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The anharmonic oscillator as a Bloch-Siegert model

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A model of a quantum anharmonic oscillator interacting with a classical coherent field of a single carrier frequency is considered. It is shown that the application of algebraic perturbation theory in the case of resonant interaction leads to a two-level atom model in which the Stark shift of the resonant levels differs depending on the resonant quantum transition and differs in the general case from both the Bloch-Siegert shift and the Stark shift of an ordinary multilevel atom. The constructed model allow describing coherent effects in an ensemble of anharmonic oscillators, which is shown by the example of nutation oscillations.

Keywords: anharmonic oscillator, coherent electromagnetic wave, resonant interaction, algebraic perturbation theory, the Stark shift, optical nutation.

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1. Introduction

Quantum oscillator is an important quantum and nonlinear optics model that is behind the theoretical concepts of interaction between electromagnetic radiation and optical resonators, superconducting structures in the Josephson effect conditions, and of the electromagnetic radiation itself when its quantum state is considered. However, in case of harmonic quantum oscillator in ensembles of such oscillators, it is impossible to form the main nonlinear-optical effects such as optical nutation, photon echo, superradiation [1], that are demanded for spectroscopy as well as for data processing and quantum computing [2,3]. The reason for such picture of interaction between the electromagnetic radiation andquantum harmonic oscillators is in the structure of creation and annihilation operator algebra that describes the Hamiltonian of a problem — this problem is mathematically solvable [1]. To avoid this „obstacle“, either combined systems are used in applications, for example, resonant atoms are placed inside the resonator [4,5], or anharmonicity is considered [6].

Consideration of quantum oscillator anharmonicity in nonlinear and quantum optics problems is usually limited to a rotating wave model where a pair of resonance levels from the non-equidistant spectrum of an anharmonic oscillator interacts with an electromagnetic field, and the system, thus, becomes equivalent to a two-level atom model that is in resonant interaction with a classical coherent wave [7].

This paper focuses on considerable difference between the two-level system model obtained from the anharmonic quantum oscillator and that obtained from an atom. The difference is in the second-order processes of interaction between the objects in question and the electromagnetic wave. Besides, if we restrict ourselves to the case of classical coherent wave of one carrier frequency and anharmonicity of two lowest degrees, then we have only three types of resonance with absorption of one quantum from an electromagnetic field

Itis known that atomic energy levels in a high-frequency electromagnetic field acquire a shift that is known as the Stark shift owing to the high-frequency Stark effect [8–10]. The magnitude of the Stark shift of a resonance level is contributed to by not only resonance levels, but also by other atomic levels among which the contribution of resonance levels is usually small [9,10]. A total absence of the contribution from other oscillator levels is a feature that has been found in the two-level model of the resonant interaction between the anharmonic oscillator and coherent field. As a result of this feature, the relation between two shifts of resonance levels is well defined and cannot be arbitrary as in the case of two-level model obtained from the problem of resonant interaction with a multilevel atom. This difference governs the behavior of collective processes in ensembles of excited two-level particles, for example, when a superrariation pulse is generated [11].

To obtain the two-level model of the resonant interaction between the anharmonic oscillator and coherent wave, we use algebraic perturbation theory and sequentially reduce the problem to the two-level model and compare it with the Bloch–Siegert model known from magnetic resonance theory [12]. In case of multilevel atoms and molecules, the resonance level energy shift in the Bloch–Siegert model is a small quantity in the total Stark shift of levels. Nevertheless, many studies [13–18], including experimental ones [16–19] have been still devoted to the investigation of the Bloch–Siegert shift in various resonance process models. Note that quantum-mechanical oscillators are not addressed here, though there are many nonlinear models for them.

The feature of our result may be restated as follows: the Stark shift in a multilevel system such as the anharmonic

oscillator differs in case of resonant interaction with a single-frequency coherent wave, generally speaking, from the Bloch–Siegert shift as well as from the cases of resonant interactions with multilayer atoms. The Bloch–Siegert relation for the Stark shifts can be achieved only iin one of three possible types of resonance that implies excitation of the anharmonic oscillator from the ground state to the third one during the absorption of one electromagnetic field quantum. In terms of the effective Hamiltonian, this resonance case is equivalent to the abstract two-level model of a quantum particle. Contrary to the previous studies [12–19], we use algebraic perturbation theory to obtain the resonance energy level shift because this theory allows coherent and broadband quantum fields to be addressed in a consistent manner. In the latter case, the use of algebraic perturbation theory is justified [20]. The differences of algebraic perturbation theory from the Magnus method [21] and the Floquet approach [22] that were previously used for the analysis of other resonance physics problems are explained in [23]. A perturbation theory version from [24,25] is also sometimes used with the same typical inaccuracies. Note that algebraic perturbation theory may be treated in this problem as generalization of the Kylov-Bogolyubov-Mitropolsky averaging method described in [8] for optical problems. Algebraic perturbation theory is laid in [9,10,26–29].

The application features of algebraic perturbation theory are discussed first, then the effective Hamiltonian is derived fro the anharmonic quantum oscillator in the resonance coherent wave field in three typical resonance cases. absorption of one coherent field quantum in these cases induces one-quantum or two-quantum or three-quantum transitions in the anharmonic oscillator. Algebraic perturbation theory made it possible to use the results of previous studies performed within this framework directly for description of optical nutation on the ensemble of anharmonic oscillators. Possibilities of the developed approach for describing other nonlinear optical effects in the quantum oscillator systems are thus demonstrated. In case of nutation oscillations, the presented results give an idea of the anharmonic oscillator parameters by means of the experimental study of optical nutation. It is the optical nutation in various systems that is often investigated experimentally [17–19].

2. Features of algebraic perturbation theory

Consider the anharmonic quantum oscillator in the coherent resonance wave field. The resonance conditions will be prescribed below. The benchmark for description of optical effects is the Schrödinger equation for the state vector $|\Psi\rangle$ of the anharmonic oscillator with the Hamiltonian H of the problem written as

$$i\hbar\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle,\ H = H_{osc} + V_{int},$$

$$H_{osc} = \hbar\Omega_c\left(c^\dagger c + \alpha(c + c^\dagger)^3 + \beta(c + c^\dagger)^4\right)$$

$$= H_{osc-Diag} + H_{osc-Non-D},$$

$$V_{int} = g\left(\mathscr{E}_{cl}\exp(-i\omega_{cl}t - i\Phi)\right.$$

$$\left. + \mathscr{E}^*_{cl}\exp(i\omega_{cl}t + i\Phi)\right)\left(c + c^\dagger\right).$$

Here, H_{osc} is the Hamiltonian of the oscillator of frequency Ω_c taking into account the anharmonicity of the third-/fourth-orders defined by the constants α and β. Operator of interaction between the oscillatior and coherent wave V_{int} is defined by the parameter g (for example, in case of a resonator, g defines the coupling of external and intracavity fields on the mirror). $\mathscr{E}_{cl}\exp(-i\omega_{cl}t - i\Phi) + \mathscr{E}^*_{cl}\exp(i\omega_{cl}t + i\Phi)$ is the electric field strength of the coherent wave carrying the frequencies ω_{cl} and amplitudes $\mathscr{E}_{cl}$, Φ is the electric field phase of the coherent wave in the oscillator position point. It is assumed that the coherent wave amplitude $\mathscr{E}_{cl}$ varies slowly over time compared with $\exp(\pm i\omega_{cl}t)$.

Note that, apart from anharmonicity, the given model with the interaction operator V_{int} has been repeatedly addressed before [30–33]. The difference of the anharmonicity consideration is in that there is both diagonal $H_{osc-Diag}$ and non-diagonal ... small corrections: $H_{osc-Non-D}$:

$$H_{osc-Diag} = \hbar\Omega_c N + V_1,\ H_{osc-Non-D} = V_2 + V_3,$$

$$N = c^\dagger c,\ V_1 = \hbar\Omega_c 6\beta\left(N + N^2\right),$$

$$V_2 = \hbar\Omega_c\alpha\left((3cN + c^3) + H.c.\right),$$

$$V_3 = \hbar\Omega_c\beta\left((c^4 - 2c^2 + 4c^2N) + H.c.\right).$$

Letters $H.c.$ usually replace the Hermitian conjugates of the previous.

It is convenient to assume β in the diagonal term V_1 as „independent“ from β in the term V_3 in the following sense.

According to the principles of algebraic perturbation theory, we perform the unitary transformation of the initial state vector $|\Psi\rangle$ of the open system and environment:

$$|\tilde{\Psi}\rangle = \hat{T}|\Psi\rangle,\ \hat{T} = e^{-iS},\ S^\dagger = S$$

and expand the Hermitian generator of the unitary transformation S as series in the available coupling constants:

$$S = S^{(1,0,0,0)} + S^{(0,1,0,0)} + S^{(0,0,1,0)} + S^{(0,0,0,1)}$$

$$+ S^{(2,0,0,0)}\ldots, \qquad (1)$$

where each of the coupling constants, i.e. constants in the interaction operators V_{int}, V_1, V_2 and V_3, is corresponded by one place in the four superscripts. The system Hamiltonian is transformed into the Hamiltonian $\tilde{H}$ according to the expression

$$\tilde{H} = \hat{T}H\hat{T}^\dagger - i\hbar\hat{T}\frac{d}{dt}\hat{T}^\dagger,$$

is expanded in the similar (1) series

$$\tilde{H} = \tilde{H}^{(0,0,0,0)} + \tilde{H}^{(1,0,0,0)} + \tilde{H}^{(0,1,0,0)} + \tilde{H}^{(0,0,1,0)} + \tilde{H}^{(0,0,0,1)} + \tilde{H}^{(2,0,0,0)} + \dots \quad (2)$$

and defines the transformed Schrödinger equation

$$i\hbar \frac{d}{dt} |\tilde{\Psi}\rangle = \tilde{H}|\tilde{\Psi}\rangle.$$

The fact that expansions in four constants (β was used twice — in the first and third superscripts) were used in expansions (1) and (2) with three interaction constants, i.e. α, β and g, exactly reflects the above-mentioned „independency“ of constants and the specifics of analysis using the algebraic perturbation theory methods.

Principles used for selection of terms in the transformation generator and transformed Hamiltonian are clearly defined when using the interaction representation. The key difference of the algebraic perturbation theory method from the above-mentioned Magnus and Floquet methods (and also other methods based on the Baker-Hausdorff formula [9,10,24,25]) is in the principle used for selection of terms in series (1) and (2). The principle is formulated more compactly in the interaction representation — the effective Hamiltonian terms in the interaction representation shall not contain any rapidly time-varying terms compared with $\exp(\pm i\Omega_c t)$ and $\exp(\pm i\omega_{cl} t)$. Then, on the contrary, the transformation generator terms include only rapidly time-varying terms. Assignment of operators to the interaction representation will be marked through explicit writing of the time argument (except the time dependence of the coherent field amplitude). The whole series can be summarized here by the first superscript because it is responsible for the diagonal addition to the Hamiltonian and therefore the whole series of terms, but for the first one, will be null. This facts will be further used as follows. The representation will be assumed hereinafter as a representation that is defined by the harmonic oscillator Hamiltonian with the diagonal correction:

$$H(t) = \exp(iH_{osc-Diag}t/\hbar) H_{Non-D} \exp(-iH_{osc-Diag}t/\hbar) = V_2(t) + V_3(t) + V_{int}(t),$$

$$V_{int}(t) = g\Big(\mathscr{E}_{cl} \exp(-i\omega_{cl}t - i\Phi) + \mathscr{E}^*_{cl} \exp(i\omega_{cl}t + i\Phi)\Big)\Big(c(t) + c^\dagger(t)\Big),$$

$$c(t) = \exp(iH_{osc-Diag}t/\hbar) c \exp(-iH_{osc-Diag}t/\hbar).$$

Wherein

$$i\hbar \frac{d}{dt} |\tilde{\Psi}\rangle = \tilde{H}|\tilde{\Psi}\rangle, \;\; |\tilde{\Psi}\rangle = \hat{T}(t)|\Psi(t)\rangle,$$

$$\hat{T}(t) = e^{-iS(t)}, \;\; S^\dagger(t) = S(t),$$

$$\tilde{H}(t) = \hat{T}(t)V(t)\hat{T}^\dagger(t) - i\hbar\hat{T}(t)\frac{d}{dt}\hat{T}^\dagger(t),$$

$$V_2 = \hbar\Omega_c\alpha\Big((3c(t)N + c(t)^3) + H.c.\Big),$$

$$V_3(t) = \hbar\Omega_c\beta\Big((c(t)^4 - 2c(t)^2 + 4c(t)^2N) + H.c.\Big),$$

$$\tilde{H}(t) = H(t) - i\Big[S(t), H(t)\Big] - \frac{1}{2}\Big[S(t), \Big[S(t), H(t)\Big]\Big] - \dots - i\hbar e^{-iS(t)}\frac{d}{dt}e^{iS(t)}.$$

Now, the transformation generator and transformed Hamiltonian ae expanded as series in coupling constants defining only interactions $V_{int}(t)$, $V_2(t)$ and $V_3(t)$. Accordingly, these interactions are met by the following expansions:

$$S(t) = S^{(1,0,0)}(t) + S^{(0,1,0)}(t) + S^{(0,0,1)}(t) + S^{(2,0,0)}(t) + \dots, \quad (3)$$

$$\tilde{H}(t) = \tilde{V}(t)^{(1,0,0)} + \tilde{V}(t)^{(0,1,0)}\tilde{V}(t)^{(0,0,1)} + \tilde{V}(t)^{(1,1,0)} + \tilde{V}(t)^{(1,0,1)}\tilde{V}(t)^{(0,1,1)} + \tilde{V}(t)^{(2,0,0)} + \dots \quad (4)$$

It is easy to obtain:

$$\tilde{V}(t)^{(1,0,0)}(t) = \hbar\frac{dS^{(1,0,0)}(t)}{dt} + V_{int}(t),$$

$$\tilde{V}(t)^{(0,1,0)}(t) = \hbar\frac{dS^{(0,1,0)}(t)}{dt} + V_2(t),$$

$$\tilde{V}(t)^{(0,0,1)}(t) = \hbar\frac{dS^{(0,0,1)}(t)}{dt} + V_3(t),$$

$$\tilde{V}(t)^{(2,0,0)}(t) = \hbar\frac{dS^{(2,0,0)}(t)}{dt} - \frac{i}{2}\Big[S^{(1,0,0)}(t), V_{int}(t)\Big]\dots$$

It is convenient, but not mandatory, to proceed to the next representation of bosonic operators of the oscillators through the projectors:

$$c \leftrightarrow \sum_{n=1}^{\infty}\sqrt{n}|n-1\rangle\langle n|, \;\; c^\dagger \leftrightarrow \sum_{n=1}^{\infty}\sqrt{n}|n\rangle\langle n-1|,$$

$$c^2 \leftrightarrow \sum_{n=2}^{\infty}\sqrt{n(n-1)}|n-2\rangle\langle n|, \dots \;\; |E_n\rangle = |n\rangle,$$

$$E_n = \hbar\Omega_c\Big(n + 6\beta(n+n^2)\Big), \;\; \Omega_{n,k} = \frac{E_{n,k}}{\hbar}, \;\; E_{n,k} = E_n - E_k.$$

Then the problem operators may be rewritten in a form similar to the operators defining the problem of interaction between the multilayer atom and resonance coherent field:

$$V_{int}(t) = g\Big(\mathscr{E}_{cl}\exp(-i\omega_{cl}t - i\Phi) + \mathscr{E}^*_{cl}\exp(i\omega_{cl}t + i\Phi)\Big) \times \Big(\sum_{n=1}^{\infty} d_{n-1,n}|E_{n-1}\rangle\langle E_n|e^{i\Omega_{n-1,n}t} + H.c\Big),$$

$$V_2(t) = \hbar\alpha\Omega_c\Big((\sum_{n=1}^{\infty} h_{n-1,n}|E_{n-1}\rangle\langle E_n|e^{i\Omega_{n-1,n}t} + \sum_{n=3}^{\infty} h_{n-3,n}|E_{n-3}\rangle\langle E_n|e^{i\Omega_{n-3,n}t}) + H.c.\Big),$$

$$V_3(t) = \hbar\beta\Omega_c\Big(\Big(\sum_{n=4}^{\infty} h_{n-4,n}|E_{n-4}\rangle\langle E_n|e^{i\Omega_{n-4,n}t} + \sum_{n=2}^{\infty} h_{n-2,n}|E_{n-2}\rangle\langle E_n|e^{i\Omega_{n-2,n}t}\Big) + H.c.\Big),$$

$$d_{n-1,n} = d_{n,n-1} = \sqrt{n},\ \ h_{n-1,n} = h_{n,n-1} = 3n^{3/2},$$

$$h_{n-3,n} = h_{n,n-3} = \sqrt{n(n-1)(n-2)},$$

$$h_{n-2,n} = h_{n,n-2} = (4n-2)\sqrt{n(n-1)},$$

$$h_{n-4,n} = h_{n,n-4} = \sqrt{n(n-1)(n-2)(n-3)}. \quad (6)$$

In notations (5), the problem of the resonant interaction between the coherent field and anharmonic oscillator becomes quite similar to the problem of resonant interaction between the coherent field and multilayer atom. Therefore, considerable difference in results here is believed to be important. Unlike the case with multilayer atoms, constants (6) that define the intensity of transitions between the levels are rigorously defined. For a multilayer atom, only sum rule conditions may be relied upon [34,35], though the same constants may be also defined in some cases.

$\tilde{V}^{(0,1,0)}(t)$ and $\tilde{V}^{(0,0,1)}(t)$ may be determined independently of the resonant interaction conditions because they are rapidly time-varying quantities. Therefore

$$\tilde{V}^{(0,1,0)}(t) = \tilde{V}^{(0,0,1)}(t) = 0.$$

This condition suggests that there are not harmonic generation processes in the first order of algebraic perturbation theory. Values for $S^{(0,1,0)}$ and $S^{(0,0,1)}$ that define the interference processes and harmonic generation are also derived from it, however, they are not shown here because harmonic generation is not addressed in this paper.

3. One-quantum resonance condition

Depending on the initial state of the anharmonic oscillator (before the interaction with the coherent field), one-quantum resonance may be implemented with different transitions where nonzero population of one energy level from the pair is the main condition. In the first order, the resonance levels shall differ in their quantum number per unit. In two-quantum resonances and higher-order resonances, the difference in the quantum number of the quantum levels is 2 or more. These cases are not addressed in this work.

Let the population of the quantum state $|n\rangle$, $n \le 1$ be nonzero. Frequencies of transitions to the neighboring energy levels are as follows

$$\Omega_{n,n-1} = \Omega_c(1 + 12\beta n),\ \ \Omega_{n+1,n} = \Omega_c(1 + 12\beta(n+1)).$$

Assume that the widths $\gamma_{n+1,n}$ and $\gamma_{n,n-1}$ of the spectral line of quantum transitions $|n+1\rangle \to |n\rangle$ and $|n\rangle \to |n-1\rangle$ are small:

$$\gamma_{n+1,n}, \gamma_{n,n-1} \ll 12\beta\Omega_c. \quad (7)$$

If the spectral line of coherent radiation and Rabi frequency are small too, then, when the harmonic oscillator is exposed to the coherent field of the carrier frequency ω_{cl}, two different one-quantum resonances, when $\omega_{cl} \cong \Omega_{n,n-1}$ or when $\omega_{cl} \cong \Omega_{n+1,n}$, are suggested. During exposure to a two-frequency field, double resonance with „cascade" configuration is possible. With the growth of the number of excitations of the quantum oscillator, the adopted anharmonicity model shall be replaced with another one, for example, use the Morse potential [36].

Transitions $|n+2\rangle \to |n\rangle$, $|n\rangle \to |n-2\rangle$, etc., during absorption of only one electromagnetic field quantum are an important feature of the one-photon resonance in the anharmonic oscillator. These processes become possible due to the interactions $V_2(t)$ and $V_3(t)$. They define the harmonic generation processes. The two-level model emerging here will be discussed using the transitions $|2\rangle \leftrightarrow |0\rangle$ as an example. Another process to be discussed is the resonance transition $|3\rangle \leftrightarrow |0\rangle$ during absorption of one coherent electromagnetic field quantum.

Thus, the following conditions of the one-quantum resonance will be discussed:

$$\omega_{cl} \cong \Omega_{n+1,n}, \quad n \ge 0, \quad (8)$$

$$\omega_{cl} \cong \Omega_{2,0}, \quad (9)$$

$$\omega_{cl} \cong \Omega_{3,0}. \quad (10)$$

4. Transformation to the two-level model. Resonance (8)

In conditions (8), the rapidly time-varying terms of the generator $S^{(1,0,0)}(t)$ are as follows:

$$S^{(1,0,0)}(t) = -\frac{g}{i\hbar}\Sigma_n''\Big(\frac{\mathscr{E}_{cl}d_{n-1,n}\exp(-i\omega_{cl}t - i\Phi + i\Omega_{n-1,n}t)}{-\omega_{cl} + \Omega_{n-1,n}} + \frac{\mathscr{E}^*_{cl}d_{n-1,n}\exp(i\omega_{cl}t + i\Phi + i\Omega_{n-1,n}t)}{\omega_{cl} + \Omega_{n-1,n}}\Big)|E_{n-1}\rangle\langle E_n| - \frac{g}{i\hbar}\Sigma_n''\Big(\frac{\mathscr{E}_{cl}d_{n,n-1}\exp(-i\omega_{cl}t - i\Phi + i\Omega_{n,n-1}t)}{-\omega_{cl} + \Omega_{n,n-1}} + \frac{\mathscr{E}^*{}_{cl}d_{n,n-1}\exp(i\omega_{cl}t + i\Phi + i\Omega_{n,n-1}t)}{\omega_{cl} + \Omega_{n,n-1}}\Big)|E_n\rangle\langle E_{n-1}|.$$

Double prime denotes the selection condition of terms written in a general form — there shall be no slowly time-varying terms and accordingly varying denominators among them.

The operator of the one-quantum resonance transition $|n+1\rangle \to |n\rangle$ is written as

$$\tilde{V}^{(1,0,0)}(t) = g\mathscr{E}^*{}_{cl}d_{n,n-1}|E_{n-1}\rangle\langle E_n|e^{i(\omega_{cl}-\Omega_{n,n-1})t} + H.c.,$$

$$\tilde{V}^{(0,1,0)}(t) = \tilde{V}^{(0,0,1)}(t) = 0. \quad (11)$$

The next operator is generally called the Stark shift operator due to the high-frequency Stark effect [8–10]:

$$\tilde{V}^{(2,0,0)}(t) = H^{St}_{n,n+1},$$

$$H^{St}_{n,n+1} = g^2|\mathscr{E}_{cl}|^2 \sum_{j=n,n+1} |E_j\rangle\langle E_j|, \tag{12}$$

$$\Pi_n(\omega_{cl}) = \frac{|d_{n,n-1}|^2}{\hbar}\Big(\frac{1}{\Omega_{n,n-1}+\omega_{cl}}\Big) + \frac{|d_{n,n+1}|^2}{\hbar}\Big(\frac{1}{\Omega_{n,n+1}-\omega_{cl}}\Big),$$

$$\Pi_{n+1}(\omega_{cl}) = \frac{|d_{n+1,n}|^2}{\hbar}\frac{1}{\Omega_{n+1,n}+\omega_{cl}} + \frac{|d_{n+1,n+2}|^2}{\hbar} \times \Big(\frac{1}{\Omega_{n+1,n+2}+\omega_{cl}} + \frac{1}{\Omega_{n+1,n+2}-\omega_{cl}}\Big),\ n \geq 1.$$

When $n = 0$

$$\tilde{V}^{(1,0,0)}(t) = g\mathscr{E}^*_{cl}|E_0\rangle\langle E_1|e^{i(\omega_{cl}-\Omega_{1,0})t} + H.c., \tag{13}$$

$$H^{St}_{0,1} = g^2|\mathscr{E}_{cl}|^2\Big(\Pi_0(\omega_{cl}|E_0\rangle\langle E_0| + \Pi_1(\omega_{cl})|E_1\rangle\langle E_1|\Big), \tag{14}$$

$$\Pi_0(\omega_{cl}) = \frac{|d_{01}|^2}{\hbar}\frac{1}{\Omega_{0,1}-\omega_{cl}},$$

$$\Pi_1(\omega_{cl}) = \frac{|d_{12}|^2}{\hbar}\Big(\frac{1}{\Omega_{1,2}+\omega_{cl}} + \frac{1}{\Omega_{1,2}-\omega_{cl}}\Big) + \frac{|d_{10}|^2}{\hbar}\frac{1}{\Omega_{1,0}+\omega_{cl}}.$$

Whereby in rigorous resonance $\omega_{cl} = \Omega_c(1+12\beta)$ and considering $12\beta \ll 1$

$$\Pi_0(\omega) \approx -\frac{1}{2\hbar\Omega_c},\ \Pi_1(\omega) \approx -\frac{1}{6\beta\hbar\Omega_c},$$

so that

$$|\Pi_0(\omega)| \ll |\Pi_1(\omega)|, \tag{15}$$

which distinguishes this situation both from the „pure“ Bloch–Siegert model where $\Pi_0(\omega) = -\Pi_1(\omega)$ and from the multilayer atom case where $|\Pi_0(\omega)| \approx |\Pi_1(\omega)|$ is possible.

Thus, the resonant interaction between the coherent wave and anharmonic oscillator in conditions (8) distinguishes a pair of resonance levels $|n\rangle$ and $|n+1\rangle$ whose Stark shift is defined by the resonance levels themselves as well as by the nearby levels $|n-1\rangle$ and $|n+2\rangle$. In case when $n = 0$, the Stark shift of the level $|0\rangle$ may be neglected. The effective two-level Hamiltonian of the resonant interaction in conditions (8) in the interaction representation may be written as

$$V^{Eff}(t) = \tilde{V}^{(1,0,0)}(t) + H^{St}_{n,n+1}(t). \tag{16}$$

5. Transformation to the two-level model. Resonance (9)

In conditions (9), the main relations of algebraic perturbation theory are as follows

$$\tilde{V}^{(1,0,0)}(t) = \tilde{V}^{(0,1,0)}(t) = \tilde{V}^{(0,0,1)}(t) = 0,$$

$$S^{(0,1,0)}(t) = -\frac{\alpha\Omega_c}{i}\Big(\sum_{n=1}^{\infty}\frac{h_{n-1,n}|E_{n-1}\rangle\langle E_n|e^{i\Omega_{n-1,n}t}}{\Omega_{n-1,n}} + \sum_{n=3}^{\infty}\frac{h_{n-3,n}|E_{n-3}\rangle\langle E_n|e^{i\Omega_{n-3,n}t}}{\Omega_{n-3,n}}\Big) + H.c.,$$

$$\tilde{V}^{(1,1,0)}(t) = -\frac{i}{2}\Big[S^{(0,1,0)}, V_{int}(t)\Big]' - \frac{i}{2}\Big[S^{(1,0,0)}, V_2(t)\Big]'.$$

Prime denotes the selection condition of terms written in a general form — there shall be no rapidly time-varying terms among them. The calculations give the following expressions:

$$\tilde{V}^{(1,1,0)}(t) = -\frac{9}{2}\sqrt{2}g\alpha\Omega_c\mathscr{E}_{cl}|E_2\rangle\langle E_0|e^{-i\omega_{cl}-i\Phi+i\Omega_{2,0}t} + H.c., \tag{17}$$

$$H^{St}_{0,2} = g^2|\mathscr{E}_{cl}|^2\Big(\Pi_0(\omega_{cl}|)E_0\rangle\langle E_0| + \Pi_2(\omega_{cl})|E_2\rangle\langle E_2|\Big), \tag{18}$$

$$\Pi_0(\omega_{cl}) = \frac{|d_{01}|^2}{\hbar}\frac{1}{\Omega_{0,1}-\omega_{cl}},$$

$$\Pi_2(\omega_{cl}) = \frac{|d_{21}|^2}{\hbar}\Big(\frac{1}{\Omega_{2,1}+\omega_{cl}} + \frac{1}{\Omega_{2,1}-\omega_{cl}}\Big) + \frac{|d_{23}|^2}{\hbar}\Big(\frac{1}{\Omega_{2,3}+\omega_{cl}} + \frac{1}{\Omega_{2,3}-\omega_{cl}}\Big).$$

In case of precise resonance (9), $\omega_{cl} = 2\Omega_c(1+18\beta)$ and $18\beta \ll 1$,

$$\Pi_2(\omega_{cl}) \approx -2\Pi_0(\omega_{cl}). \tag{19}$$

takes place The effective two-level Hamiltonian of the resonant interaction in conditions (9) in the interaction representation may be written as

$$V^{Eff}(t) = \tilde{V}^{(1,1,0)}(t) + H^{St}_{0,2}(t). \tag{20}$$

6. Transformation to the two-level model. Resonance (10)

In conditions (10), excitation of the anharmonic oscillator from the ground $n = 0$ to the third $n = 3$ energy quantum level takes place during absorption of one quantum from the electromagnetic coherent field. The main relations of algebraic perturbation theory are the same:

$$\tilde{V}^{(1,0,0)}(t) = \tilde{V}^{(0,1,0)}(t) = \tilde{V}^{(0,0,1)}(t) = 0,$$

However, now to calculate the effective Hamiltonian of the resonant interaction responsible for the resonance $\omega_{cl} \cong \Omega_{3,0}$, the unitary transformation generator $S^{(0,0,1)}(t)$ must be known:

$$S^{(0,1,0)}(t) = -\frac{\beta\Omega_c}{i}\Big(\sum_{n=1}^{\infty}\frac{h_{n-4,n}|E_{n-4}\rangle\langle E_n|e^{i\Omega_{n-4,n}t}}{\Omega_{n-4,n}}$$

$$+\sum_{n=3}^{\infty}\frac{h_{n-2,n}|E_{n-2}\rangle\langle E_n|e^{i\Omega_{n-2,n}t}}{\Omega_{n-2,n}}\Big) + H.c..$$

Compared with the expression for $S^{(1,0,0)}(t)$, $S^{(0,0,1)}(t)$ contains no primes because all terms in the written equation are rapidly time-varying functions.

Then, the resonance transition operator $\tilde{V}^{(1,0,1)}(t)$ is defined as follows

$$\tilde{V}^{(1,0,1)}(t) = -\frac{i}{2}\Big[S^{(0,0,1)}, V_{int}(t)\Big]' - \frac{i}{2}\Big[S^{(1,0,0)}, V_3(t)\Big]'. \tag{21}$$

They give the following results:

$$\tilde{V}^{(1,0,1)}(t) = -8\sqrt{6}g\beta\mathscr{E}_{cl}|E_3\rangle\langle E_0|e^{-i\omega_{cl}t-i\Phi+i\Omega_{3,0}t} + H.c., \tag{22}$$

$$H^{St}_{0,3} = g^2|\mathscr{E}_{cl}|^2\Big(\Pi_0(\omega_{cl})|E_0\rangle\langle E_0| + \Pi_3(\omega_{cl})|E_3\rangle\langle E_3|\Big), \tag{23}$$

$$\Pi_0(\omega_{cl}) = \frac{|d_{01}|^2}{\hbar}\frac{1}{\Omega_{0,1}-\omega_{cl}},$$

$$\Pi_3(\omega_{cl}) = \frac{|d_{32}|^2}{\hbar}\Big(\frac{1}{\Omega_{3,2}+\omega_{cl}} + \frac{1}{\Omega_{3,2}-\omega_{cl}}\Big)$$

$$+\frac{|d_{34}|^2}{\hbar}\Big(\frac{1}{\Omega_{3,4}+\omega_{cl}} + \frac{1}{\Omega_{3,4}-\omega_{cl}}\Big).$$

It should be emphasized that the equation for $\Pi_0(\omega_{cl})$ is no formally different from the equations in the previous sections, however, another quantity for the coherent field carrier ω_{cl} is substituted into it. Therefore, in case of precise resonance (10), $\omega_{cl} = 3\Omega_c(1+24\beta)$ and $24\beta \ll 1$, the following relation takes place

$$\Pi_3(\omega_{cl}) \approx -\Pi_0(\omega_{cl}), \tag{24}$$

i.e. the relation between the resonance level frequency shifts of the effective two-level system here coincides with that for the Bloch–Siegert shifts in the initial purely two-level system.

The effective two-level Hamiltonian of the resonant interaction in conditions (10) in the interaction representation may be written as

$$V^{Eff}(t) = \tilde{V}^{(1,0,1)}(t) + H^{St}_{0,3}(t). \tag{25}$$

7. Conclusion

The main results of the work are effective Hamiltonians (16), (20) and (25) describing the two-level quantum system isolated from the anharmonic quantum oscillator spectrum by the resonant interaction between the oscillator and classical coherent wave with the same carrier frequency. A situation has been found - resonance (10) — when we may speak of a „purely“ two-level system with an accuracy up to terms of the second order. This fact distinguishes the results described here from the majority of works addressing the behavior of both harmonic and anharmonic oscillators. For example, study [37] has shown the difference in behavior of the quantum oscillator and classical oscillator in case of excitation by unipolar short pulses. In this case, the oscillator state is formed with a wide set of states of the latter due to the excitation broadbandness. Focus shall be also made on study [38], where the nonlinear behavior of the anharmonic oscillator is treated as a quantum equivalent of the classical renormalization method [6]. The identified quantum corrections to the classical motion of the anharmonic oscillator were found in the coherent state conditions of the initial oscillator, rather than in coherent excitation conditions of the initial oscillator by the external resonance quasi-monochromatic light field. The main results in the cited article pertain to the behavior of the second and next harmonics that appear in the oscillator motion exactly due to consideration of the quantum additions.

The effective Hamiltonian was obtained using algebraic perturbation theory with an accuracy up to the second order in the coherent field. It consists of the resonance transition operator with absorption or emission of the coherent field quantum and operator of the Stark interaction between the anharmonic oscillator and coherent field. This is the difference from the standard explanation based on the non-equidistance of the anharmonic oscillator spectrum where a pair of levels of the two-level model is isolated by a simple resonance condition. Further analysis of such two-level system leads to the Bloch–Siegert model with relation between the parameters of type (23) Stark shift where $\Pi_0(\omega_{cl})$ corresponds to the lower level g of the two-level system and $\Pi_3(\omega_{cl})$ corresponds to the upper level e. Some other cases where these relations are different have been also addressed. It should be emphasized that these relations define the temporal dynamics of collective disintegration of a two-level particle ensemble in a quantum field [11].

The analysis conducted using algebraic perturbation theory makes it possible to apply the results of standard description of the nonlinear optical effects based on the optical nutation obtained within the framework of the same algebraic perturbation theory directly to the anharmonic oscillator systems [10]. For example, the frequency of nutation oscillations of he resonance light wave intensity in the vicinity of the leading edge (or after a short-time perturbation of its amplitude or phase) Ω_N is given [10] by equation:

$$\Omega_N = \sqrt{(\Delta-\Delta^{St})^2 + \Omega_R^2},$$

where Δ is the deviation from the resonance, $\Delta^{St} = g^2|\mathscr{E}_{cl}|^2\Big(\Pi_e(\omega_{cl}) - \Pi_g(\omega_{cl})\Big)$ is the transition fre-

quency shift due to the high-frequency Stark effect, and Ω_R is the Rabi frequency:

$$\Omega_R = 2g\sqrt{n}|\mathscr{E}_{cl}|\hbar^{-1},$$

$n = 1, 2$ in case of resonance (8),

$$\Omega_R = 9\sqrt{2}g\alpha|\mathscr{E}_{cl}|\hbar^{-1}$$

in case of resonance (9),

$$\Omega_R = 16\sqrt{6}g\beta|\mathscr{E}_{cl}|\hbar^{-1}$$

in case of resonance (10).

Nutation oscillations are one of the important objects of the performed experimental investigations [17–19]. The shown equations give an idea of the anharmonic oscillator parameters in the examined model using the results of the experimental investigation of the optical nutation.

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Conflict of interest

The authors declare that they have no conflict of interest.

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