films containing air inclusions with a unimodal distribution of their sizes with a large mean-square deviation σ comparable to their average value r_0: $\sigma \propto r_0$. We also assume that the density of air inclusions is so low that size l of the terminals through which voltage is applied to the planes of the layer is much smaller than average distance l_0 between the air inclusions, $l \ll l_0$. This makes it fair to assume that the electrical breakdown of the layer is caused by a single electron avalanche originating from the surface of the terminal with electrical voltage applied to it.

2. Construction of the theoretical model

Let us consider a dielectric layer with thickness d. The electrical strength of the material is denoted as u so that ud is the corresponding electrical breakdown voltage. Let air inclusions be distributed randomly and, on average, uniformly within the layer volume. We assume that they have a spherical shape, which is due to the influence of surface tension of an air bubble inside the layer of a polymer material that was in the process of solidifying from a liquid state. The geometric state of each individual random inclusion is then characterized completely by a single positive random variable $\tilde{r}$ (bubble radius). To construct a probabilistic model, we introduce set $\{\tilde{\mathbf{x}}_k;\, k = 1-N\}$ of centers of spherical air inclusions, where $N \gg 1$ is the number of inclusions in a certain finite part of the layer. Points $\tilde{\mathbf{x}}_k;\, k = 1-N$ are random three-dimensional vectors. In mathematical terms, set $\{\tilde{\mathbf{x}}_k;\, k = 1-N\}$ is a realization of a point random field (see, e.g., [9]). It is also assumed that average density $\rho \propto l_0^{-3}$ of inclusions is low (i.e., average distance l_0 between them is much greater than their average radius r_0). This smallness of the average density makes it fair to assume that the inclusions do not exert any influence on each other (i.e., are statistically independent). Owing to the on-average uniformity of the distribution of inclusions inside the layer, $\{\tilde{\mathbf{x}}_k;\, k = 1-N\}$ may be regarded as a uniform Poisson random field with density ρ within the dielectric volume. Random points $\tilde{\mathbf{x}}_k$, $k = 1-N$ are in a one-to-one relation with random variables $\tilde{r}_k$, which are the radii of inclusions with centers at the corresponding points $\tilde{\mathbf{x}}_k$,

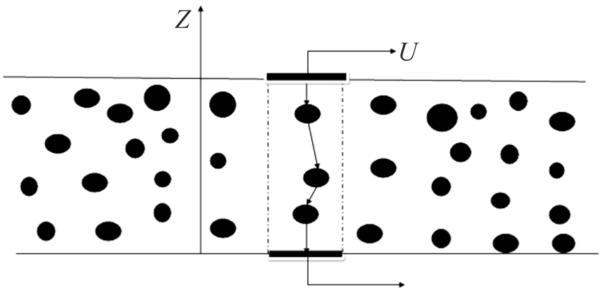

Figure 1. Dark spots in this schematic diagram represent air inclusions. A chain of $M = 3$ inclusions is highlighted, and arrows indicate the propagation of the breakdown electron avalanche.

$k = 1, 2, 3, \ldots, N$. They form a set of identically distributed positive continuous random variables. Let us denote their aggregate distribution density as $h(r)$. Moreover, since the centers of inclusions are statistically independent in aggregate, it may be assumed that all random variables $\tilde{r}_k$, $k = 1, 2, 3, \ldots$ are also statistically independent in aggregate.

Let electrical voltage U be applied to the planes of the layer. When the value of U exceeds a certain (fairly high and, generally speaking, random) level $\tilde{U}$, an electrical breakdown occurs. It proceeds by means of an electron avalanche between the planes of the layer surface, which leads to degradation of the material. Since the electrical strength of air is significantly lower than the electrical strength of the dielectric material, it is natural to assume that the electron avalanche tends to choose the path of least resistance through certain air inclusions within the layer volume. It is assumed in the present study that the density of inclusions is so low that electrical breakdown is induced by just a single avalanche; in other words, it is unlikely that two or more avalanches propagate simultaneously between the terminals supplying electrical voltage.

The propagation of this electron avalanche is represented by a broken line with straight segments connecting a series of centers of inclusions. Start $\tilde{\mathbf{x}}_0$ and end $\tilde{\mathbf{x}}_{M+1}$ points of this broken line, which are located on the layer surface, should be covered by the terminals supplying electrical voltage to the polymer layer. To simplify further analysis, we assume that these terminals have a square shape with side length l. The centers of these squares are positioned opposite to each other on opposite planes so that these squares may be combined by parallel translation in the direction perpendicular to the layer.

Let u_0 be the electrical strength of air. When the electron avalanche propagates along the broken line segment (Fig. 1) drawn from inclusion center $\mathbf{x}_{k_m}$, which is the mth to be passed in the course of breakdown, to center $\mathbf{x}_{k_{m+1}}$ of the next $(m + 1)$ inclusion, the breakdown voltage is the sum of breakdown voltage of the mth air bubble, which is equal to $2u_0\tilde{r}_{k_m}$, and the breakdown voltage of the polymer material along this broken line segment; i.e., it is written as

$$[2u_0\tilde{r}_{k_m} + u(|\tilde{\mathbf{x}}^{(m)}_{k_m} - \tilde{\mathbf{x}}^{(m+1)}_{k_{m+1}}| - 2\tilde{r}_m)].$$

The electrical breakdown voltage in the case of propagation of the breakdown avalanche along the considered randomly chosen broken line is then equal to

$$\tilde{U} = \sum_{m=0}^{M} \Big[2u_0\tilde{r}_m + u(|\tilde{\mathbf{x}}^{(m)}_{k_m} - \tilde{\mathbf{x}}^{(m+1)}_{k_{m+1}}| - 2\tilde{r}_m)\Big]. \qquad (1)$$

Here, $k_1, \ldots, k_M$ are the numbers of points of a random field and $\tilde{\mathbf{x}}_{k_0}$, $\tilde{\mathbf{x}}_{k_{M+1}}$ are points on the exterior planes.

Note that formula (1) is not related in any way to the kinetics of generation of an electron avalanche at the microscopic level. It provides a general physical description of the trajectory of propagation of an already formed avalanche that penetrates through the entire film. This is exactly the kind of propagation that leads to subsequent degradation of the film material, making it ill-suited for the role of an electrical insulating element. Thus, formula 1) does not provide a comprehensive description of the phenomenon of electrical breakdown; it serves just to solve the problem that is analyzed in the present study. A microscopic theory of electrical breakdown is needed to obtain a detailed description of the process of avalanche initiation and its evolution through the liberation of electrons from the outer shells of atoms of the material and their consequent ionization. In our view, such a theory will provide nothing fundamentally new in the context of characterizing the statistics of trajectories of the breakdown electron avalanche.

It is assumed below that $l \ll l_0$. The size of the terminals may be neglected in this case, and it may be assumed that points $\mathbf{x}_1$ and $\mathbf{x}_{M+1}$ are positioned precisely at the centers of the terminals in a way that makes them non-random. Thus, electrical breakdown voltage $\tilde{U}$, the randomness of which is associated with the random arrangement of inclusions and their random sizes, is given by formula (1). It is obvious that an avalanche must evolve in such a way that the centers of inclusions through which it propagates do not deviate greatly from each other in the direction transverse to the axis vertical with respect to the surface planes. This implies that the avalanche trajectory is almost straight. Average radius r_0 of an inclusion is mush smaller than average distance l_0 between the inclusions, which, in turn, is much smaller than layer thickness d; therefore, $r_0 + l_0 \ll d$. The difference between the length of each segment of a broken line and its projection onto the mentioned vertical axis may be neglected in this case.

Let us introduce probability density function $f(U)$ of random variable (1) at fixed density ρ of positioning of points of a random field $\{\tilde{\mathbf{x}}_k; k = 1{-}N\}$ and distribution density $h(r)$ under the condition that average inclusion radius $r_0 = \int\limits_0^\infty rh(r)dr$ is much smaller than l_0. With these assumptions, the sum in (1) may be presented as $u_0\tilde{s} + u(d-\tilde{s})$, where random distance

$$\tilde{s} = \sum_{m=1}^{M} \tilde{r}_m$$

is the total length of all M air inclusions through which the electron avalanche propagates. In terms of random variable $\tilde{s}$, formula (1) is written as $\tilde{U} = ud - 2(u-u_0)\tilde{s}$.

Let us denote the doubled difference between the electrical strengths of the dielectric material and air as $\nu = 2(u-u_0)$. The basic formula specifying the random value of electrical breakdown voltage $\tilde{U}$ within the considered probabilistic model then takes the form (cf. [2–4])

$$\tilde{U} = ud - \nu\tilde{s},$$

where random variable $\tilde{s}$ is the sum of independent random variables distributed identically with density $h(r)$.

Let us assume that an electrical breakdown occurs when random variable $\tilde{U}$ exceeds electrical voltage U applied at a specific spot on the dielectric layer surface. Let us set $s = (ud-U)/\nu$. Since inequality $U > ud - \nu\tilde{s}$ is equivalent to $s < \tilde{s}$, probability distribution function $F(U) = \Pr\{\tilde{U} < U\}$ of occurrence of an electrical breakdown is given by

$$F(U) = \Pr\{\tilde{U} < U\} = 1 - \Pr\{\tilde{s} < s\},$$

due to the continuity of the probability distribution of random variable $\tilde{s}$ at $s > 0$ or, what is the same, at $U < ud$. Let us introduce distribution density

$$g(s) = \frac{d}{ds}\Pr\{\tilde{s} < s\}.$$

Therefore, density $f(U)$ of the distribution of random variable $\tilde{U}$ is

$$f(U) = \frac{d}{dU}\Pr\{\tilde{U} < U\} = \nu^{-1} g(s)\big|_{s=(ud-U)/\nu},$$

since $ds/dU = -\nu^{-1}$.

Probability $\Pr\{\tilde{s} < s\}$ is the sum of probabilities of random events $\{\tilde{s} < s,\, \tilde{m} = m\}$ over all possible values $m = 0, 1, 2, \ldots$ of random variable $\tilde{m} < M$, where $\tilde{m}$ is the random number of air inclusions through which the electron avalanche propagates when an electrical breakdown occurs. Number M cannot exceed a level inherently lower than $d/2r_0$, since $d \gg l_0 + r_0$. A stronger statement may also be made. Owing to the smallness of density ρ, maximum value M of random variable $\tilde{m}$ cannot be large in the event of breakdown. Since the inclusions are distributed independently of each other, the propagation of an avalanche through different inclusions should also be considered as a sequence of random events independent in aggregate. Therefore, probability distribution $\Pr\{\tilde{m} = m\}$ of the random number of inclusions through which an avalanche propagates at a fixed value of M is determined by a sequence of independent trials (see, e.g., [9])

$$\Pr\{\tilde{m} = m\} = \binom{M}{m} \upsilon^m (1-\upsilon)^{M-m}, \qquad (2)$$

where „success“ probability $\upsilon > 0$ is proportional to $\rho^{1/3}$. Thus, if $M\upsilon$ is very small, then, according to the Poisson distribution, probability $\lambda^M \exp(-\lambda)/M!$, $\lambda = M\upsilon$ of occurrence of large values m of random variable $\tilde{m}$ is very low. This is the reason why one may allow this random variable to vary from zero to infinity and may replace its probability distribution (2) with the Poisson distribution

$$\Pr\{\tilde{m} = m\} = \frac{\lambda^m}{m!}\exp(-\lambda). \qquad (3)$$

Density $g(s)$ should then be assumed independent of M.

Since $\{\tilde{s} < s, \tilde{m} = m\}$ is a product of random events $\{\tilde{s} < s\}$ and $\{\tilde{m} = m\}$, it follows with (3) taken into account that

$$\Pr\{\tilde{s} < s,\ \tilde{m} = m\} = \frac{\lambda^m}{m!}\exp(-\lambda)\Pr\{\tilde{s} < s | \tilde{m} = m\}.$$

At $m = 0$, conditional probability $\Pr\{\tilde{s} < s | m = 0\}$ is the same as Heaviside step function $\theta(s)$, since $\tilde{s} = 0$ in this case. Therefore, the formula of overall probability [9] yields the following expression:

$$\Pr\{\tilde{s} < s\} = \theta(s)\exp(-\lambda) + \sum_{m=1}^{\infty}\Pr\{\tilde{s} < s, \tilde{m} = m\}. \quad (4)$$

Thus, the distribution function of random variable $\tilde{s}$ is determined completely by the probability distribution of random variable $\tilde{s}$. At a fixed value of m, the latter is the probability distribution of a sum of m independent random variables distributed identically in accordance with density $h(r)$. Therefore (see, e.g., [9]), it is equal to

$$\Pr\{\tilde{s} \le s | \tilde{m} = m \ne 0\} = \int_{-0}^{s} \underbrace{(h * \ldots * h)}_{m}(r)dr \equiv \int_{-0}^{s} (h_*^m)(r)dr, \qquad (5)$$

where

$$h(r) = \frac{d}{dr}\Pr\{\tilde{r} < r\}, \qquad \int_0^d h(r)dr = 1$$

and symbol $*$ denotes the binary operation of convolution of probability distribution densities [9], so that the reapplication of this operation to density $h(r)$ is represented as

$$(h_*^{m+1})(r) = \int_{0+}^{r} (h_*^m)(r')h(r-r')dr'.$$

According to (4), (5), the $g(s)$ distribution density is written as

$$g(s) = \exp(-\lambda)\left[\delta(s) + \sum_{l=1}^{\infty}\frac{\lambda^l}{l!}h_*^l(s)\right], \qquad (6)$$

where it was taken into account that $d\theta(s)/ds = \delta(s)$ is the Dirac function.

The δ-function singularity in the $g(s)$ distribution density at point $s = 0$ translates into a δ-function singularity in the $f(U)$ distribution density at point $U = ud$. The corresponding maximum, which is proportional to $\exp(-\lambda)$, is distinguishable in the histograms of experimental data only at small values of λ (i.e., at very low values of inclusion density ρ).

3. Distribution density of the electrical strength

In practice, the form of density $h(r)$ and the value of parameter λ are unknown. Therefore, mathematical analysis of the model described above requires choosing a certain class of densities $h(r)$, comparing the obtained predictions with statistical data, and selecting a specific density model $h(r)$ and parameter λ that provide the best fit to experimental data.

In the present study, we analyze the behavior of density $f(U)$ in the case of model densities $h(r)$ that are unimodal with a nonzero maximum $r_{\max}$ at $r_* > r > 0$, but are not localized in the vicinity of this maximum with a small dispersion; on the contrary, the mean-square deviation corresponding to this density is comparable in order of magnitude to r_0. It is assumed that $r_* \ll d$.

Let us introduce dimensionless variable $x = r/r_*$ and set $h(r) = w(x, \eta)/r_*$, where each density $w(x, \eta)$ is concentrated on [0, 1]. Parameter $\eta > 0$ here characterizes the delocalization of density on the definition interval. Let one-parameter family of densities $w(x, \eta)$ be such that $w(x, \eta) = w(x) + 0(\eta)$ at $\eta \to +0$, where $w(x) = \theta(x)\theta(1 - x)$. An example of such a family is

$$w(x, \eta) = Z_\eta\big[1 - \exp\big(-v(x)/\eta\big)\big],$$

where function $v(x) > 0$ is defined on (0, 1), has one minimum, and $\lim\limits_{x\to+0} v(x) = \lim\limits_{x\to1-} v(x) = \infty$; $Z_\eta > 0$ is the normalization constant.

In the present study, we analyze the behavior of the probability distribution of electrical strength in the case suitable for any one-parameter family of densities $w(x, \eta)$ of the specified type in the $\eta \to +0$ limit. As was already noted, inequality $l_0 \ll d$ is satisfied in order of magnitude; therefore, $\lambda \ll 1$.

Let us write formula (6) in the leading approximation at $\eta \to 0$ in terms of dimensionless variable x:

$$g(s) = \frac{1}{r_*}\exp(-\lambda)\left[\delta(x) + \sum_{l=1}^{\infty}\frac{\lambda^l}{l!}w_*^l(x)\right].$$

It follows from Theorem 3, which is proven in the Appendix, that functions $w_*^m(x)$ are localized on [0, m], respectively. They are continuous and differentiable at $m \geq 2$ (see the Appendix, formula (A4)). The $w(x)$ function is evidently discontinuous and localized on [0, 1]. It is also proven in the Appendix that when parameter λ satisfies inequality

$$\sum_{l=2}^{\infty}\frac{\lambda^{l-1}}{(l!)^2} < 1, \tag{7}$$

which is fulfilled at small values of λ, the function represented by the sum in square brackets has, at $x > 1$, a single maximum at point $x_* = 1$, which is accompanied by a jump in density at this point to the right of the maximum. The discontinuity of this type is purely model in nature. It is manifested only in the $\eta \to +0$ limit so that density $g(s)$ is continuous and has a maximum at point $x_* = 1$ at finite values of parameter $\eta > 0$ for the considered class of densities $w(x; \eta)$. Thus, with the specified relations between the physical parameters of the model in the analyzed case where the unimodal model $h(r)$ distribution density is „smeared" over segment $[0, r_*]$ of its definition, density $g(s)$ always has two peaks, which coincide with $s = 0$ and $s = r_*$. Therefore, distribution density $f(U)$ of the electrical breakdown voltage of the dielectric layer also features two peaks under such conditions. One of them matches breakdown voltage ud of the defect-free material, and the second one corresponds to $(ud - r_*\nu) = u(d - r_*) + r_*u_0$. An additional peak emerges in the $g(s)$ distribution density (Fig. 2) at λ values violating condition (7).

Conclusion

A statistical model of evolution of an electrical breakdown avalanche in a polymer material layer with randomly distributed air inclusions, which have mesoscopic random sizes, was constructed on the basis of general physical concepts and analyzed. It was demonstrated that exactly two peaks should form in the experimental histograms of electrical strength under the conditions of smallness of average size r_0 of an air defect in the dielectric material layer in comparison with its thickness d and smallness of the defect density. One of these peaks coincides with the electrical breakdown voltage of the defect-free material, and the second one matches electrical voltage $u(d - r_*) + r_*u_0$, where u and u_0 are the electrical strengths of the material and air, respectively, and r_* is the characteristic size of air inclusions. This should be observed with physical characteristics of the dielectric varying within fairly wide ranges.

It would be instructive to clarify the physical cause of violation of the unimodality of density $f(U)$ at $U < ud$ in future studies by examining its behavior within the proposed model with the considered type of model distribution densities $h(r)$ of random inclusion sizes in the case when density ρ of the distribution of air defects in the material is not very low. In addition, it is important to investigate the model of electrical breakdown development with another class of unimodal $h(r)$ densities considered in [3].

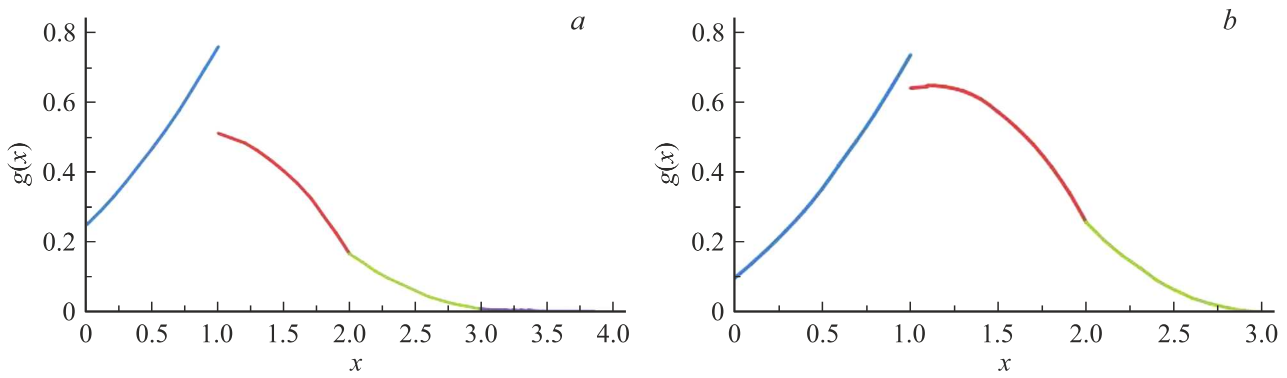

Figure 2. Plots of density $g(x)$: *a*) — satisfying condition (7), $\lambda = 1$; *b*) — violating this condition, $\lambda = 3$. The jump at $x = 1$ is due to the fact that calculations were performed in limit case $\eta \to \infty$.

Conflict of interest

The authors declare that they have no conflict of interest.

Appendix

The operation of convolution of two distribution densities $f_1(x)$ and $f_2(x)$ concentrated on $[0, \infty)$ is given by

$$(f_1 * f_2)(x) = \int_{-0}^{\infty} f_1(y) f_2(x - y) dy. \qquad \text{(A1)}$$

It may be viewed as commutative multiplication on the set of all such densities. From a mathematical point of view, a set of densities fitted with such an operation is a semigroup with a unit, which is density $\delta(x)$. With this definition of multiplication, the corresponding mth degree $(f_*^m)(x)$ of arbitrary density $f(x)$ on $[0, \infty)$ is specified by recurrence relation

$$(f_*^m)(x) = \int_{-0}^{\infty} (f_*^{m-1})(y) f(x - y) dy.$$

Let us examine the qualitative behavior of degrees $w_*^m(x)$ of model distribution density $w(x) = \theta(x)\theta(1 - x)$. First of all, let us prove the following theorem.

Theorem 1. *The following formula is valid for densities* $w_*^m(x)$*:*

$$w_*^m(x) = w_*^m(m - x). \qquad \text{(A2)}$$

Let us define

$$\bar{u}_j(k) = \int_{-\infty}^{\infty} u_j(x) e^{ikx} dx, \quad j \in \{1, 2\}$$

Fourier transforms of densities $u_1(x)$ and $u_2(x)$. According to (A1),

$$(\overline{u_1 * u_2})(k) = \int_{-\infty}^{\infty} e^{ikx} (u_1 * u_2)(x) dx = \int_{-\infty}^{\infty} e^{ikx}$$

$$\times \left(\int_{-\infty}^{\infty} u_1(y) u_2(x - y) dy \right) dx = \int_{-\infty}^{\infty} e^{iky} u_1(y)$$

$$\times \left(\int_{-\infty}^{\infty} e^{ik(x-y)} u_2(x - y) dx \right) dy = \bar{u}_1(k) \bar{u}_2(k).$$

Therefore, the Fourier transform of the mth degree of density $u(x)$

$$(\overline{u_*^m})(k) = \int_{-\infty}^{\infty} (u_*^m)(x) e^{ikx} dx, \quad m = 1, 2, 3, \ldots$$

satisfies relation $(\overline{u_*^m})(k) = (\overline{u_*^{m-1}})(k)\bar{u}(k)$ and, consequently,

$$(\overline{u_*^m})(k) = \bar{u}^m(k). \qquad \text{(A3)}$$

Equality $w(x) = w(1 - x)$ holds true for density $w(x)$. Therefore, the following equalities are valid for its Fourier transform:

$$\bar{w}(k) = \int_{-\infty}^{\infty} e^{ikx} w(x) dx = \int_{-\infty}^{\infty} e^{ikx} w(1 - x) dx$$

$$= e^{ik} \int_{-\infty}^{\infty} e^{ik(x-1)} w(1 - x) dx = e^{ik} \bar{w}(-k),$$

i.e.,

$$\bar{w}(k) = e^{ik} \bar{w}(-k).$$

It follows from this equality and (A3) that

$$\bar{w}^m(k) = e^{ikm} \bar{w}^m(-k).$$

Applying the inverse Fourier transform to both sides of the last equality, we find

$$w_*^m(x) = \frac{1}{2\pi}\int e^{-ikx}\bar{w}^m(k)dk = \frac{1}{2\pi}\int e^{-ik(x-m)} \times \bar{w}^m(-k)dk = w_*^m(m-x).$$

Corollary 1. *The maximum of function $w_*^m(x)$ is at point $x = m/2$.*

This follows from (A2).

The next statement is a refinement of the well-known Ibragimov theorem (see [10,11]) on the so-called strictly unimodal functions as applied to density $w(x)$.

Theorem 2. *If the maximum point of non-negative continuous function $u(x)$ on $[0, \infty)$ is unique and function $u(x)$ has no constancy intervals, the maximum point of function $(w * u)(x)$ is also unique and this function has no constancy intervals.*

Since function

$$(w * u)(x) = \int_{\max\{0,x-1\}}^{x} u(y)dy$$

is continuously differentiable, each of its extremum points x_*, which are characterized by the vanishing of its derivative, may only be found at $x > 1$ and should satisfy equation

$$\frac{d}{dx}(w * u)(x) = u(x) - u(x-1) = 0. \tag{A4}$$

Let us assume the contrary: function $(w * u)(x)$ has two maximum points x_j, $j \in \{1, 2\}$ $(x_1 < x_2)$ on $(0, \infty)$. They are the solutions of Eq. (A4). Since the maximum point of function x_* is unique, $x_j \geq x_*$, $j \in \{1, 2\}$. Equality is impossible here, since it would imply that point x_* is not unique. It is also evident that $x_j - 1 < x_*$, $j \in \{1, 2\}$. If $u(x_2) = u(x_1)$, function $u(x)$ has constancy interval (x_1, x_2), which contradicts the hypothesis of theorem. Thus, the only possibility is that $u(x_2) < u(x_1)$. Following the same line of reasoning, we find that $u(x_2 - 1) > u(x_1 - 1)$. Subtracting equalities $u(x_j) = u(x_j - 1)$ from each other, we arrive at a contradiction: $u(x_2) - u(x_1) < 0$ and, at the same time, $u(x_2) - u(x_1) = u(x_2 - 1) - u(x_1 - 1) > 0$.

Corollary 2. *Functions $w_*^m(x)$ at $m \geq 2$ have a unique maximum point.*

Function $w_*^2(x)$ explicitly has a unique maximum point at $x = 1$. The general statement is derived from Theorem 2 via induction on $m > 2$.

Sequential calculation of the degrees of the $w(x)$ density convolution operation is performed in accordance with formula

$$w_*^{m+1}(x) = \int_0^x w(x-y)w_*^m(y)dy = \int_0^x \theta(x-y) \times \theta(1-x+y)w_*^m(y)dy. \tag{A5}$$

It is evident that $w_*^m(x) = \theta(x)w_*^m(x)$; moreover, the following may be proven based on formula (A5) via induction on m.

Theorem 3. *Each density $w_*^m(x)$ is concentrated on $[0, m]$, $m = 1, 2, 3, \ldots$; i.e., formula $w_*^m(x) = \theta(m - x)w_*^m(x)$ is valid.*

Let us substitute, in accordance with the induction hypothesis, density $w_*^m(y)$ in the integration term with $w_*^m(y)\theta(m - y)$. At $x>m+1$ and $y<m$, $1+y>x$, $1 + m > 1 + y > x > m + 1$ should hold true, but this is impossible; i.e., $\theta(m - y)\theta(1 - x + y)\theta(x - m - 1) = 0$. Therefore, the integral in (A5) is proportional to $\theta(m + 1 - x)$.

Taking (A4) and the fact that $w_*^{m+1}(x)$ is concentrated on $[0, m + 1]$ into account, we write the following expression for it on this interval:

$$w_*^{m+1}(x) = \int_0^x \theta(x-y)\theta(1-x+y)w_*^m(y)dy = \int_0^x w_*^m(y)dy + \theta(x-1)\int_{x-1}^x w_*^m(y)dy. \tag{A6}$$

It follows from (A4) via induction on m that functions $w_*^m(x)$ are continuous from $m = 2$ onward and differentiable at $m > 2$.

Let us use formula (A6) to prove the following assertion.

Theorem 4. *The degrees of $w_*^m(x)$ may be presented as*

$$w_*^m(x) = \sum_{k=0}^{m-1}\theta(x-k)\theta(k+1-x)P_{m,k}(x), \tag{A7}$$

where polynomials $P_{m,k}(x)$, $k = 0, 1, \ldots, m - 1$, $m = 1, 2, 3, \ldots$ satisfy recurrence relations

$$P_{m+1,0}(x) = \int_0^x P_{m,0}(y)dy, \quad x \in [0, 1), \tag{A8}$$

$$P_{m+1,m}(x) = \int_{x-1}^{m} P_{m,m-1}(y)dy, \quad x \in [m, m+1), \tag{A9}$$

$$P_{m+1,k}(x) = \int_{x-1}^{k} P_{m,k-1}(y)dy + \int_k^x P_{m,k}(y)dy, \quad x \in [1, m), \quad k = 1 - m - 1. \tag{A10}$$

Expression (A7) is valid at $m = 1$ with $P_{1,0}(x) = 1$. Let us construct the induction step from m to $m + 1$. Inserting (A7) at $m + 1$ into the right-hand part of formula (A7) at $x \in [0, m + 1]$, we obtain equality

$$w_*^{m+1}(x) = \theta(1-x)\int_0^x P_{m,0}(y) + \theta(x-1)$$

$$\times \sum_{k=0}^{m-1}\int_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy, \qquad \text{(A11)}$$

where it is taken into account that only the term with polynomial $P_{m,0}(y)$ produces a nonzero contribution to the first integral. Let us present the last integral at $k = 0, 1, 2, \ldots$ in the form

$$\int_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy = \theta(x-k)\theta(k+2-x)$$

$$\times \int_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy,$$

since it is zero at $x - 1 > k + 1$ and $x < k$.

At $k < m$, $k < x - 1 < k + 1$ if $k + 1 < x < k + 2$. In this case,

$$\int_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy = \int_{x-1}^{k+1} P_{m,k}(y)dy;$$

if $k < x < k + 1$, $x - 1 < k$ and

$$\int_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy = \int_{k}^{x} P_{m,k}(y)dy.$$

Therefore,

$$\int_{x-1}^{x} \theta(y-k)\theta(k+1-y)P_{m,k}(y)dy = \theta(x-k-1)$$

$$\times \theta(k+2-x)\int_{x-1}^{k+1} P_{m,k}(y)dy + \theta_k\theta(x-k)\theta(k+1-x)$$

$$\times \int_{k}^{x} P_{m,k}(y)dy,$$

where $\theta_k = 1 - \delta_{k,0}$. Inserting the obtained expressions for integrals into (A11), we find that

$$w_*^{m+1}(x) = \theta(x)\theta(1-x)\int_0^x P_{m,0}(y) + \sum_{k=0}^{m-1}\Big[\theta_k\theta(x-k)$$

$$\times \theta(k+1-x)\int_k^x P_{m,k}(y)dy + \theta(x-k-1)\theta(k+2-x)$$

$$\times \int_{x-1}^{k+1} P_{m,k}(y)dy\Big] = \sum_{k=1}^{m-1}\theta(x-k)\theta(k+1-x)\Big[\int_k^x P_{m,k}(y)dy$$

$$+ \int_{x-1}^{k} P_{m,k-1}(y)dy\Big] + \theta(x)\theta(1-x)\int_0^x P_{m,0}(y)dy$$

$$+ \theta(x-m)\theta(m+1-x)\int_{x-1}^{m} P_{m,m-1}(y)dy.$$

Defining functions $P_{m+1,0}(x)$, $P_{m+1,m}(x)$, and $P_{m+1,k}$, $k = 1{-}m-1$, in accordance with (A8), (A9), and (A10), we obtain the sought-for representation of density $w_*^{m+1}(x)$,

$$w_*^{m+1}(x) = \sum_{k=0}^{m}\theta(x-k)\theta(k+1-x)P_{m+1,k}(x).$$

Corollary 3. *Polynomials* $P_{m+1,k}(x)$, $k = 0, 1, \ldots, m-1$ *satisfy equalities*

$$P_{m,k}(x) = P_{m,m-1-k}(m-x). \qquad \text{(A12)}$$

Inserting expansions (A7) for functions $w_*^m(x)$ and $w_*^m(m-x)$ into equality (A2), we find that the following must be satisfied:

$$\sum_{k=0}^{m-1}\theta(x-k)\theta(k+1-x)P_{m,k}(x) = \sum_{k=0}^{m-1}\theta(m-x-k)$$

$$\times \theta(k+1-m+x)P_{m,k}(m-x).$$

Substituting summation variable $m - 1 - k$ with k in the sum in the right-hand part of the equality, we arrive at equality

$$\sum_{k=0}^{m-1}\theta(x-k)\theta(k+1-x)P_{m,k}(x) = \sum_{k=0}^{m-1}\theta(x-k)$$

$$\times \theta(k+1-x)P_{m,-1-k}(m-x),$$

which demonstrates that (A12) is valid at $x \in [k, k+1]$.

The following assertion is an addition to Theorem 2 in the case when function $u(x)$ has a peak that is accompanied by a discontinuity of the first kind.

Lemma. *Let $u(x)$ be a left-continuous distribution density on $[0, \infty)$ with a unique maximum point $x_* = 1$ that is a point of discontinuity of the first kind. Then, if $u(x_* + 0) \le u(0)$, the $(w * u)(x)$ distribution density has a single maximum at point x_*.*

Density $(w * u)(x)$ has peak z. It necessarily satisfies inequality $z \ge x_*$. In the proof of Theorem 2, the continuity of function $u(x)$ was used only to obtain an equation for the maximum point of density $(w * u)(x)$. Therefore, if $z > 1$, $u(z) = u(z-1)$ should be true for this point. However, since $u(x_* + 0) \le u(0)$ and point x_* is the only peak of density $u(x)$, $u(z) < u(x_* + 0) < u(x_* - 1) < u(z-1)$ and the indicated equality is infeasible. The only possible location $x_* = 1$ then remains for the $(w * u)(x)$ density peak.

Let us now apply this statement to the analysis of function

$$W(x) = \sum_{l=1}^{\infty} \frac{\lambda^l}{l!} w_*^l(x),$$

which is, up to normalization, the distribution density and has a discontinuity of the first kind at point $x_* = 1$ generated by the term with $l = 1$.

Theorem 5. *If positive number λ is such that $I_0(2\sqrt{\lambda}) \le 1 + 2\lambda$, where I_0 is the modified zero-order Bessel function, function $W(x)$ is unimodal and its peak is located at point $x_* = 1$.*

It follows from the restriction on parameter λ in the formulation of the theorem that

$$\sum_{l=2}^{\infty} \frac{\lambda^l}{(l!)^2} \le \lambda.$$

In particular, $\lambda < 4$. Let us introduce functions

$$W_N^{(m)}(x) = \sum_{l=1}^{N} \frac{\lambda^{l+m}}{(l+m)!} w_*^l(x), \quad m = 0, 1, 2, \ldots .$$

Since density $w_*^2(x)$ concentrated on $[0, 2]$ is unimodal with peak at x_*, all functions $W_2^{(m)}(x) = \lambda^{m+1} w(x)/(m+1)! + +\lambda^{m+2} w_*^2(x)/(m+2)!$, $m = 0, 1, 2, \ldots$ are unimodal, and their peaks are located at $x_* = 1$. Each of these functions has a discontinuity of the first kind at this point.

Since $P_{l,1}(x_*) = \int_0^1 P_{l,0}(y)dy$ according to (A9) and $P_{l,0}(x) = x^{l-1}/(l-1)!$ according to (A7) at $P_{1,0}(x) = 1$, $P_{l,1}(1) = 1/l!$ and, consequently,

$$W(x_* + 0) = \sum_{l=2}^{\infty} \frac{\lambda^l}{l!} w_*^l(x_*) = \sum_{l=2}^{\infty} \frac{\lambda^l}{l!} P_{l,1}(1) = \sum_{l=2}^{\infty} \frac{\lambda^l}{(l!)^2}.$$

Under the hypothesis of Theorem 5, $\lambda \le 4$; therefore, $\lambda/2(m+2) \le 1$, $m = 0, 1, 2, \ldots$. Then, $W_2^{(m)}(x_* + 0) < W_2^{(m)}(0) = \lambda^{m+1}/(m+1)!$ and, according to the statement of Lemma, each function $(w * W_2^{(m)})(x)$, $m \in \mathbb{N}$ is continuous and unimodal with a peak at point $x_* = 1$. Therefore, since each of these functions increases within the $(0, 1)$ interval, each function $\lambda^m w(x)/m! + (w * W_2^{(m)})(x) = W_3^{(m-1)}(x)$, $m = 1, 2, 3, \ldots$, has the same feature (specifically, at $m \ge 2$).

Assume that all functions $W_N^{(m)}(x)$, $m = 0, 1, 2, \ldots$ are unimodal, have a unique peak at point $x_* = 1$, and have a discontinuity of the first kind at this point at certain fixed N. Let us construct the induction step from N to $N + 1$. According to the above assumption and the hypothesis of the theorem, the following equality should hold:

$$W_N^{(m)}(x_* + 0) = \sum_{l=2}^{N} \frac{\lambda^{l+m} w_*^l(1)}{(l+m)!} = \sum_{l=2}^{N} \frac{\lambda^{l+m}}{(l+m)!l!}$$

$$< \sum_{l=2}^{N} \frac{\lambda^{l+m}}{(l!)^2(m+1)!} < \frac{\lambda^{m+1}}{(m+1)!} = W_N^{(m)}(0),$$

since inequality $(l+m)! > l!(m+1)!$ is valid at $l \ge 2$. It follows from the statement of Lemma that each function $(w * W_N^{(m)})(x)$, $m = 0, 1, 2, \ldots$, is continuous and unimodal with a peak at point $x_* = 1$. Since all of them increase within interval $(0, 1)$, each function $\theta(1-x)\lambda^m w(x)/m! + (w * W_N^{(m)})(x) = W_{N+1}^{(m-1)}(x)$, $m = 1, 2, 3, \ldots$, has the same feature. Thus, according to the induction step, we may conclude that all functions $W_N^{(m)}(x)$ are unimodal with a peak at point $x_* = 1$ at arbitrary $N = 2, 3, 4, \ldots$ and $m = 0, 1, 2, \ldots$. In particular, this is true at $m = 0$. Passing to the $N \to \infty$ limit, we find that limit function $W(x) = \lim_{N\to\infty} W_N^{(0)}(x)$ is unimodal with a peak at point $x_* = 1$, since the limit of unimodal functions is a unimodal function [10].

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Translated by D.Safin