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Formation of an attached cavity with fixed separation points upon impact of a floating circular cylinder

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The plane problem of the vertical and separation impact of a circular cylinder under the free surface of an ideal incompressible heavy fluid is considered. It is assumed that after the impact the cylinder moves deep into the liquid at a constant speed. The dynamics of an attached cavity formed behind a body is studied under the assumption that the separation points of the internal free boundary of the liquid are motionless. The characteristic physical quantities — the Froude number and pressure in the cavern are chosen in such a way that the Kutta-Zhukovsky condition is satisfied at the separation points.

Keywords: Key words: circular cylinder, separation impact, attached cavity, separation points, short times.

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Introduction

The classical model of an impact with a separation plays an important role in the studies of cavitation in a liquid caused by the impact of a floating body [1]. Being of great independent interest for analytical research, this model provides the initial conditions for solving a more complex dynamic cavitation problem. The study of problems of such a plan was carried out earlier at small times on the basis of the first two terms of the asymptotics [2,3]. At the same time, the dynamics of the separation points of the internal free boundary of the liquid (the boundaries of the cavity) was taken into account. The position of the separation points at each moment of time was determined from the Kutta-Joukowski condition. It was shown that there is a linear dependence of the angular coordinates of these points on time in the subject asymptotic approximation. Finding the following terms of the asymptotics is associated with serious technical difficulties, which are explained by the need to use a special variable substitution that transfers information about the dynamics of separation points into the equation and boundary conditions of the problem. For this reason, the question remains open about the possibility of finding the following terms of the asymptotic expansion in degrees of small time based on the Kutta-Joukowski condition. In this paper, it is proposed to study this issue under the additional assumption that the points of separation of the internal free boundary of the liquid remain stationary after impact, at least for some small period of time. This assumption allows analyzing a larger number of terms of the asymptotics in a short time, since a direct asymptotic method is used in this case without special replacement of variables. However, at the same time, it is necessary to impose serious restrictions on the choice of characteristic physical quantities. In this paper, it is shown that when the cylinder moves along the gravity vector, it is possible to specify such a law of artificial gas supply into the cavity, in which the solution of the problem is represented as a power-law asymptotic expansion in a short time (taking into account the first three terms) satisfying the Kutta-Joukowski condition. In the general case (without the assumption made about the immobility of the separation points), it is necessary to take into account the dynamics of the separation points and take into account the possibility of changing the structure of the asymptotics itself for its younger members, starting with the third one. It is expected that similar conclusions can be drawn for a number of similar tasks (for example, when studying rapid acceleration or rapid braking of a body in a liquid). It should be noted that, in addition to the Kutta-Joukowski condition, another important physical condition must also be fulfilled — the positivity of pressure on the wetted surface of the body. If both of these conditions are met, then the solution of the problem built on small times is correct and fully corresponds to the given physical situation. The pressure condition is checked after solving the problem and, in the case of artificial cavitation (when the pressure in the cavity is of the order of atmospheric and higher), as a rule, is always fulfilled [3].

The general principles of cavitation flows in the interaction of solids with a liquid are described in various monographs and articles (see, for example, [4,5]). A review of works on similar problems of penetration of bodies into a liquid, taking into account the phenomenon of separation of liquid particles from their surfaces, is given in [6]. Some results obtained in the study of the underwater launch of rockets by cavitation method are given in [7]. The problems of wave generation in case of a separation-free impact and acceleration of a floating circular cylinder have been studied at short times in [8,9]. Recent studies in this field are presented in [10]. Modern studies are also conducted on the classical model of a separation-free impact, which indicates a great interest in the impact topic [11–14].

1. General problem formulation

The plane problem of a vertical and a separation of a circular cylinder under the free surface of an ideal incompressible heavy liquid is considered [1]. It is assumed that after impact, the cylinder moves deep into the liquid at a constant speed. An attached cavity is formed behind the cylinder, the shape of which depends on the physical and geometric parameters of the problem. It is necessary to study the dynamics of the cavity at short times with the additional assumption that the points of separation of the internal free boundary of the liquid remain stationary after impact at some initial stage of the cylinder movement.

The mathematical formulation of the problem, written in dimensionless variables in a movable coordinate system associated with a cylinder, has the form (Fig. 1–4):

$$\Delta\Phi = 0, \quad r \in \Omega(t), \tag{1}$$

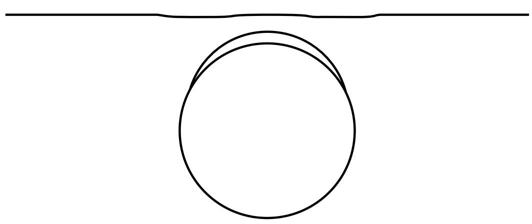

Figure 1. The shape of the cavern at $t = 0.1$.

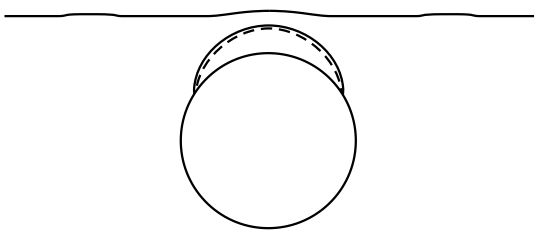

Figure 2. The shape of the cavern at $t = 0.2$; the stroke shows solutions for the boundary layer.

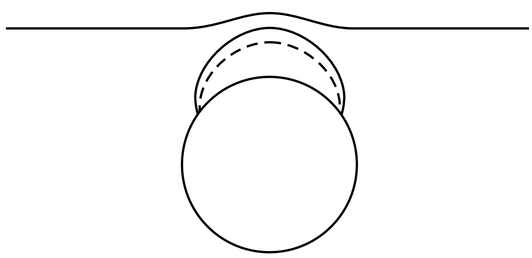

Figure 3. The shape of the cavity at $t = 0.3$; the stroke shows solutions for the boundary layer.

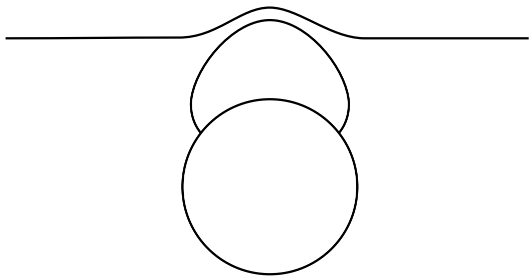

Figure 4. Cavity shape at $t = 0.4$.

$$\frac{\partial\Phi}{\partial n} = -n_y, \quad r \in S_{11}, \tag{2}$$

$$\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial y} + 0.5(\nabla\Phi)^2 + Fr^{-2}(y - t - H) - 0.5\chi = 0, \quad r \in S_{12}(t), \tag{3}$$

$$\frac{\partial\Phi}{\partial r} + \sin\theta = \frac{\partial\eta}{\partial\theta}\dot{\theta}(t) + \frac{\partial\eta}{\partial t}, \quad r \in S_{12}(t), \tag{4}$$

$$\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial y} + 0.5(\nabla\Phi)^2 + Fr^{-2}\xi(x, t) = 0, \quad r \in S_2(t), \tag{5}$$

$$\frac{\partial\Phi}{\partial y} = \frac{\partial\xi}{\partial x}\frac{\partial\Phi}{\partial x} + \frac{\partial\xi}{\partial t}, \quad r \in S_2(t), \tag{6}$$

$$\frac{\partial\Phi}{\partial y} = 0, \quad y = -H_b + t; \quad \frac{\partial\Phi}{\partial x} = 0, \quad x = \pm H_R, \tag{7}$$

$$\Phi(x, y, 0) = \Phi_0(x, y), \quad \xi(x, 0) = 0, \quad \eta(\theta, 0) = 0. \tag{8}$$

The potential of the velocities $\Phi_0(x, y)$ acquired by the liquid particles at the moment immediately following the impact and the initial separation zone are based on the solution of the classical model of an impact with a separation [1]:

$$\Delta\Phi_0 = 0, \quad r \in \Omega(0), \tag{9}$$

$$\frac{\partial\Phi_0}{\partial n} = -n_y, \quad \Phi_0 \le 0, \quad r \in S_{11}, \tag{10}$$

$$\frac{\partial\Phi_0}{\partial n} \ge -n_y, \quad \Phi_0 = 0, \quad r \in S_{12}(0), \tag{11}$$

$$\Phi_0 = 0, \quad y = H, \tag{12}$$

$$\frac{\partial\Phi_0}{\partial y} = 0, \; y = -H_b; \quad \frac{\partial\Phi_0}{\partial x} = 0, \; x = \pm H_R. \tag{13}$$

Due to the unknown separation zone $S_{12}(0)$, the problem (9)–(13) is nonlinear and belongs to the class of problems with free boundaries. According to [15], its solution exists and is unique. Let's pay attention to the inequalities that are formulated in the contact and separation zones (formulas (10) and (11)). These inequalities are quite equivalent to the Kutta-Joukowski condition at the points of separation. The inequality in the formula (10) means that the impulsive pressure $p_\tau = -\rho\Phi_0$ must be non-negative everywhere on the wetted surface of the body (in the contact zone). The inequality in the formula (11) prohibits liquid particles from entering the solid body, although it does not prevent their separation from the solid surface. These inequalities are convenient to use when solving the problem of impact with separation by direct numerical methods. The Kutta-Joukowski condition turns out to be effective in constructing analytical solutions.

Note an important property of the regularity of the solution of the problem (9)−(13), which consists in the fact that the function Φ_0 in the vicinity of the point of separation of boundary conditions behaves as $r^{1.5}$, where r — radius is the vector of the corresponding point (in the usual mixed problem of potential theory, the exponent is 0.5). This property ensures that the Kutta-Joukowski condition is fulfilled at the initial moment of time.

Dimensionless variables are introduced using equalities

$$t' = \frac{a}{V_0}t, \ x' = ax, \ y' = ay, \ \Phi' = aV_0\Phi, \ p' = \rho V_0^2 p,$$

where dimensional quantities are marked with strokes.

The fixed X, Y coordinates are related to the movable x, y ratios: $\mathrm{X} = x$, $\mathrm{Y} = y + h(t)$, where $h(t)$ is the law of motion of the cylinder. It is assumed that the axis y is directed against the gravity vector, the origin is in the center of the cylinder. The fluid flow pattern is symmetric with respect to the axis y.

The main characteristic quantities in the problem are the Froude number Fr and the dimensionless pressure difference χ:

$$Fr = \frac{V_0}{\sqrt{ga}}, \quad \chi = 2\,\frac{p_a - p_c}{\rho V_0^2},$$

where p_a — atmospheric pressure; p_c — pressure in the cavity (with artificial cavitation, dependence on coordinates and time is allowed); g — acceleration of gravity; ρ — liquid density; V_0 — speed acquired by the cylinder as a result of impact, a — cylinder radius.

The following designations are also used in this paper: Φ — the velocity potential of the absolute motion of the fluid, recorded relative to the moving coordinate system; $\Omega(t)$ — the fluid flow region; S_{11} — the part of the cylinder surface on which there is no separation of liquid particles; $S_{12}(t)$ — internal free boundary (cavity boundary); $S_2(t)$ — external free surface of the liquid ($y = H$ — its undisturbed level); θ_s, $\pi - \theta_s$ — angular coordinates of separation points; $\mathbf{V_0} = (0, -V_0)$ — the velocity vector of the cylinder ($V_0 > 0$), $h(t) = -t$ — its dimensionless law of motion; $y - H_b$ — the bottom of the pool; $x = \pm H_r$ — symmetrical side walls; r — radius vector with coordinates (x, y); r, θ — cylindrical coordinates ($x = R\cos\theta$, $y = R\sin\theta$).

Dynamic and kinematic conditions are formulated on previously unknown free boundaries. Dynamic conditions are written on the basis of the Cauchy-Lagrange integral in a moving coordinate system. It is assumed that atmospheric pressure acts on the outer free boundary ($p = p_a$), and the pressure distribution on the inner free boundary depends on the method of artificial gas supply to the cavity ($p = p_c$). The kinematic condition is that liquid particles never leave the free boundary. The forms of the internal and external free boundaries are determined using the equalities:

$$R = 1 + \eta(\theta, t); \quad y = H + \xi(x, t) + t.$$

The kinematic equations (4), (6) are obtained by differentiating these equalities in time along the trajectory of motion of a liquid particle located on an internal or external free boundary. Polar coordinates R, θ are used to derive the kinematic equation of the internal free boundary of the fluid.

The Kutta-Joukowski condition is set at the points of intersection of the inner free boundary with the cylinder surface (at the points of separation) meaning that the velocity of the liquid at these points should be finite.

2. Asymptotics of solving the problem at small times

First of all, it should be noted that the points of separation of the internal free boundary of the liquid can be kept stationary after impact, mainly due to artificial cavitation. It is assumed that the pressure of the gas entering the cavity from the side of the body has a linear dependence on time. As a result, χ is represented in the form

$$\chi = \chi_0 + \chi_1 t + f(\theta)t, \quad f(\theta) = \chi_2 f_1(\theta) + \chi_3 f_2(\theta),$$

where χ_0, χ_1, χ_2, χ_3 are assumed to be constant values. The functions $f_1(\theta)$ and $f_2(\theta)$ are even relative to the point 0.5π, and their derivatives have root and logarithmic singularities at the separation point, respectively (in this case, the coefficients χ_1, χ_2, χ_3 can be chosen in such a way that the third asymptotic approximation satisfies the Kutta-Joukowski condition).

The terms „root singularity" and „logarithmic singularity" in derivatives mean that the asymptotic formulas ($\theta \to \theta_s + 0$) are valid in a small neighborhood of the separation point:

$$f_1'(\theta) \sim \mathrm{const} \cdot (\theta - \theta_s)^{-0.5}, \quad f_2'(\theta) \sim \mathrm{const} \cdot \ln(\theta - \theta_s).$$

Next, we proceed to the solution of the main problem, which consists in constructing a regular asymptotic expansion of the velocity potential by degrees of small time with a special choice of these coefficients, as well as the Froude number. Such a choice of parameters justifies the assumption of the immobility of the separation points.

Here and further, the term „regular function" means that the first derivatives of this function should be continuous at the separation points.

The solution of the problem (1)−(13) will be sought in the form of the following asymptotic expansions:

$$\Phi = \Phi_0 + t\Phi_1 + t^2\Phi_2 + \cdots, \tag{14}$$

$$\eta(\theta, t) = t\eta_0(\theta) + t^2\eta_1(\theta) + t^3\eta_2(\theta) + \cdots, \tag{15}$$

$$\xi(x,t) = t\xi_0(x) + t^2\xi_1(x) + t^3\xi_2(x) + \cdots, \tag{16}$$

where the ellipsis denotes members of a higher order of smallness by t.

Substituting the decompositions (14)−(16) in the equation and boundary conditions of the problem (1)−(13), transferring the boundary conditions from the disturbed sections of the boundary of the region $\Omega(t)$ to the initially undisturbed level using the Taylor formula and then equating the coefficients with the same powers t, we arrive to mixed boundary value problems of potential theory in the initially undisturbed domain to define the functions Φ_1 and Φ_2 $\Omega(0)$. The solutions of the latter problems are presented in the form:

$$\Phi_1 = 0.5\chi_0 u + v, \quad \Phi_2 = 0.25\chi_1 u + w,$$

where the functions u, v, w are harmonic in the domain $\Omega(0)$. These functions satisfy the condition that the normal derivative is equal to zero on the wetted surface of the body ($r = 1$, $-\pi - \theta_s < \theta < \theta_s$) and on the side walls ($x = \pm H_R$). The boundary conditions of the first kind are met for them in the separation zone ($r = 1$, $\theta_s < \theta < \pi - \theta_s$) and on the outer boundary ($y = H$) (formulas (17), (18) corresponds to the separation zone):

$$u = 1, \quad v = Fr^{-2}(H - \sin\theta) - \frac{\partial \Phi_0}{\partial r}\sin\theta - 0.5\Big(\frac{\partial \Phi_0}{\partial r}\Big)^2, \tag{17}$$

$$w = -\frac{\partial \Phi_1}{\partial r}\eta_0(\theta) + 0.5\frac{\partial \Phi_0}{\partial r}[\eta_0^2(\theta) + \cos^2\theta]$$

$$- 0.5Fr^{-2}[\eta_0(\theta)\sin\theta - 1 - \cos^2\theta] + 0.25 f(\theta), \tag{18}$$

$$u = 0. \quad v = -\frac{\partial \Phi_0}{\partial y} - 0.5\Big(\frac{\partial \Phi_0}{\partial y}\Big)^2,$$

$$w = -0.5Fr^{-2}\frac{\partial \Phi_0}{\partial y} - \frac{\partial \Phi_1}{\partial y} - \frac{\partial \Phi_0}{\partial y}\frac{\partial \Phi_1}{\partial y}, \quad y = H.$$

In addition, the following ratios are valid at the bottom:

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = \frac{\partial^2 \Phi_0}{\partial x^2},$$

$$\frac{\partial w}{\partial y} = 0.5\frac{\partial^3 \Phi_0}{\partial y \partial x^2} + \frac{\partial^2 \Phi_1}{\partial x^2}, \quad y = -H_b.$$

The value $\eta_0(\theta)$ in the formula (18), as well as other decomposition coefficients (15) are represented as:

$$\eta_0(\theta) = a_1 + \sin\theta, \quad 2\eta_1(\theta) = a_4 - a_1\eta_0(\theta) - \eta_0'(\theta)\cos\theta,$$

$$\begin{aligned}3\eta_2(\theta) =& a_6 + (1.5a_1^2 + a_2^2)\eta_0(\theta) - a_4(1.5a_1 + \sin\theta)\\ &+ a_2\cos\theta(5.5a_1 + 3\sin\theta + Fr^{-2})\\ &+ a_1(3\cos^2\theta - Fr^{-2}\sin\theta) + 0.5a_3(\eta_0^2(\theta)\\ &+ \cos^2(\theta)) - 0.5a_5\cos\theta + Fr^{-2}\cos 2\theta.\end{aligned}$$

The coefficients of the asymptotic formula (16) are determined by the equalities

$$\xi_0(x) = b_1, \quad 2\xi_1(x) = b_4,$$

$$3\xi_2(x) = (b_1 + 1)(0.5b_3(b_1 + 1) + b_2^2) + b_5.$$

The following notation is used in the last formulas:

$$a_1 = \frac{\partial \Phi_0}{\partial r}, \quad a_2 = \frac{\partial^2 \Phi_0}{\partial r \partial \theta}, \quad a_3 = \frac{\partial^3 \Phi_0}{\partial r \partial \theta^2},$$

$$a_4 = \frac{\partial \Phi_1}{\partial r}, \quad a_5 = \frac{\partial^2 \Phi_1}{\partial r \partial \theta}, \quad a_6 = \frac{\partial \Phi_2}{\partial r},$$

$$b_1 = \frac{\partial \Phi_0}{\partial y}, \quad b_2 = \frac{\partial^2 \Phi_0}{\partial x \partial y}, \quad b_3 = \frac{\partial^3 \Phi_0}{\partial y \partial x^2},$$

$$b_4 = \frac{\partial \Phi_1}{\partial y}, \quad b_5 = \frac{\partial \Phi_2}{\partial y},$$

where derivatives by r and θ are calculated at $r = 1$, and differentiations by x and y are carried out at $y = H$.

Further discussion is based on the statement that the derivatives of the functions u, v, w at the angular coordinate θ at $\theta \to \theta_s - 0$ ($r = 1$) have root features. This allows selecting the coefficients χ_0 and χ_1 in such a way that the functions Φ_1 and Φ_2 are regular at the separation points (coefficients are calculated with growing terms). Drawing an analogy with the papers [2,3], in which the regularity of the velocity potential was ensured by choosing the position of the separation points (with fixed physical parameters of the problem), we come to the following expressions for the coefficients χ_0 and χ_1:

$$\chi_0 = -\frac{2d_2}{d_1}, \quad \chi_1 = -\frac{4d_3}{d_1}, \tag{19}$$

$$d_1 = \lim_{\theta \to \theta_s - 0} \frac{\partial u}{\partial \theta}\sqrt{\theta_s - \theta},$$

$$d_2 = \lim_{\theta \to \theta_s - 0} \frac{\partial v}{\partial \theta}\sqrt{\theta_s - \theta},$$

$$d_3 = \lim_{\theta \to \theta_s - 0} \frac{\partial w}{\partial \theta}\sqrt{\theta_s - \theta}.$$

It is important to note that for the validity of the last formulas, it is necessary to require that the boundary functions in (17), (18) be regular at the separation points (i.e. derivatives of these functions by θ must be continuous at the specified points). The singularity of the corresponding derivative in θ at $\theta \to \theta_s - 0$ may not be a root singularity in case of violation of this condition, and the reasoning leading to formulas (19) may be incorrect. This situation is most easily explained by the example of a mixed problem for a half-plane, where there is an exact solution. It can be obtained using the Keldysh-Sedov formula or based on the method of paired integral equations. The analysis of this solution shows that if the derivative of the boundary function in the condition of the first kind has a root singularity, then the singularity of the corresponding derivative of the velocity potential is stronger than the root one (the coefficient for the root singularity has the form of a divergent integral). Another logarithmic multiplier is introduced in this case. We will also provide physical considerations that indicate the need for such regularity. Since the internal free boundary approaches the separation point tangentially (this follows from solution (21) for the boundary layer), it can be assumed that in the small neighborhood of this point, liquid particles located on the free boundary actually lie on the arc of the circle (i.e., on the boundary of the body). It follows from this that violation of the regularity condition of boundary functions formulated above means unlimited velocities of liquid particles located on the inner free boundary near the separation point.

The boundary condition for the function v in (17) automatically satisfies the formulated requirement. In fact, the derivative of the function v by θ ($r = 1$, $\theta_s < \theta < \pi - \theta_s$) has the form

$$\frac{\partial v}{\partial \theta} = -\frac{\partial^2 \Phi_0}{\partial r \partial \theta}\eta_0(\theta) - \frac{\partial \Phi_0}{\partial r}\cos\theta - Fr^{-2}\cos\theta.$$

The first term in this expression has no singularity at the point of separation, since, due to the regularity of solving the classical problem of a blow with separation, the function $\eta_0(\theta)$ at $\theta \to \theta_s + 0$ behaves like $\mathrm{const} \cdot (\theta - \theta_s)^{0.5}$, and the second derivative of the function Φ_0 by r and θ as $\mathrm{const} \cdot (\theta - \theta_s)^{-0.5}$. The second term also contains no singularity due to the continuity of the first derivatives of the function Φ_0 at this point. Thus, the regularity of the function Φ_1 is ensured only by a special choice of the coefficient χ_0 (formula (19)). At the same time, χ_0 depends on the Froude number, which at this stage (taking into account only the first two terms of the asymptotics) can be chosen arbitrarily.

Now let's analyze the boundary condition for the function w (formula (18)). Unlike (17), its regularity is no longer obtained automatically. This function will be continuous at the point of separation, and its derivative by θ in general will have root and logarithmic singularities there. An expression in which all the terms containing these features are collected is provided below:

$$\frac{\partial^2 \Phi_0}{\partial r \partial \theta}\left[-2\frac{\partial \Phi_0}{\partial r}\sin\theta - \left(\frac{\partial \Phi_0}{\partial r}\right)^2 - 1 + Fr^{-2}\sin\theta\right] - 0.5\chi_2 f_1'(\theta),$$

$$2\,\frac{\partial^2 \Phi_1}{\partial r \partial \theta}\,\eta_0(\theta) + 2\,\frac{\partial^2 \Phi_0}{\partial r \partial \theta}\frac{\partial \Phi_1}{\partial r} - 0.5\chi_3 f_2'(\theta).$$

The presence of a root singularity in the first formula follows from the properties of the solution of the classical model of an impact with a separation. Therefore, we will focus on the explanation of the logarithmic singularity in the second formula. Let's find out how the normal derivative of the function Φ_1 behaves near the separation point. Since this problem is local, it can be explained by a specially selected example that has an analytical solution. Consider a mixed boundary value problem of potential theory in a half-plane with an ejected semicircle, where there are two symmetric points of separation of the boundary conditions of the first and second kind on the arc of the circle (the boundary functions here are the same as for Φ_1). This problem is reduced to a mixed boundary value problem in a half-plane using a conformal mapping for which an exact analytical solution is constructed. The analysis of this solution shows that the normal derivative of the function Φ_1 near the point of separation of the boundary conditions (from the side of the domain of setting the condition of the first kind) has the following representation: first comes the root feature (which is occupied in the main problem by choosing the parameter χ_0), and then the first small term having the form $\mathrm{const}(\theta - \theta_s)^{0.5}\ln(\theta - \theta_s)$. Based on the reasoning carried out, it can be argued that the first two terms in the last formula have purely logarithmic features.

Now the coefficients χ_2, χ_3 can be chosen in such a way that the boundary function (18) is continuously differentiable at the separation points. As a result, all the terms of the asymptotic expansion of the velocity potential will satisfy the Kutta-Joukowski condition.

The case of $\chi_2 = 0$, $\chi_3 = 0$ is considered separately (the simplest law of artificial cavitation). Here, the root singularity in the expression for the derivative of the boundary function can be eliminated by a special choice of the Froude number. To do this, it is necessary to require that the expression in square brackets tends to zero at $\theta \to \theta_s + 0$. As a result, we arrive at the following equality:

$$Fr^2 = \frac{\sin\theta_s}{\cos^2\theta_s}. \tag{20}$$

At the same time, a weak logarithmic feature remains. However, the solution obtained for the third approximation can be successfully used as an approximation. In this regard, we note that the logarithmic multiplier in the expression for the normal derivative of the function Φ_1 cannot be detected numerically. At the same time, the ratio of this normal derivative to the square root behaves relatively stable (with an error of only a few percent) in the range $\theta - \theta_s = 0.005{-}0.1$. This can only be explained by the presence of a very small coefficient before the logarithm (a coefficient of the order of one would be immediately noticeable). It is not possible to carry out numerical calculations in a smaller neighborhood of the separation point. Note also that the derivative of the boundary function containing a logarithmic singularity can be smoothed in a small neighborhood of the separation point. Such smoothing can be justified by a slight change in the law of artificial cavitation. As a result, the third approximation will strictly satisfy the Kutta-Joukowski condition. At the same time, the difference between strict and approximate solutions will be practically invisible.

Now let's focus on determining the shape of the internal free boundary of the liquid. Analysis of the asymptotic formula (15) shows that the coefficients $\eta_0(\theta)$, $\eta_1(\theta)$, $\eta_2(\theta)$

behave as follows near the separation point:

$$\eta_0(\theta) \sim \beta(\theta - \theta_s)^{0.5},$$

$$\eta_1(\theta) \sim -0.25\beta \cos\theta_s (\theta - \theta_s)^{-0.5},$$

$$\eta_2(\theta) \sim -24^{-1}\beta \cos^2\theta_s (\theta - \theta_s)^{-1.5},$$

where β is determined numerically based on the first of these equalities.

Thus, the decomposition (15) is not applicable near the separation point. These features can be smoothed out at short times using a special solution for the boundary layer, which is constructed by analogy with [2,3] (i.e., such a solution that is effective near the separation point). Its final form is provided below:

$$\eta(\theta, t) = \beta t^{1.5} F(\tau) + \cdots, \quad \tau = \frac{\theta - \theta_s}{t}, \tag{21}$$

$$F(\tau) = \frac{2}{3\cos\theta_s}\tau^{1.5}, \quad 0 < \tau < \cos\theta_s,$$

$$F(\tau) = \frac{2}{3\cos\theta_s}\left[\tau^{1.5} - (\tau - \cos\theta_s)^{1.5}\right], \quad \cos\theta_s < \tau < \infty.$$

A slight difference from the works [2,3] consists in the fact that for $\tau \to \infty$, the alignment with the external decomposition is based on the first three (and not two) terms of the asymptotics. In addition, the methods of boundary layer theory are applied here to the original, and not to the transformed problem.

3. Numerical implementation and analysis of results

For the numerical solution of the classical problem of impact with separation, a special iterative method is used to sequentially refine the zones of separation and contact of liquid particles unknown in advance [2,3]. According to this method, the nonlinear problem (9)–(13) is reduced to the sequential solution of linear boundary value problems (with fixed points of the boundary conditions section), which are solved numerically by the finite element method, using the package FreeFem++ [16]. Linear boundary value problems arising for functions u, v, w are also solved using this package.

When considering specific numerical examples, the following geometric parameters of the problem were fixed: $H = 1.2$, $H_b = 5$, $H_R = 5$, and the coefficients χ_2, χ_3 were assumed to be zero (the simplest law of artificial cavitation is considered). For the angular coordinate of the separation point, as well as the Froude number and coefficients χ_0 and χ, the following approximate values were obtained: $\theta_s = 0.584$, $Fr = 0.89$, $\chi_0 = -2.99$, $\chi_1 = -9.81$.

Figure 1–4 shows the dynamics of the cavity at time points $t = 0.1$, 0.2, 0.3, 0.4. Fig. 2 demonstrates a good agreement of solutions for the boundary layer with external decomposition in almost the entire range of variation of the angular coordinate θ. This makes it possible to describe the shape of the cavern at small times ($0 < t < 0.2$) using simple analytical formulas. Note that the constructed solution for the boundary layer does not depend on the values Fr and χ. Consequently, the good agreement noted above suggests that the physical parameters begin to affect the shape of the internal free boundary of the liquid only at $t > 0.2$. Note also that the difference between the forms of free boundaries obtained on the basis of two and three terms of the asymptotics becomes noticeable for times greater than $t = 0.3$. This is the basis for the application of the proposed method in the specified time range.

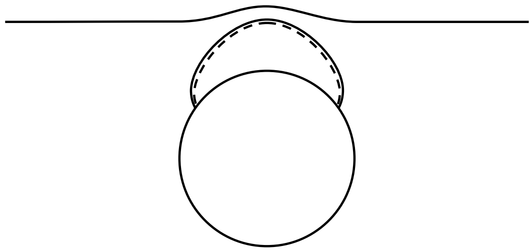

Figure 5. The shape of the cavity at $t = 0.3$; the stroke shows the solutions obtained by the formula (21) at $\beta = 2.0$.

Let us pay attention to the following interesting fact obtained by numerical experiments. The coefficient β in the formula (21) can be chosen in such a way that for $t = 0.3$, 0.4, the boundary layer solution approximates the external decomposition very well over the entire range of the angular coordinate θ. Figure 5 shows the alignment of these decompositions at $t = 0.3$ ($\beta = 2.0$, cf. with Figure 3).

It is important to note that the numerical value found for the value χ_0 can be verified in another way. To this end, we will consider a similar problem taking into account the dynamics of separation points. Previously, the following asymptotic formula was obtained for the angular coordinate of the separation point [3]:

$$\theta_s = \theta_{s0} + c_1 t.$$

If the value χ_0 is found correctly, then the coefficient c_1, determined from the regularity condition, should go to zero (due to the assumption of the immobility of the separation point). Numerical calculations carried out using the method proposed in [2,3] show that the coefficient $c_1 \approx 0.002$. Given the small error given by the numerical program, we can assume that $c_1 = 0$. Similar checks were performed for various other values Fr and $\chi_0 = \chi_0(Fr)$: $\chi_0(0.5) = -7.10$; $\chi_0(1) = -2.38$; $\chi_0(5) = -0.87$; $\chi_0(10) = -0.82$. It should be noted that it is necessary to fix the Froude number and determine χ_0 using the formula (19) to obtain the functional dependence $\chi_0(Fr)$.

An alternative another way to find the coefficient χ_0 can be provided (19):

$$\chi_0 = -\frac{2d_5}{d_4}, \quad d_4 = \lim_{\theta \to \theta_s + 0} \frac{\partial u}{\partial r}\sqrt{\theta - \theta_s},$$

$$d_5 = \lim_{\theta \to \theta_s + 0} \frac{\partial v}{\partial r} \sqrt{\theta - \theta_s}, \quad r = 1. \tag{22}$$

χ_0 is determined in it from the continuity condition of the normal derivative of the function Φ_1 for $r = 1$, $\theta = \theta_s$. In other words, the coefficient is calculated for the root feature of this derivative at $r = 1$, $\theta \to \theta_s + 0$. Numerical calculations have shown a good agreement of the approximate values of this coefficient obtained in two different ways (according to formulas (19) and (22)). Similarly, the coefficient χ_1 can be determined.

Finally, it should be noted that for an arbitrary Froude number, the following functions can be selected as $f_1(\theta)$, $f_2(\theta)$ (the derivative of r is taken for $r = 1$):

$$f_1(\theta) = \eta_0(\theta), \quad f_2(\theta) = \frac{\partial \Phi_1}{\partial r} \eta_0(\theta).$$

In this case, the coefficients χ_2 and χ_3 will have the form

$$\chi_2 = 2Fr^{-2} \sin\theta_s - 2\cos^2\theta_s, \quad \chi_3 = 4.$$

Conclusion

The planar problem of the impact with a separation of a circular cylinder and its subsequent movement deep into the liquid at a constant velocity is investigated. It is assumed that after impact, the separation points of the internal free boundary of the liquid remain stationary, at least for some small period of time. It is shown that the solution of such a problem can be constructed in the form of a power-law asymptotic expansion over a short time (taking into account the first three terms of the asymptotics) only with a special choice of characteristic physical quantities (Froude number and pressure in the cavity). An asymptotic analysis of the problem is carried out taking into account the solutions for the boundary layer at the separation points. Concrete examples with numerical solutions are considered.

Conflict of interest

The author declares that he has no conflict of interest.

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