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Self-similar vortex

© V.P. Koverda, V.N. Skokov

Institute of Thermal Physics, Ural Branch of the Russian Academy of Science, Yekaterinburg, Russia
E-mail: koverda@itp.uran.ru

Received January 29, 2024
Revised February 20, 2024
Accepted March 4, 2024

An analytical approach is proposed to describe a computer model of a „sand heap"that describes positive $1/f$ fluctuations. The solutions to the proposed equations are random processes with power-law behavior of power spectra and distribution functions of random fluctuations.

Keywords: self-similar random processes, stochastic equations, vortices, power spectrum, $1/f$-noise.

DOI: 10.61011/TPL.2024.06.58474.19876

Random processes are called self-similar if they have no preferred spatial or time scales. Power-law dependences of the spectral density and distribution functions are characteristic features of such processes. Self-similar random processes (self-similar vortex structures included) are found in physics, geophysics, biology, and other fields of science [1–6]. Power-law dependences enable the emergence of prominent spikes and large-scale vortices, which are observed in ocean currents, atmospheric phenomena related to cloud and tornado formation, and astrophysical phenomena involved in the formation of galaxies [7–9]. Self-similar random processes associated with non-equilibrium phase transitions were discovered experimentally in critical heat and mass transfer regimes: in transition from bubble boiling of a liquid to film boiling [10], critical flow of a boiling-up liquid [11], and acoustic cavitation [12].

Computer „sand heap" models employing the concept of self-organized criticality are applied in the interpretation of numerous random processes with strong self-similar fluctuations [1]. Such a system evolves without fine tuning of parameters, and the distribution is similar to fluctuations in the critical state. Self-similar random processes are often characterized on the basis of fractional integration of white noise (fractional diffusion equation, fractal-structure diffusion [13–16]). Random processes obtained via modeling of this type require vast computational resources and are generally non-stationary.

We have proposed an analytical approach to characterization of self-similar random processes with strong fluctuations that shortens and simplifies considerably the calculation procedure (compared to fractional integration). This approach involves the use of a system of nonlinear stochastic equations that characterizes the stochastic dynamics in the context of interacting non-equilibrium phase transitions.

The system of stochastic equations is written as

$$\frac{d\varphi}{dt} = -\varphi\psi^2 + \psi + \xi_1(t),$$

$$\frac{d\psi}{dt} = -\psi\varphi^2 + 2\varphi + \xi_2(t), \qquad (1)$$

where φ and ψ are dynamic variables and ξ_1 and ξ_2 are Gaussian δ-correlated noises.. System of equations (1) characterizes random walks in potential corresponding to the superposition of interacting subcritical and postcritical phase transitions. This system was derived with the use of the Landau phase transition theory by expanding the thermodynamic potential in order parameters of subcritical and postcritical phase transitions with account for their interaction and rotating the coordinate axes by $\pi/4$ [17].

In the case of numerical finite-difference integration, system (1) takes the following form [17]:

$$\varphi_{i+1} = (\varphi_i + \psi_i \Delta t)(1 + \psi_i^2 \Delta t)^{-1} + p_i \Delta t^{0.5},$$

$$\psi_{i+1} = (\psi_i + 2\varphi_i \Delta t)(1 + \varphi_i^2 \Delta t)^{-1} + q_i \Delta t^{0.5}, \qquad (2)$$

where p_i and q_i are vectors of Gaussian random numbers with zero mean and variance σ^2 and Δt is the time interval (integration step). At $\Delta t \to 0$, system (2) matches system (1), which has a non-stationary solution. At a finite Δt, the system has a stationary solution [18]. Stochastic equations are specific in that the time differential is a second-order infinitesimal with respect to the stochastic variable differential [19]. The time interval is raised to a power of 0.5 in Eqs. (2).

The second equation in systems (1) and (2) is the master one, while the first equation is the slave. The solution of the master equation characterizes the evolution of fluctuations that have Gaussian „tails" of the distribution function (variable ψ). This allows one to use the classical expression for entropy in estimation of the solution stability [20]. The solution of the slave equation yields a self-similar random process with a power-law distribution (variable φ). At a critical noise intensity ($\sigma_1 = \sigma_2 \approx 1$), the φ fluctuation power spectrum assumes the shape of the spectrum of flicker noise $S_\varphi \sim 1/f$. The spectrum of variable ψ has the form of $S_\psi = 1/f^2$.

Positive and negative fluctuations are the solution of systems (1) and (2). Positive values of fluctuations are considered in the computer „sand heap“ model. System of equations (1) and its numerical counterpart (2) characterize random walks in two hyperbolic valleys of potential of interacting phase transitions that lead to fluctuations of different signs. If we introduce mirror reflection of phase trajectories from the ordinate axis, only positive values of variables will remain. This corresponds to the computer „sand heap“ model. Mirror reflection of trajectories is effected by introducing the modulus sign into equations. System (2) then takes the form

$$\varphi_{i+1} = (|\varphi_i| + |\psi_i|\Delta t)(1 + \psi_i^2\Delta t)^{-1} + p_i\Delta t^{0.5},$$
$$\psi_{i+1} = (|\psi_i| + 2|\varphi_i|\Delta t)(1 + \varphi_i^2\Delta t)^{-1} + q_i\Delta t^{0.5}. \qquad (3)$$

The calculation of φ_i and ψ_i realizations was performed in 16 384 integration steps with magnitude $\Delta t = 0.02$. An ensemble of 128 realizations was used to determine the distribution functions and power spectra. Numerical solutions demonstrate that product $\varphi(t)\psi(t) \approx 1$ is preserved; consequently, the power spectrum of the quantity reciprocal to $\psi(t)$ (i.e., variable $1/\psi(t)$) matches the spectrum of self-similar random processes $\varphi(t)$ and has the form of $S_{1/\psi} \sim 1/f$ [20,21]. This property provides an opportunity to derive a master stochastic equation for variable $\psi(t)$ based on system (1) and define variable $\varphi(t)$ as a quantity reciprocal to ψ:

$$\varphi = \frac{\psi}{\varepsilon + \psi^2},$$
$$\frac{d\psi}{dt} = \frac{1}{\psi} - \sigma_\theta^2\psi + \xi(t), \qquad (4)$$

where ε is a small constant that excludes divergence of reciprocal function $1/\psi$ when ψ accidentally approaches zero. With the properties of white noise taken into account, one may write $\sigma_\theta^2 = \sigma^2\Delta t$ in numerical calculations [19,21]. The second equation of system (4) characterizes random walks in a force field with a logarithmic potential. In the case of numerical integration, system (4) takes the following form (with mirror reflection of phase trajectories taken into account):

$$\varphi_i = \frac{\psi_i}{\varepsilon + \psi_i^2},$$
$$\psi_{i+1} = |\psi_i| + \frac{|\psi_i|\Delta t}{\varepsilon + \psi_i^2} - \sigma^2\psi_i\Delta t^2 + q_i\Delta t^{0.5}. \qquad (5)$$

Figure 1 presents (in logarithmic coordinates) the $P(\psi)$ and $P(\varphi)$ distribution functions averaged over an ensemble of 128 realizations. At large values of ψ, the $P(\psi)$ distribution function has a Gaussian „tail,“ while the $P(\varphi)$ distribution function decreases as a power function at large φ. The Gaussian behavior of „tails“ of variable ψ allows one to estimate the stability of a random process with the use of the formulae of classical statistics utilizing the Gibbs–Shannon principle of maximum entropy [22–24]. The maximum entropy corresponds to the critical noise value. Power spectra of fluctuations were determined from calculated realizations by fast Fourier transform with

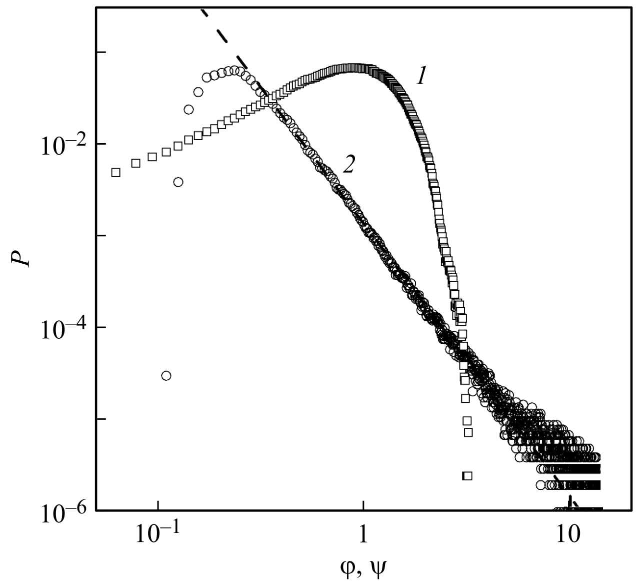

Figure 1. $P(\psi)$ (*1*) and $P(\varphi)$ (*2*) distribution functions derived from the numerical solutions of system (5). The dashed curve represents the $P \sim \varphi^{-3}$ dependence.

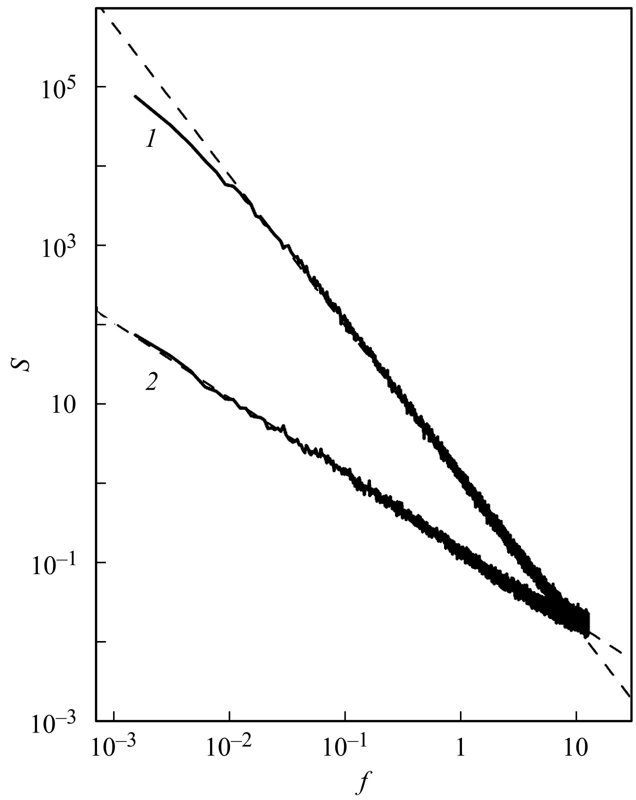

Figure 2. Power spectra of variables $\psi(t)$ (*1*) and $\varphi(t)$ (*2*) derived from the numerical solutions of system (5).

subsequent averaging over an ensemble. Averaged power spectra of variables φ and ψ are shown in Fig. 2. The power spectrum of variable φ takes the form of $S_\varphi \sim 1/f$ at a critical noise intensity ($\sigma_c \approx 1.4$) that, owing to the one-

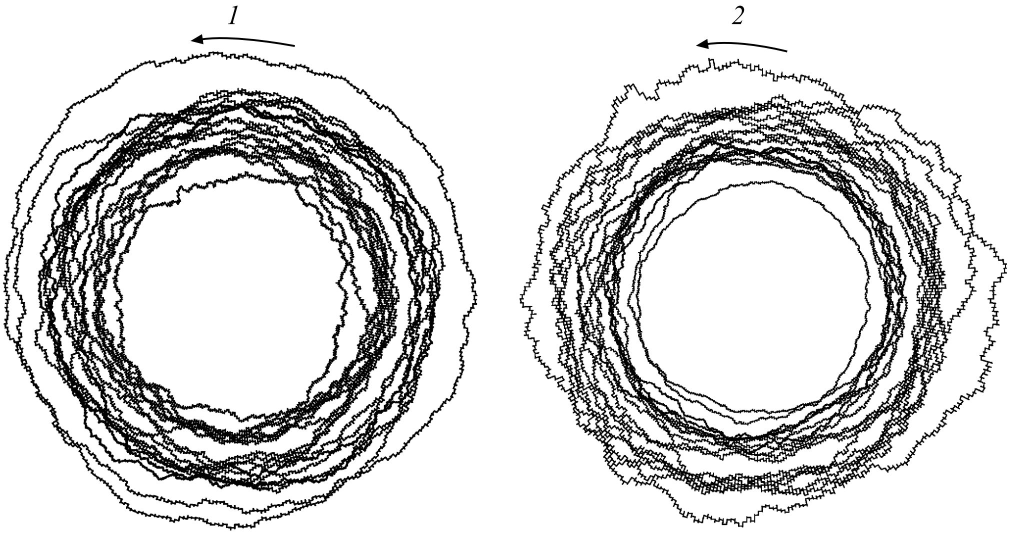

Figure 3. Vortex trajectories of random processes $\psi(t)$ (*1*) and $\varphi(t)$ (*2*) in polar coordinates.

dimensionality of the second equation of system (5), is $\sqrt{2}$ times higher than the critical intensity in two-dimensional systems (1) and (2). The power spectrum of variable ψ takes the form of $S_\psi = 1/f^2$. The power spectra of variables calculated from Eqs. (5) match the corresponding spectra calculated based on system (3).

Positive power-law fluctuations from system (5) provide an opportunity to characterize a self-similar vortex if one switches over to the system of equations in polar coordinates: $r = \psi(t)$ and $r = \varphi(t)$, angle $\theta = 2\pi\Delta t^{-1}t$. The obtained vortices are shown in Fig. 3. Fragment *1* in Fig. 3 corresponds to a vortex lacking self-similarity $r = \psi(t)$, since the $P(\psi)$ distribution function has Gaussian „tails." Fragment *2* in Fig. 3 illustrates a self-similar vortex $r = \varphi(t)$ with a power-law distribution function.

Thus, an analytical approach to description of the computer „sand heap" model, which characterizes a self-similar random process, was proposed. This approach involves the use of a system of nonlinear stochastic equations, which characterizes the stochastic dynamics in the context of interacting non-equilibrium phase transitions, and its viability was demonstrated through the example of a random vortex. Fluctuation processes with a power-law behavior of power spectra and distribution functions of fluctuations are the solutions of proposed stochastic differential equations.

Conflict of interest

The authors declare that they have no conflict of interest.

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Translated by D.Safin