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Methods for detecting phase transitions in complex dynamic systems

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The paper proposes two methods for detecting transitions between laminar and turbulent behavior using the example of unidirectionally coupled Ressler systems in the band chaos regime. Both methods are based on calculating the probability of diagnosing the turbulent phase and introducing Poincar sections. Depending on the choice of method, you can either reduce the time it takes to find phase transitions or increase the accuracy of determining the moment of transition between two modes.

Keywords: generalized synchronization, intermittent behavior, unidirectionally coupled systems, Poincare section.

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Synchronization [1] and the accompanying intermittency mechanisms [2], which emerge near the boundary of synchronization, are common to dynamic systems of various nature. A large number of types of synchronous behavior are already known. For example, full, generalized, phase, and noise-induced synchronizations, which are found in radiophysics, chemistry, geology, and other branches of modern science and in living systems, are distinguished, and all of them are associated with different types of intermittent behavior [3,4]. Specifically, eyelet intermittency is observed near the phase synchronization boundary in the case of fairly weak detuning between chaotic systems [5], and ring intermittency is found under the conditions of strong frequency detuning [6]. At the same time, jump intermittency is observed at the boundary of generalized synchronization in systems with a relatively complex (two-sheeted) structure [7], while on-off intermittency occurs in the case of a simpler attractor topology [4].

Intermittency emerges when laminar and turbulent behavior phases alternate in complex systems. In intermittent synchronization regimes, laminar phases correspond to the synchronous behavior of interacting systems, and turbulent outbursts manifest themselves when the synchronization regime breaks down. Statistical characteristics of length of laminar and turbulent phases are one of the key tools for distinguishing between different types of intermittent behavior [8]. Therefore, it is important to determine correctly the moment of transition between laminar and turbulent phases.

In the case of intermittent generalized synchronization, the auxiliary system approach [9] is the most efficient method for identifying laminar and turbulent phases in the dynamics of coupled systems, although this method has a significant constraint: it is applicable only to unidirectionally coupled systems [10]. It allows one to detect synchronous and asynchronous behavior of unidirectionally coupled chaotic oscillators (driving and driven) at each point in time. The auxiliary system approach requires the introduction of an additional system, which is identical to the driving one in all respects except for the initial conditions [9]. By definition, a unique functional dependence is established between the states of driving and driven systems and driving and auxiliary systems in the generalized synchronization regime [9,11], resulting in identity of states of driven and auxiliary systems; in the case of asynchronous dynamics (turbulent phase), driving and auxiliary systems have different states [4]. In addition, it was demonstrated in [12] with the use of the auxiliary system method that the intermittent generalized chaotic synchronization regime is characterized by multistability: depending on the initial conditions for the driven system, both synchronous and asynchronous dynamics may be observed at one and the same point in time. A „driven–auxiliary“ pair of systems is not sufficient for the examination of multistability, since only one state (synchronous or asynchronous) out of two possible ones is observed in this case even in a multistable state. The authors of [12] have proposed a way to characterize this phenomenon and determine more accurately the moment of switching between laminar and turbulent phases: one should consider an ensemble of auxiliary systems, which may be regarded as an ensemble of driven systems, and calculate the probability of observation of synchronous (laminar) and asynchronous (turbulent) behavior of unidirectionally coupled systems at each point in time. The greater the number of systems N in an ensemble is, the higher is the accuracy of calculation of the probability value at each point in time; however, the amount of computational resources needed to solve the discussed problem also increases. At the same time, an excess accuracy of probability calculation is often unnecessary for the solution. Practically speaking, an ensemble of several hundred systems is sufficient to obtain valid results. The time dependence of probability of detection of a turbulent (or laminar) phase may be used to identify the corresponding phases in a more correct fashion and, consequently, obtain more accurate statistical characteristics of intermittent behavior.

However, the implementation of this approach evidently requires a large amount of calculations. First, the states of at least several hundred dynamic systems need to be calculated simultaneously, while only three of them were considered in the classical auxiliary system method. Second, the probability of diagnosing a turbulent (laminar) phase also needs to be calculated at each point in time based on the states of all systems. Since the duration of transition from a laminar phase to a turbulent one (and vice versa) in the intermittent generalized chaotic synchronization regime exceeds considerably the length of a characteristic period of oscillations of interacting oscillators and the flow systems themselves may be reduced to lower-dimensional maps with the use of the Poincaré section [13,14], one may estimate the probability of detection of a turbulent or laminar behavior phase only at section points, making it unnecessary to perform such calculations at each point in time and reducing the overall computation time.

The aim of the present study is to develop new methods for identifying the transitions from a turbulent phase to a laminar one (and vice versa). The proposed techniques provide a higher sensitivity to interphase transitions or simplify the calculation of states of examined systems.

The object under examination is an ensemble of unidirectionally coupled weakly non-identical Rössler systems in the band chaos regime:

$$\begin{aligned} \dot{x}_d &= -\omega_d y_d - z_d, \\ \dot{y}_d &= \omega_d x_d + a y_d, \\ \dot{z}_d &= p + z_d(x_d - c), \\ \dot{x}_r^i &= -\omega_r y_r^i - z_r^i + \varepsilon(x_d - x_r^i), \\ \dot{y}_r^i &= \omega_{ri} x_r^i + a y_r^i, \\ \dot{z}_r^i &= p + z_r^i(x_r^i - c), \end{aligned} \tag{1}$$

where indices d and r correspond to driving and driven systems, respectively; index $i = 1, \ldots, N$ ($N = 500$ was considered in the present study) denotes the driving system number; these systems have different initial conditions; $a = 0.15$, $p = 0.2$, $c = 10$, $\omega_d = 0.93$, and $\omega_r = 0.95$ are the control parameters for interacting systems; and ε is the coupling parameter. To avoid the „numerical calculation trap“ with the states of driven systems in the synchronous motion phase appearing to be identical within the number format precision in random access memory (if this occurs, the systems are characterized by identical dynamics even in the turbulent phase), we used the standard technique of random selection of slightly different parameter ω_{ri} values for driven systems: $\omega_{ri} = \omega_r + \Delta\omega_i$, where $|\Delta\omega_i| < 10^{-7}$. The generalized synchronization regime is established in system (1) at the chosen values of control parameters and

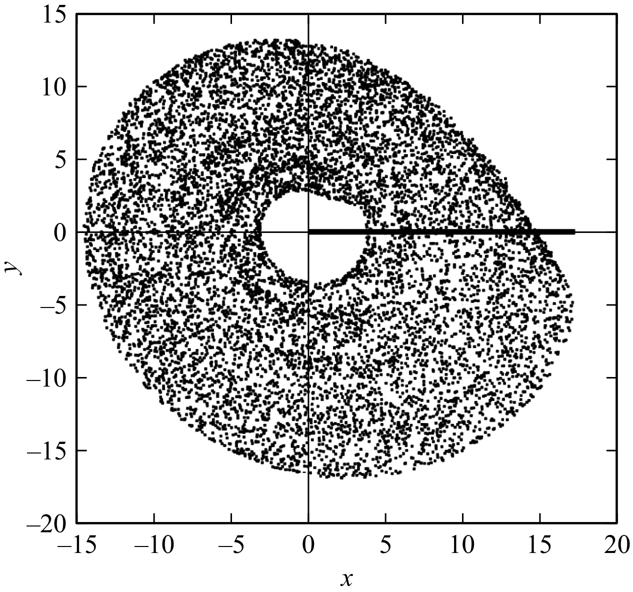

Figure 1. Example (x, y) projection of the phase portrait of the driving system with a Poincaré section line indicated.

$\varepsilon = 0.178$. The probability of observation of asynchronous behavior [12] is given by

$$P_a = 1 - \sum_{i=1}^{N} \frac{n(x_r^i)}{N(N-1)}, \tag{2}$$

where $n(x_r^i)$ is the number of systems in synchrony with the i-th oscillator and N is the number of oscillators in the ensemble. When probability P_a is near-unity, the turbulent behavior phase is detected in the examined system; when P_a is close to zero, the laminar phase is observed. Setting the threshold of distinction between laminar and turbulent phases, one may distinguish characteristic phases of behavior of interacting systems. $P_L = 0.1$ was chosen as the threshold value.

A Rössler system attractor with the Poincaré section introduced is shown in Fig. 1. In order to illustrate the feasibility of the proposed approach, we compared the probabilities of diagnosing the turbulent behavior regime calculated in various ways for the examined systems within an extended time interval corresponding to the turbulent phase. The first technique relied on the method from [12] with probability P_a determined at each trajectory point. The second technique for determination of the probability of diagnosing the turbulent regime involved P_a calculation only at Poincaré section points.

Figure 2, a presents the results obtained when the probability was calculated only in the Poincaré section (method 1). The data corresponding to techniques 1 and 2 are represented by the gray curve and black dots, respectively. As expected, all dots fit the curve corresponding to the first calculation technique. It is evident that these dots are quite sufficient to determine accurately the boundaries of the

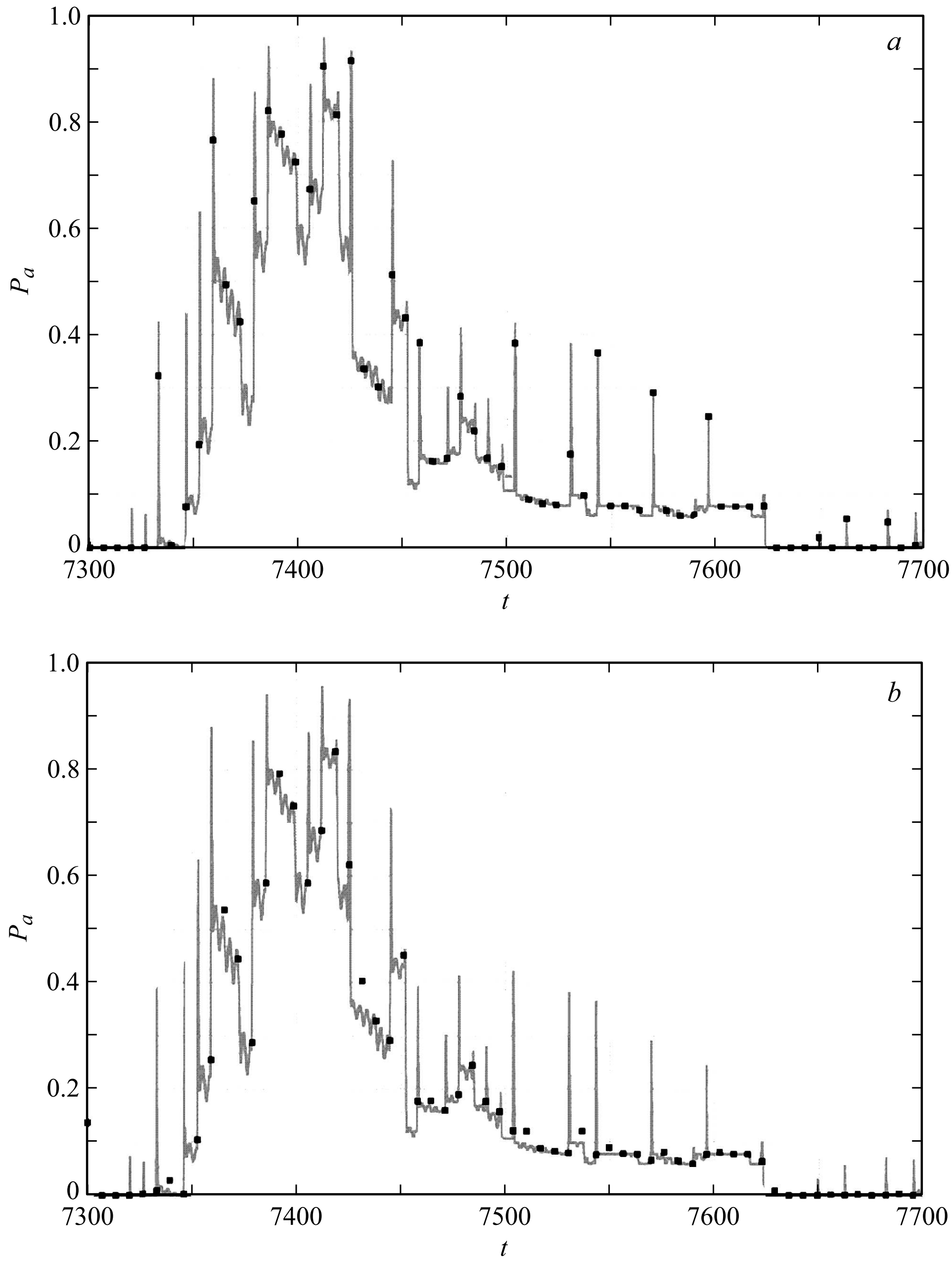

Figure 2. Example time dependences of probability of detection of a turbulent phase $P_a(t)$ calculated in different ways. The gray curve represents the probability calculated at each point in time, and black dots are the results of probability determination with the use of the Poincaré section. a — Probability is calculated in the Poincaré section only (method 1); b — probability is calculated at each point in time and averaged between Poincaré sections (method 2).

turbulent phase. It should also be noted that „spikes," which complicate the search for an exact moment of transition from the turbulent regime to the laminar one (and vice versa), are observed in the $P_a(t)$ dependence obtained with the probability calculated at each point. Although the use of the Poincaré section helps reduce the time of probability calculation, these spikes are still inherited partially from the first calculation technique.

One may use a different modification of the initial method to get rid of the mentioned spikes in dependence $P_a(t)$: calculate the probability for each moment in time and perform averaging within the time intervals between two Poincaré sections (method 2). The corresponding results are shown in Fig. 2, b. The amount of calculations is not reduced in this case, but it is evident that average probability values smooth out the spikes of probability of detection of the turbulent phase, allowing one to identify more accurately and easily the moments of transition between different regimes.

Thus, applying the proposed approaches in scenarios involving the analysis of collective dynamics of systems (e.g., in diagnostics of epileptic discharges [15,16]), one may identify efficiently the temporal boundaries of turbulent and laminar phases in less time (method 1) or enhance the accuracy of detection of moments of transition by removing artifacts (method 2).

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Conflict of interest

The authors declare that they have no conflict of interest.

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