Graphene in the magnetic field with constant gradient

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The possibility of the carriers' confinement in graphene by the magnetic field with constant gradient is considered. The obtained results are compared with the classical description of the plasma motion within magnetically neutral sheet in the earth geomagnetic tail. The consideration is carried out within the original strictly gauge invariant approach making use of the additional integral of motion, so called pseudo-momentum. The essential role of the tunneling effect distinguishing quantum case from classical behavior is revealed.

Keywords: graphene, magnetic field, tunneling effect, classical and non-relativistic quantum cases.

1. Introduction

Graphene is considered to serve as a promising basis for electronic and optoelectronic devices. The necessary condition for the implementation of this task is the possibility to control the electron behavior. The commonly achieved control using the electric fields presents a problem for graphene due to the Klein effect [1–3] implying the Dirac electron tunneling through arbitrary high and wide potential barriers upon normal incidence. The considered alternative is to use inhomogeneous magnetic fields for carriers confining by magnetic barriers and magnetic dots created by ferromagnetic microstructures or by superconducting stripes. The various magnetic field configurations were proposed and considered in the number of the papers [4–8]. In our work we consider the graphene carrier behavior in the perpendicular static magnetic field with the constant gradient as compare with its classical counterpart problem of plasma motion within neutral sheet in the earth geomagnetic tail. The problem of the gauge invariance of the formulated equations determining the eigenwave function received primary emphasis in our work. The ensuing qualitative analysis of the possible solutions allows to reveal the peculiarities of the carrier's behavior in graphene in the vicinity of the magnetic field neutral line. Also, we compare obtained results with the predictions for 2D non-relativistic quantum motion.

2. Motion in classical and non-relativistic quantum cases

The heliospheric current sheet is the boundary between the oppositely directed magnetic fields regions originating in the nature due to the Sun magnetic dipole field. The problem of the plasma motion within such magnetic field configuration has been first formulated by Störmer [9] and had received subsequently much consideration in connection with the description of the auroral phenomena and cosmic rays [10]. For the qualitative understanding of the peculiarities of the behavior of the charged particles propagating within and near the current sheet very often the simplified variant of magnetic field configuration with constant gradient is used. Following [11,12] we consider the inhomogeneous magnetic field of the form

$$\mathbf{B}(\mathbf{r}) = B_0 \frac{x}{L}\,\hat{\mathbf{e}}_z. \tag{1}$$

Where B_0 is a constant and L is the constant gradient length scale. This seemingly oversimplified example of the inhomogeneous magnetic field nevertheless reveals the main peculiarities of the stated problem leading to the formation of the current sheet confining the particle motion along neutral magnetic field lines. Moreover, as it has been pointed in [11] that „practically any two-dimensional magnetic field can be locally approximated by the magnetic field with constant gradient", the proposed problem is important in elucidating the possibility of the electron control in graphene by inhomogeneous magnetic field of this kind. Before we start with the analysis of the carrier behavior in graphene, it is instructive to consider the classical description of the plasma motion and compare it with the quantum non-relativistic problem of the charge movement in such a field. The classical equations of motion for the particle with the charge $|q|$ are ($\hbar = c = 1$)

$$m\frac{dv_x}{dt} = \frac{x}{Ll_{B_0}^2}\,v_y, \quad m\frac{dv_y}{dt} = -\frac{x}{Ll_{B_0}^2}\,v_x. \tag{2}$$

Here $l_{B_0}^2 = 1/|q|B_0$ is square of the „magnetic length". We are interested only in result of the integration of the latter equation (as the classical variant of this problem has been discussed in detail in [12,13]) which is

$$\pi_y = -\frac{x^2}{2Ll_{B_0}^2} + k_y. \tag{3}$$

Where $\pi_y = mv_y$ and $\partial_t k_y = 0$. The emerging constant of the integration of momentum dimensions k_y has its origin in the problem of the particle movement in homogeneous magnetic field where it corresponds through simple scaling to the X component of the radius-vector of the gyration center [14]. The guiding center approximation play essential role in the description of the classical plasma behavior in strong fields enabling to decouple fast helical motion about a local magnetic line from the slow bounce and drift along and across it [15–17]. With the help of the conserved quantity k_y it can be shown [11,12] that in the considered field configuration there exist two variants of the charged particle drift in the opposite directions depending on whether the particle trajectory crosses neutral line or not [11,18]. The quantum mechanical behavior of the charged particle confined to $X-Y$ plane in the considered magnetic field is described in the non-relativistic case by the following Hamiltonian

$$\hat{H} = (\hat{\pi}_x^2 + \hat{\pi}_y^2)/2m. \qquad (4)$$

Where $\hat{\pi}_i = -i\partial_y - |q|A_i$. The classical quantity k_y transforms according to quantum mechanical rules into the corresponding operator $k_y \to \hat{k}_y$ while remaining extra conserved quantity due to the easily checked property $[\hat{H}, \hat{k}_y] = 0$. These operators are known as the pseudo momentums operators [19] connected through scaling to the guiding center operators [20,21] which are intensively used in quantum mechanical description of the particle motion in the homogeneous static magnetic field [22,23]. The problem arising within quantum mechanical description is an ambiguity in the necessary choice of the particular vector potential form. The only constrain upon the possible variant is the condition that $\mathbf{B}(\mathbf{r}) = [\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})]$. In our case we can rely on the symmetry of the problem and fix $\mathbf{A}(\mathbf{r}) = (0, (x^2B_0)/2L)$ thus assuming that the wave functions must describe the free propagation along Y axis. Nevertheless, the question arises whether we are to proceed with this fixed gauge, or are we can start with any suitable one. The fixing of the vector potential in the problem of homogeneous constant magnetic field $\mathbf{B}(\mathbf{r}) = B_0\hat{\mathbf{e}}_z$ in the Landau gauge $\mathbf{A}(\mathbf{r}) = (0, xB_0)$ or in the symmetric gauge $\mathbf{A}(\mathbf{r}) = (-yB_0/2, xB_0/2)$ leads to the conflicting conclusions. In the former case we obtain free propagation along Y axis, while in the latter the solutions are the set of the bounded wave functions. Following the commonly accepted statement that we are free in the choice of $\mathbf{A}(\mathbf{r})$ specific spatial dependence, let us try in our problem the so called Poincare gauge [24] which appears to be best suited for analyzing behavior in the inhomogeneous fields as it is defined readily only by the magnetic field spatial dependence

$$\mathbf{A}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 u\mathbf{B}(\mathbf{r}u)du. \qquad (5)$$

The vector potential components expressed in the explicit form in this gauge for chosen gradient field are

$$|q|A_x = -\frac{yx}{3Ll_{B_0}^2}, \quad |q|A_y = \frac{x^2}{3Ll_{B_0}^2}. \qquad (6)$$

Notice that written down vector potential components do not take into account the symmetry of the problem, while correctly reproducing the spatial behavior of the considered field (1). Now the introduced before pseudo-momentum operator $\hat{k}_y$ comes into play. As this extra conserved quantity commutes with the 2D Hamiltonian we can label the eigenfunctions $\Psi(\mathbf{r})$ by its eigenvalues

$$\hat{k}_y\Psi(\mathbf{r}) = \left[-i\partial_y - |q|A_y + \frac{x^2}{2l_{B_0}^2 L}\right]\Psi(\mathbf{r}) = \lambda\Psi(\mathbf{r}). \qquad (7)$$

Correspondently, the wave functions can be presented as

$$\Psi(\mathbf{r}) = e^{i\lambda y}e^{-ix^2y/6Ll_{B_0}^2}\varphi(x). \qquad (8)$$

Where $\varphi(x)$ in the non-relativistic case is the solution of the equation

$$\left[-\partial_x^2 + \left(\lambda - \frac{x^2}{2Ll_{B_0}^2}\right)^2\right]\varphi(x) = \varepsilon\varphi(x). \qquad (9)$$

Thus we arrive at the equation we would obtain by choosing the gauge from symmetry considerations. It can be shown that this result remains true for any vector potential form if we make use of the outlined $\hat{k}_y$ operator property.

The solutions of the Eq. (9) demonstrate qualitative difference in the quantum mechanical predictions as compare with the classical description. As in classical picture there also exist two distinct types of the motion depending on the λ sigh. For $\lambda < 0$ we are to deal with the problem of the quantum anharmonic oscillator [25,26]. For $\lambda > 0$ emerges no less well-investigated problem of the double-well potential [27]. As it follows from (), for $\lambda < 0$, the possible solutions describe a carrier motion only along the neutral line within the confines of the 1D anharmonic „magnetic“ potential $(x^2/2Ll_{B_0}^2 + |\lambda|)^2$. For $\lambda > 0$, the double-well potential $(x^2/2Ll_{B_0}^2 - |\lambda|)^2$ also confines a particle to the neutral line due to its „instanton“ property [28]. It is interesting that in the both cases the particle cannot leave the current sheet. In the double-well case this is due to the tunneling effect which prevents the particle localization in one of the effective potential minimums permissible under classical consideration. Thus, the drifting in the opposite directions particles within classical consideration differ additionally in that are their trajectories crossing neutral line or not. It is interesting to investigate if quantum born tunneling effect can play any role in the observed plasma behavior in the heliospheric current sheet but this task lies far beyond the scope of the present article. There remains the question of the possibility to interpret $\hat{k}_y$ as observable. We cannot ascribe to its eigenvalue λ/m in the given state the meaning of the particle average velocity describing its flow along Y axis direction. The problem is that the physically meaningful gauge invariant quantity is the current density defined as [29]

$$j_y(\mathbf{r}) = \frac{i}{2m}\left[(\partial_y\Psi^*(\mathbf{r}))\Psi(r) - \Psi^*(\mathbf{r})\partial_y\Psi(\mathbf{r})\right] - \frac{|q|}{m}A_y\Psi^*(\mathbf{r})\Psi(\mathbf{r}). \quad (10)$$

In considered problem the quantum average of the current operator $\hat{J}_y$ according to (10) is (see (8), (9))

$$\langle \hat{J}_y\rangle = \int j_y(\mathbf{r})d\mathbf{r} = \frac{1}{m}\left(\lambda - \frac{1}{2Ll_{B_0}^2}\langle x^2\rangle\right). \quad (11)$$

Where $\langle x^2\rangle = \int x^2|\varphi(x)|^2dx()$. It follows from (11) that for $\lambda < 0$ the sign of pseudo-momentum points to the right direction of particle propagation but for $\lambda > 0$ this is true if and only if $\lambda > \langle x^2\rangle/2Ll_{B_0}^2$. In the both cases the absolute value of the average current operator is not determined only by λ value as it could be expected but depends essentially on the transverse sideways bounce $\langle x^2\rangle$. Two limiting situations stand out. The presence of the current for $\lambda = 0$, and its absence (localized state) for $\lambda = \langle x^2\rangle/2Ll_{B_0}^2$. The first case is interesting in that λ viewed upon as the characteristics of the particle drift along the neutral line is absent altogether from the current expression and particle propagation is determined solely by its „x“ behavior. The issue of the „proper“ choice of the vector potential arises in many problems. It is worth mention e. g. the paper [30]. The authors argued that in the perpendicular magnetic field with the constant asymptotics $B(r \to \infty) = B_0$ there can exist the zero energy solution depending only on the sign of the magnetic field in the infinite boundary. Their argumentation is based largely on the choice of the symmetric gauge in describing carrier behavior in the asymptotic region. At the same time, following common prescription we are free to choose linear Landau gauge. But this choice (which is commonly used for the description of the states in the homogeneous magnetic field) if used as such, leads to un-renormalized states and destroys their derivation. The way out is in the gauge invariant approach outlined above leading unambiguously to uniqueness of the symmetry form of the vector potential choice.

3. Current sheet magnetic field configuration in graphene

The introduced above in the non-relativistic problem the gauge invariant pseudo-momentum operator $\hat{k}_y$ remains valid as the extra conserving quantity for the description of the low-energy envelope states in the single layer graphene in the chosen gradient magnetic field. This results from the commutation properties of this pseudo-momentum operator. It is easy to verify that $\hat{k}_y$ commutes not only with $\hat{\pi}_x^2 + \hat{\pi}_y^2$ but obeys more strict commutation relations $[\hat{k}_y, \hat{\pi}_i] = 0$. This ensures $\hat{k}_y$ commutation with the Dirac-like Hamiltonian describing carriers behavior in graphene within $\mathbf{k}\cdot\mathbf{p}$ approach and thus it can be of use as the corresponding extra quantum number. Due to the valley degeneracy of graphene Hamiltonian it suffices to restrict our consideration to the one of the valleys (say K valley) described by the Hamiltonian $\hat{H} = v_F(\hat{Q}^+\sigma^+ + \hat{Q}^-)$ [31]. Here $\hat{Q} = \hat{\pi}_x + i\hat{\pi}_y$, $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$, and σ_i are Pauli matrixes [1,31]. First, let us consider behavior of the zero-mode states (if existing) described by the first-order partial differential equation

$$(\hat{\pi}_x + i\hat{\pi}_y)\Psi(\mathbf{r}) = 0. \quad (12)$$

As discussed above, we can choose any appropriate gauge subjected only to the condition $\mathbf{B}(\mathbf{r}) = [\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r})]$ if in what follows we adhere to the outlined gradient invariant approach. In deciding on the gauge $\mathbf{A}(\mathbf{r}) = (0, B_0x^2/2L)$ we arrive at the simplest form for $\hat{k}_y = -i\partial_y$ (see discussion above). Labeling the eigenfunctions $\Psi_\lambda^T(\mathbf{r}) = (\varphi_\lambda(x), 0)\exp(i\lambda y)$ by its eigenvalue λ we obtain

$$[-i\partial_x + i(\lambda - x^2/2Ll_{B_0}^2)]\varphi_\lambda(x) = 0. \quad (13)$$

It follows from (13) that $\varphi_\lambda(x) \sim \exp(\lambda x - x^3/6Ll_{B_0}^2)$. Contrary to the behavior in the considered above non relativistic Schrodinger case where a particle is always tied to the neutral line in accord with the classical behavior crossing it hither and thither, the zero-mode carriers in K valley are to escape to $x \to -\infty$ thus destroying the current sheet. The destruction of the zero-modes states by the considered magnetic field configuration is the predictable result. It is caused by the specific property of the zero energy states in graphene. Consider the well-known problem of the charged particle movement in the static homogeneous magnetic field. In distinction to the behavior of the curriers in the higher Landau levels spreading over the both graphene sub-lattices, the motion of the particles in the zero-mode state is confined to the one of the sub-lattices only [1,32]. The choice of the sub-lattice over which to jump is dictated by the magnetic field direction and at which of the two valleys (K or K') the consideration is carried out. It is interesting to note that thus only the zero-energy states demonstrate the „hidden“ C_3 symmetry of graphene smeared by the higher energy states to C_6. It is clear from this consideration that in the gradient field the particle is to change A-lattice sites to the B-sites after crossing the magnetic field neutral line. Moreover, the electron-like character of the wave function must change to the hole-like one. So, the particle at crossing the neutral line finds itself in the „hostile“ environment which leads to the disappearance of the square-integrable zero-modes. Nevertheless, the current sheet conditions can be set up for zero energy states also by choosing the specific configuration of the inhomogeneous magnetic field. The symmetric magnetic field profile relative to the neutral line must be employed. The simplest magnetic field configuration confining the particle current can be chosen as

$$\mathbf{B}(\mathbf{r}) = B_0 \frac{|x|}{L} \mathbf{e}_z. \tag{14}$$

If we choose the vector potential gauge as $A = (0, B_0\theta(x)x^2/2L - B_0\theta(-x)x^2/2L)$ where $\theta(x)$-Heaviside function) and use the corresponding expression for $\hat{k}_y$, we arrive to the following equation for zero-energy states

$$[-i\partial_x + i(\lambda - \theta(x)x^2/(2Ll_{B_0}^2) + \theta(-x)x^2/(2Ll_{B_0}^2)]\varphi_\lambda(x) = 0. \tag{15}$$

Now the asymptotic behavior of $\varphi_\lambda(x) \sim \exp(-|x|^3/6Ll_{B_0}^2)$ is of the desired form. Pay attention that in the both cases the eigenvalue λ lost completely physical interpretation as the quantity describing the particle drift. Due to the absence of one of the pseudo-spinors component in the zero-mode wave function, the corresponding average of the current matrix $I = i[\hat{H}, y] = v_F\sigma_y$ elements vanish identically [33,34]. The general Landau state ($E \neq 0$) is given by the solution $\Phi(\mathbf{r})^T = (\varphi_1(\mathbf{r}), \varphi_2(\mathbf{r}))$ of the matrix equation

$$(-EI + v_F\hat{Q}^+\sigma^+ + v_F\hat{Q}^-)\Phi(\mathbf{r}) = 0. \tag{16}$$

These equations of the first order can be transformed to the equations of the second order by applying to the Eq. (16) the operator $(EI + v_F\hat{Q}^+\sigma^+ + v_F\hat{Q}^-)$

$$(EI + v_F\hat{Q}^+\sigma^+ + v_F\hat{Q}^-) \times (-EI + v_F\hat{Q}^+\sigma^+ + v_F\hat{Q}^-)\Phi(\mathbf{r}) = 0. \tag{17}$$

As a result, we are to solve two Schrodinger-like equations

$$(-E^2 + v_F^2\hat{Q}^+\hat{Q})\psi_1(\mathbf{r}) = 0,$$
$$(-E^2 + v_F^2\hat{Q}\hat{Q}^+)\psi_2(\mathbf{r}) = 0. \tag{18}$$

Which are super-symmetry (SUSY) connected [35,36] as the solutions $\Psi^T(\mathbf{r}) = (\psi_1(\mathbf{r}), \psi_2(\mathbf{r}))/\sqrt{2}$ are subjected to the condition $\psi_2(\mathbf{r}) = v_F\hat{Q}\psi_1(\mathbf{r})/E$ (it is assumed that $\psi_1(\mathbf{r})$ is normalized). With such choice $\Psi(\mathbf{r})$ simultaneously satisfies not only the system (18) but the system of the first-order Eq. (16) also.

As we are left with the solution of the one Schrodinger-like equation, the procedure outlined above for non-relativistic problem can be at once applied for the analysis of carrier spectrum in graphene. For the chosen gradient magnetic field, the explicit form of the corresponding Schrodinger — like operator is

$$v_F^2\hat{Q}^+\hat{Q} = v_F^2(\hat{\pi}_x - i\hat{\pi}_y)(\hat{\pi}_x + i\hat{\pi}_y) = v_F^2[\hat{\pi}_x^2 + \hat{\pi}_y^2 + x/Ll_{B_0}^2]. \tag{19}$$

The difference with the non-relativistic case discussed above resides in the linear in x term (compare with (4)). It is interesting that the addition of this term in our pseudo-Schrodinger problem makes it similar to the non-relativistic problem of the motion in crossed static magnetic and electric fields. The linear in x term can be considered as an effective electrostatic potential leading to constant spatially homogeneous effective „electric“ field $\mathbf{E}_{\text{eff}} = \mathbf{e}_x/|q|Ll_{B_0}^2$. Accordingly, we can state that the carriers in graphene must follow the predictions for nonrelativistic case considered above, demonstrating two distinct types both confined to neutral line, plus the contribution of the drift induced by $\mathbf{E}_{\text{eff}}$. We will not proceed further with the analysis of the wave function solutions which will be considered in detail elsewhere. Our task in the presented paper has been to pave gradient invariant road to the formulation of „proper“ wave equations with the help of pseudo-momentum aka guiding center operator and reveal the main features of the quantum mechanical description of the current sheets distinguished it from the classical motion in the earth geomagnetic tail. Once more we must return to the question of interpretation of $\hat{k}_y$ as observable and its physical meaning. First we must note that $\hat{k}_y$ origin can be traced to the problem of the particle motion in homogeneous field [18,19] where it is shown that it is in one-to one correspondence with the X component of the guiding center operator. In this connection we must once more raise the issue of the physical meaning of $\hat{k}_y$ and its interpretation as an observable. Consider the average $\langle I\rangle = v_F\langle\sigma_y\rangle$ of the current matrix which for the discussed eigenstates is

$$\langle I\rangle = \frac{v_F^2}{E}\left(\lambda - \frac{1}{2Ll_{B_0}^2}\int x^2\varphi_1^2(x)dx\right). \tag{20}$$

It is assumed that in this expression $\varphi_1(x)$ is real. This result predicts non-zero value of the current. This is a peculiar statement for the equilibrium steady states. The authors [37] who has arrived to the same result analyzing carriers' behavior in the step-like magnetic configuration reminiscing our gradient one, propose the existing of the compensating boundary currents. We can assume that in our case this conundrum can be solved if the following restrictions were imposed upon the allowed λ values

$$\lambda := \frac{1}{2Ll_{B_0}^2}\int x^2\varphi_1^2(x)dx. \tag{21}$$

This condition means that only double-well potential is permissible and the corresponding Schrodinger-like equation becomes non-linear. The wave functions and corresponding Landau levels in this setting of the problem will be presented elsewhere.

4. Conclusion

We presented the qualitative description of the carriers' motion in graphene within the neutral current sheet produced by the magnetic field with constant gradient. It is shown that such magnetic field configuration has the most drastic effect upon zero energy states in graphene due to their mixed electron-hole feature. As a result, the square-integrable Landau states with zero-energy cannot exist in such „hostile“ environment in contrast to their counterpart in the homogeneous magnetic field. The specific symmetric 1D magnetic field profile with the neutral region ensuring zero energy states localization on the one of the sublattices is proposed. It is shown that the pseudo-Schrodinger SUSY connected equations describing current sheet states for $E \gtrless 0$ in gradient field are in one-to-one correspondence with the non-relativistic 2D equations describing the particle behavior in the crossed gradient magnetic and „effective“ electric“ ($\mathbf{E}_{\text{eff}} = \mathbf{e}_x/|q|Ll_{B_0}^2$) fields. Notice that in our paper the problem of constructing gauge invariant equations describing the particle motion in gradient field received much consideration. This objective was accomplished invoking the additional integral of motion (constant of integration) operator $\hat{k}_y$. With it help the „uniqueness“ of the choice of the vector potential form for the given magnetic field configuration was proved. The subsequent qualitative analysis of the obtained equations allows to reveal the main differences in the classical description of the plasma motion in the presence of the current sheets and its quantum mechanical counterpart. It is shown that the main difference is due to the tunneling effect being of the primary importance as in the 2D non-relativistic problem, so in the graphene mono-layer confining the carriers to the neutral line. Additionally, we consider the problem of the physical meaning of the pseudo-momentum $\hat{k}_y$ and its interpretation as the quantum mechanical observable.

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References

[1] A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, A.K. Gaim. Rev. Mod. Phys., **81**, 109 (2009).
[2] M.I. Katsnelson. Materials Today, **10**, 20 (2007).
[3] M.I. Katsnelson, K.S. Novoselov, A.K. Geim. Nature Physics, **2**, 620 (2006).
[4] A. De Martino, L. Dell'Anna, R. Egger. Phys. Rev. Lett., **98**, 066802 (2007).
[5] L. Oroszlany, P. Rakyta, A. Kormanyos, C.J. Lambert, J. Cserti. Phys. Rev. B, **77**, 081403 (2008).
[6] J.E. Müller. Phys. Rev. Lett., **68**, 385 (1992).
[7] H.S. Sim, K.-H. Ahn, K.J. Chang, G. Ihm, N. Kim, S.J. Lee. Phys. Rev. Lett., **80**, 1501 (1998).
[8] A. Kormányos, P. Rakyta, L. Oroszlány, J. Cserti. Phys. Rev. B, **78**, 045430 (2008).
[9] C. Störmer. Arch. Sci. Phys. Nat., **24**, 350 (1907).
[10] A.J. Dragt. Rev. Geophys., **3**, 255 (1965).
[11] K. Kabin. Phys. Plasmas, **28**, 122101 (2021).
[12] J.M. Repko, W.W. Repko, A. Saaf. Am. J. Phys., **59**, 652 (1991).
[13] A.J. Brizard. Phys. Plasmas, **24**, 042115 (2017).
[14] Y. Avron. Am. J. Phys., **54**, 659 (1986).
[15] T.G. Northrop. *The Adiabatic Motion of charged Particles* (Interscience Publishes, N.Y., 1963).
[16] T.G. Northrop. Ann. Phys., **15**, 79 (1961).
[17] J.R. Cary, A.J. Brizard. Rev. Mod. Phys., **81**, 693 (2009).
[18] P.W. Seymour. J. Phys., **12**, 309 (1959).
[19] G. Konstantinou, K. Moulopoulos. Eur. J. Phys., **37**, 065401 (2016).
[20] M.H. Jonson, B.A. Lippmann. Phys. Rev., **76**, 828 (1949).
[21] E.L. Rumyantsev, A.V. Germanenko. *The guide to the guiding center aka pseudo-momentum construction*, arXiv:2301.11641 [cond-mat. mes-hall] (2023).
[22] A. Malkin, V.I. Man'ko. JETP, **55**, 1014 (1968).
[23] R. Ferrari. Phys. Rev. B, **42**, 4598 (1990).
[24] Wesley E. Brittin, W. Rodman Smythe, Walter Wyss. Am. J. Phys., **50**, 693 (1981).
[25] C.M. Bender, T.T. Wu. Phys. Rev., **184**, 1231 (1969).
[26] A.V. Turbiner, J.C. del Valle. J. Phys. A, **54**, 295404 (2021).
[27] B. Holstein. Am. J. Phys., **54**, 338 (1988).
[28] E. Shuryak, A.V. Turbiner. Phys. Rev. D, **98**, 105007 (2018).
[29] L.D. Landau, L.M. Lifshitz. *Quantum mechanics: Non-relativistic theory* (Pergamon Press Ltd, 1977).
[30] [J.S. Ardenghi, A. Juan, V. Orazi, L. Sourrouille. Phys. E: Low-Dim. Syst. Nanostruct., **109**, 225 (2019).
[31] D.P. DiVincenzo, E.J. Mele. Phys. Rev. B, **29**, 1685 (1984).
[32] M.O. Coerbig. Rev. Mod. Phys., **83**, 1193 (2011).
[33] S. Kuru, J. Negro, L.M. Nieto. J. Phys.: Condens. Matter, **21**, 455305 (2009).
[34] L. Cohnitz, W. Hausler, A. Zazunov, R. Egger. Phys. Rev. B, **92**, 085422 (2015).
[35] F. Cooper, A. Khare, U. Sukhatme. Phys. Rep., **251**, 267 (1995).
[36] G. Junker. *Supersymmetric Methods in Quantum and Statistical Physics* (Springer Verlag Berlin-Heidelberg, Berlin, 1996).
[37] T.K. Ghosh, A. De Martino, W. Häusler, L. Dell'Anna, R. Egger. Phys. Rev. B, **77**, 081404(R) (2008).

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