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Parallel multi-qubit evolution in the quantum non-demolition interaction protocol

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To date, two-qubit quantum logic gates have become an essential part of quantum computing. The study of methods for their implementation is an important practical problem for various quantum optical and information applications. In this paper, we have considered a protocol for multimode quantum non-demolition interaction between an atomic ensemble and multimode light with orbital angular momentum in order to analyze the applicability of this protocol in discrete variables computing. We have developed a multimode interaction Hamiltonian and analyzed the dynamics of field and atomic variables depending on the structure of the driving field. We discuss in detail the procedure for separating qubit noninteracting subsystems on a set of atomic and field modes for various values of the orbital angular momentum of the driving field. Such a procedure helps to reduce the considered protocol to the parallel evolution of two-qubit closed systems.

Keywords: quantum non-demolition interaction, Laguerre-Gaussian modes, orbital angular momentum, qubits, discrete variables.

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1. Introduction

As of this date, various quantum computing schemes have been proposed for both discrete [1] and continuous variables [2], many of which have been successfully implemented experimentally [3,4]. Working with quantum systems in terms of continuous variables has both a number of undeniable advantages [5] and disadvantages, which include the finite squeezing resource of real quantum oscillators [6] and the complexity of the experimental implementation of non-Gaussian transformations for carrying out an arbitrary computational operation [7]. Calculations in terms of discrete variables are devoid of these difficulties, but they also have a key factor limiting their use, namely, the fundamentally probabilistic nature of a successful computational operation, which, when scaling the computational scheme, will obviously lead to an increase in computation time and the loss of quantum supremacy. Therefore, the efforts of many scientific groups are aimed at studying various physical systems and processes, such that performing operations on them would be most effective. To construct universal quantum computation in discrete variables, there must be a way to implement a universal set of quantum logical operations [8]. In other words, any valid computational operation should be reduced to a finite sequence of gates from the universal set. In our previous paper [9] we proposed a method for implementing quantum single-qubit gates, and also generalized qubit protocols to higher-dimensional systems — qudits. In this paper, we explore the prospects of multimode quantum non-demolition interaction for the construction of a two-qubit logic transform, with future plans to also extend the analysis to higher-dimensional systems.

One promising resource for quantum computing in discrete variables is light with orbital angular momentum (OAM) [10], since the OAM projection can take on any integer value, which allows to work in high-dimensional Hilbert space [11]. Laguerre-Gaussian modes, which have orbital momentum, also have high stability when propagating in a turbulent atmosphere, that is, they show a relatively high decoherence time [12]. Since we are talking about well-localized spatial modes, there are now reliably proven methods for generating, separating and detecting such multimode radiation, based on the use of phase holograms [13], q-plates [14] and systems of cylindrical lenses [15]. However, to perform effective transformations over modes with different OAM using such optical elements, it is required to change the system parameters specifically for each mode, which cannot be satisfactory for quantum computing schemes. The potential of a high-dimensional Hilbert space for computation in discrete variables can be assessed dually: we can either introduce several qubit systems on a set of physical states, or work with a smaller number of higher-dimensional objects-qudits. The advantages of one approach or another remain an open question, since working with multiple qubits can cause difficulties in initiating the input multiqubit state, while working with qudits is more difficult in some cases to ensure coherence of the transformation of different qudit states [9]. Constructing single-mode quantum logic gates over single photons with OAM was successfully attempted [9,16,17]. At the same time, constructing two-mode gates over such

systems is a non-trivial task due to the absence of direct (without intermediaries) interaction between light states.

In our paper, we constructed a multimode Hamiltonian of quantum non-demolition interaction and analyzed the dynamics of field and atomic variables. We discuss in detail the procedure for isolating qubit non-interacting subsystems across multiple atomic and field modes. We show that qubit coding methods, in which the system can be separated into non-interacting parts, turn out to be different for different interaction modes, switching between which is possible by changing the spatial profile of the driving field. The correct procedure for encoding the logical states of qubits helps reduce the protocol under consideration to the parallel evolution of several two-qubit closed subsystems.

2. Quantum non-demolition interaction between light with OAM and an atomic ensemble

2.1. Model and Hamiltonian of interaction

We review an atomic ensemble with 4-level atoms (Fig. 1). We believe that the ensemble of atoms is localized in a cylindrical cell with area S and length L along the axis z. We assume that the lower energy levels $|1\rangle$ and $|2\rangle$ are longlived and neglect the decay of the levels during the light-atomic interaction We believe that the atoms are initially prepared in the $|1\rangle$ state, the average collective spin of which is directed along the x axis, and the magnetic spin moment is equal to $m_x = -\frac{1}{2}$. The $|3\rangle$ and $|3'\rangle$ levels are practically not populated due to the large detuning of Δ

The ensemble interacts with the strong classical driving field $\mathbf{E}_d(\mathbf{r}, t)$, which is a quasi-monochromatic wave in the paraxial approximation at the frequency ω_0, and the quantum field $\hat{E}_s(\mathbf{r}, t)$ with the carrier frequency also ω_0. The frequency ω_0 is detuned by the amount $-\Delta$ from the frequencies of atomic transitions $\omega_{13'}$ and ω_{23}, respectively. We will review quantum and classical fields in the form of a superposition of modes with a certain angular momentum:

$$\hat{E}_s(\mathbf{r}, t) = \frac{-i\sqrt{\hbar\omega_0}}{\sqrt{8\pi c}} \sum_m \hat{a}_m(z, t) U_m^{(s)}(\boldsymbol{\rho}) e^{ik_0 z - i\omega_0 t} \mathbf{e}_y + H.c., \quad (1)$$

$$\mathbf{E}_d(\mathbf{r}, t) = -i \sum_n E_n(z, t) U_n^{(d)}(\boldsymbol{\rho}) e^{ik_0 z - i\omega_0 t} \mathbf{e}_x + c.c. \quad (2)$$

Here $\mathbf{e}_x, \mathbf{e}_y$ are polarization vectors, which are selected in such a way that the control classical field acts on the transitions $|1\rangle - |3'\rangle$ and $|2\rangle - |3\rangle$, and the quantum field acts on the transitions $|1\rangle - |3\rangle$ and $|2\rangle - |3'\rangle$, $U_m^{(s)}(\boldsymbol{\rho})$, $U_n^{(d)}(\boldsymbol{\rho})$ — functions Laguerre-Gaussian quantum and classical fields, respectively, the operators $\hat{a}_m(z, t)$ have the meaning of annihilation operators acting in a mode with a certain projection of the orbital angular momentum m onto the propagation axis m.

It is worth mentioning here that, using the results of the paper [18], we assume the cell length L to be quite

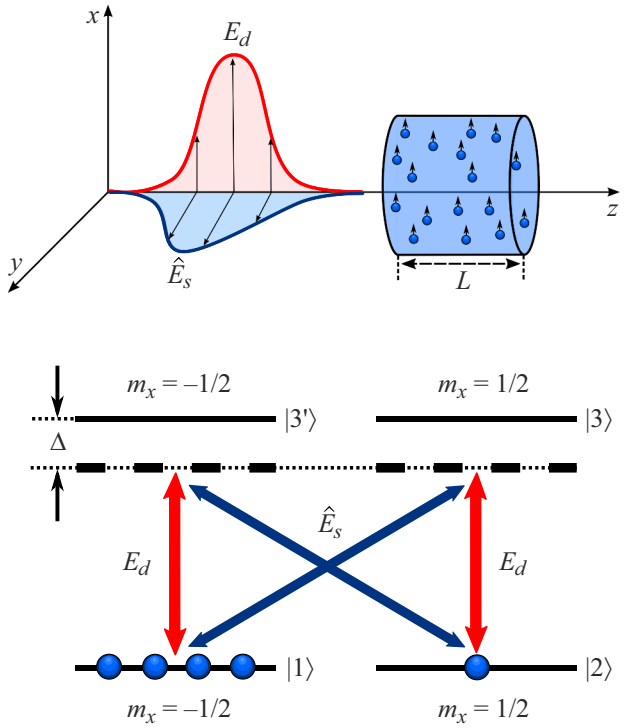

Figure 1. The geometry of interaction of the driving and quantum fields with the atomic ensemble, as well as a schematic representation of atomic levels.

small compared to the Rayleigh radius of classical and quantum beams and neglect diffraction at interaction scales. It was shown in the paper [19] that in the waist plane the transverse spatial profile of the Laguerre-Gaussian mode is a ring of radius $w\sqrt{|l|+1}/2$, where w is the beam waist radius, the index l is responsible for the projection of the orbital angular momentum. The cross-sectional area of the beam is defined as $S_l = \pi w^2 \frac{(|l|+1)}{4}$. As part of the atomic-light interaction under consideration, the presence of both the driving classical and quantum signal fields is required for the implementation of two-photon transitions. The field operators and classical field amplitude in free space are usually defined in such a way that the square of the classical field amplitude and the quantum field particle number operator have the meaning of the flux of photons through the cross-sectional area of the mode per unit time. To correctly describe the interaction of an ensemble of atoms with classical and quantum light beams with different areas, we renormalize the amplitudes, operators and mode functions to the cross-sectional area of the mode, thus moving to dimensionless amplitudes and operators that have the same normalization regardless of the mode index.

In order to ensure the best overlap of the transverse spatial profiles of the classical and quantum fields, we assume that the field waists are different, and we will further vary the ratio of these waists, controlling the overlap of the modes and, consequently, the effective interaction constants. Then the functions of the spatial distribution of amplitude

in the transverse plane for quantum and classical fields are defined as normalized Laguerre-Gaussian distributions:

$$U_l^{(j)}(\boldsymbol{\rho}) = \sqrt{\frac{(|l|+1)}{2|l|!}}\left(\frac{\boldsymbol{\rho}\sqrt{2}}{w^{(j)}}\right)^{|l|}\exp(-\frac{\boldsymbol{\rho}^2}{(w^{(j)})^2})\, e^{il\phi}, \quad (3)$$

$$\sum_l U_l^{(j)}(\boldsymbol{\rho})\, U_l^{(j)*}(\boldsymbol{\rho}\,') = \delta^{(2)}(\boldsymbol{\rho}-\boldsymbol{\rho}\,'), \quad (4)$$

$$\int d\boldsymbol{\rho}\, U_l^{(j)*}(\boldsymbol{\rho}) U_{l'}^{(j)}(\boldsymbol{\rho}) = \delta_{l,l'}\pi(w^{(j)})^2\frac{(|l|+1)}{4} = S_l^{(j)}, \quad j = s, d. \quad (5)$$

Here $w^{(j)}$ is the waist radius of the quantum field beam at $j = s$ and the waist radius of the classical field at $j = d$. The commutation relations for field operators $\hat{a}_m(z,t)$ from (1) can be written as

$$\left[\hat{a}_m(z,t), \hat{a}^\dagger_{m'}(z',t)\right] = \frac{c\delta_{m,m'}}{S_m^{(s)}}(1-\frac{i}{k_0}\frac{\partial}{\partial z})\delta(z-z'). \quad (6)$$

To describe the atomic ensemble we will use the collective variables of coherence and population:

$$\hat{\sigma}_{ij}(\mathbf{r},t) = \sum_{k=1}^{N}\hat{\xi}_{ij}^k(t)\ \delta(\mathbf{r}-\mathbf{r}_k), \quad (7)$$

$$\hat{N}_i = \hat{\sigma}_{ii}(\mathbf{r},t) = \sum_{k=1}^{N}\hat{\xi}_{ii}^k(t)\ \delta(\mathbf{r}-\mathbf{r}_k) \quad (8)$$

(index k numbers atoms, N — total number of atoms), operators $\hat{\xi}_{ij} = |i\rangle\langle j|$ — projectors of state $|j\rangle$ onto state $|i\rangle$ at time instant t. The position of the k-th atom is specified by the radius vector $\mathbf{r}_k$. The commutation relations for coherence operators have the form

$$[\hat{\sigma}_{ij}(\mathbf{r},t), \hat{\sigma}_{mn}(\mathbf{r}\,',t)] = (\delta_{jm}\hat{\sigma}_{in}(\mathbf{r},t) - \delta_{ni}\hat{\sigma}_{mj}(\mathbf{r},t)) \times \delta^{(3)}(\mathbf{r}-\mathbf{r}'). \quad (9)$$

According to [20], the Hamiltonian of quantum non-demolition (QND — Quantum non-demolition) interaction in the dipole approximation has the form:

$$\hat{H}_{QND} = \frac{1}{\sqrt{2}}(\hat{H}_1 - \hat{H}_2), \quad (10)$$

where $\hat{H}_1$ — part of the interaction Hamiltonian describing the beam splitter type transformation, and $\hat{H}_2$ describes the parametric generation process. In our case of multimode interaction, it is also possible to isolate two corresponding parts of the Hamiltonian. In the rotating wave approximation, after eliminating terms that vary at double the optical frequency, the Hamiltonians have the form

$$\hat{H}_1 = i\hbar\int d^3r\left[g\hat{\sigma}_{13}^\dagger(\mathbf{r},t)\sum_m \hat{a}_m(z)U_m^{(s)}(\boldsymbol{\rho})e^{-i\Delta t+ik_0z} + \right.$$
$$\left. +\hat{\sigma}_{23}^\dagger(\mathbf{r},t)\sum_n \Omega_n(z,t)U_n^{(d)}(\boldsymbol{\rho})\, e^{-i\Delta t+ik_0z}\right] + H.c., \quad (11)$$

$$\hat{H}_2 = i\hbar\int d^3r\left[g\hat{\sigma}_{23'}^\dagger(\mathbf{r},t)\sum_m \hat{a}_m(z)U_m^{(s)}(\boldsymbol{\rho})e^{-i\Delta t+ik_0z} + \right.$$
$$\left. +\hat{\sigma}_{13'}^\dagger(\mathbf{r},t)\sum_n \Omega_n(z,t)U_n^{(d)}(\boldsymbol{\rho})\, e^{-i\Delta t+ik_0z}\right] + H.c. \quad (12)$$

Here we introduce the notation for the coupling constant of an atom with the field g and the Rabi frequencies $\Omega_n(z,t)$, assigned to the classical field mode with the number n:

$$g = \sqrt{\frac{\omega_0}{8\pi\hbar c}}d_{13}, \quad (13)$$

$$\Omega_n(z,t) = \frac{E_n(\mathbf{r},t)d_{23}}{\hbar}, \quad (14)$$

where d_{ij} — operator matrix elements of the transition dipole moment between the levels $|i\rangle$ and $\langle j|$, which we will consider real ($d_{ij} = d_{ji}$).

Since, within the accepted approximations, the upper atomic levels are not populated during the interaction process, we have the opportunity to carry out adiabatic exclusion of these levels. In this case, only two-photon transitions occur in the system, accompanied by the transition of atoms from the $|1\rangle$ level to the $|2\rangle$ level and back. Meanwhile, since we consider the quantum field to be quite weak compared to the classical one, such transitions occur rarely, and the population of the $|1\rangle$ level can be considered unchanged and equal to N. In this case, as shown in [21], it is possible to use the Holstein-Primakov approximation and replace the spin coherence operators $\hat{\sigma}_{12}(\mathbf{r},t)$ with bosonic operators $\hat{b}(\mathbf{r},t)$. The interaction Hamiltonian can be written as follows:

$$\hat{H}_1 = -\hbar\int dz\ \frac{2g\sqrt{N}}{\Delta}\sum_{m,k,l}\Omega_k^*\hat{b}_l^\dagger\hat{a}_m\times$$
$$\times\int d\boldsymbol{\rho}\, U_k^{(d)*}(\boldsymbol{\rho})U_l^{(s)*}(\boldsymbol{\rho})U_m^{(s)}(\boldsymbol{\rho}) + H.c., \quad (15)$$

$$\hat{H}_2 = -\hbar\int dz\ \frac{2g\sqrt{N}}{\Delta}\sum_{m,k,l}\Omega_k\hat{b}_l^\dagger\hat{a}_m^\dagger$$
$$\times\int d\boldsymbol{\rho}\, U_k^{(d)}(\boldsymbol{\rho})U_l^{(s)*}(\boldsymbol{\rho})U_m^{(s)*}(\boldsymbol{\rho}) + H.c. \quad (16)$$

Here, following the approach [20], we omitted the terms responsible for the Stark effect from classical and quantum fields. New atomic operators $\hat{b}_l(z,t)$ are introduced as projections of bosonic operators $\hat{b}(\mathbf{r},t)$ onto the Laguerre-Gaussian profile with index l:

$$\hat{b}_l(z,t) = \int d\boldsymbol{\rho}\ \ \hat{b}(\mathbf{r},t)U_l^s(\boldsymbol{\rho}), \quad (17)$$

$$\left[\hat{b}_m(z,t), \hat{b}^\dagger_{m'}(z',t)\right] = \frac{\delta_{l,l'}}{S_l^{(s)}}\delta(z-z'). \quad (18)$$

Separately, it would be preferable to discuss the integrals of the Laguerre-Gauss functions that arise in (15), (16).

Using the explicit form of the functions $U_l^{(j)}$, given by expression (3), the form of the integrals can be simplified as follows:

$$\int d\boldsymbol{\rho}\, U_k^{(d)*}(\boldsymbol{\rho}) U_l^{(s)*}(\boldsymbol{\rho}) U_m^{(s)}(\boldsymbol{\rho}) =$$

$$= \int d\boldsymbol{\rho}\, |U_k^{(d)}(\boldsymbol{\rho})||U_l^{(s)}(\boldsymbol{\rho})||U_m^{(s)}(\boldsymbol{\rho})|\delta_{k,l-m} \equiv \chi_{l-m}, \quad (19)$$

$$\int d\boldsymbol{\rho}\, U_k^{(d)}(\boldsymbol{\rho}) U_l^{(s)*}(\boldsymbol{\rho}) U_m^{(s)*}(\boldsymbol{\rho}) =$$

$$= \int d\boldsymbol{\rho}\, |U_k^{(d)}(\boldsymbol{\rho})||U_l^{(s)}(\boldsymbol{\rho})||U_m^{(s)}(\boldsymbol{\rho})|\delta_{k,l+m} \equiv \chi_{l+m}. \quad (20)$$

The presence of the Kronecker symbol in expressions (19), (20) allows to formulate some selection rules for interacting modes, which we will discuss in more detail below. Substituting expressions for the overlap integrals in (15), (16), we obtain expressions for the one-dimensional multimode Hamiltonian:

$$\hat{H}_{QND} = -\frac{\sqrt{2}\hbar g\sqrt{N}}{\Delta} \int dz \sum_{m,l} \left(\chi_{l-m}\Omega_{l-m} \left[\hat{b}_l^\dagger \hat{a}_m + b_l \hat{a}_m^\dagger \right] \right.$$

$$\left. - \chi_{l+m}\Omega_{l+m} \left[\hat{b}_l^\dagger \hat{a}_m^\dagger + \hat{b}_l \hat{a}_m \right] \right). \quad (21)$$

The χ_k coupling constants are the overlap integrals of the transverse spatial profiles of the driving field, quantum field, and spin coherence modes. We have the opportunity to control the value of individual constants geometrically by changing the ratio of the waists of the light fields [9]. It is worth noting that the interaction constants are invariant with respect to changes in the sign of the index: $\chi_{l-m} = \chi_{m-l}$.

2.2. Heisenberg equations and input-output relations

Let us note that the resulting expression for the Hamiltonian (21) does not allow us to clearly see that we are really dealing with a QND interaction. Let us demonstrate that Hamiltonian (21) provides quantum non-demolition interaction of atomic and field systems similar to that described in [20]. To do this, let us move on to another set of modes, which is a superposition of modes with OAM:

$$\hat{a}_0 = \hat{a}_0, \qquad \hat{b}_0 = \hat{b}_0, \quad (22)$$

$$\hat{a}_m^{(+)} = \frac{1}{\sqrt{2}}(\hat{a}_m + \hat{a}_{-m}), \quad (23)$$

$$\hat{a}_m^{(-)} = \frac{1}{i\sqrt{2}}(\hat{a}_m - \hat{a}_{-m}), \quad (24)$$

$$\hat{b}_m^{(+)} = \frac{1}{\sqrt{2}}(\hat{b}_m + \hat{b}_{-m}), \quad (25)$$

$$\hat{b}_m^{(-)} = \frac{1}{i\sqrt{2}}(\hat{b}_m - \hat{b}_{-m}). \quad (26)$$

In addition, for further purposes it will be useful to assume the pump is symmetrical in OAM and assume $\Omega_k = \Omega_{-k}$. To avoid confusion, we will further call a pair of modes with moments k and $-k$ one mode of the driving field, since only the symmetric spectrum is considered. Hamiltonian (21) in new variables and taking into account pump symmetry is significantly simplified:

$$\hat{H}_{QND} = \hat{H}^{(+)} + \hat{H}^{(-)}, \quad (27)$$

$$\hat{H}^{(+)} = \int dz \left[\frac{\kappa_0}{2}\chi_0 \hat{p}_0 \hat{y}_0 + \sum_{k=1}^{\infty} \frac{\kappa_k}{\sqrt{2}}\chi_k \left(\hat{p}_k^{(+)} \hat{y}_0 + \hat{p}_0 \hat{y}_k^{(+)} \right) \right.$$

$$\left. + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \kappa_{|m-k|}\chi_{m-k} \hat{p}_k^{(+)} \hat{y}_m^{(+)} \right], \quad (28)$$

$$\hat{H}^{(-)} = \int dz \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \kappa_{|m-k|}\chi_{m-k} p_k^{(-)} \hat{y}_m^{(-)}, \quad (29)$$

$$\kappa_k = \frac{4\hbar\sqrt{2}g\sqrt{N}}{\Delta}\Omega_k. \quad (30)$$

Here the operators $\{\hat{q}_i^{(\pm)}, \hat{p}_i^{(\pm)}\}$ are quadrature components of the operators $\hat{b}_i^{(\pm)}$ describing the atomic system, $\{\hat{x}_i^{(\pm)}, \hat{y}_i^{(\pm)}\}$ — quadrature components of the operators $\hat{a}_i^{(\pm)}$describing the field system. It can be noted that in the variables we have chosen, the Hamiltonian splits into two non-interacting parts, since operators with different superscripts commute:

$$[\hat{q}_m^j, \hat{p}_k^i] = [\hat{x}_m^j, \hat{y}_k^i] = \frac{i}{2S_m^{(s)}}\delta_{k,m}\delta_{i,j}, \quad (31)$$

$$k, m \in [1, \infty);\ i, j = \{(+), (-)\}.$$

Thus, the Hamiltonian $\hat{H}^{(+)}$ ensures the interaction of atomic and field modes with indices 0 and (+) in the QND type through various modes of the driving field with OAM (the subscript of κ_k indicates the moment of the control field). Modes with a superscript (−) form a closed subsystem and evolve through the Hamiltonian $\hat{H}^{(-)}$ without interacting with modes with other superscripts. For definiteness, we will further, in the context of constructing qubits, follow only the subsystem developed by the Hamiltonian $\hat{H}^{(+)}$, and we will omit the superscript for simplicity of presentation. In addition, we will limit ourselves to considering only those situations where the OAM spectrum of the driving field is limited to one mode with the number k. This is due to the fact that for cases of a multimode driving field, that is, when in (28) we must sum over the index k, each mode of the atomic ensemble interacts with many modes of the quantum field. Meanwhile, the effective integral interaction time, calculated as $T_k = \int_0^T dt\, \Omega_k^2(t)/\Delta^2$, is different for different Ω_k, which leads to even greater complexity in the interaction picture.

To write the input-output relations in the presence of one mode of the driving field Ω_k, we move on to

integral dimensionless quadrature operators according to the following expressions:

$$\hat{X}_m(z) = \frac{\int_0^T \Omega_k(t)\hat{x}_m^s(z,t)dt}{\sqrt{\int_0^T \Omega_k^2(t)dt}}, \quad (32)$$

$$\hat{Y}_m(z) = \frac{\int_0^T \Omega_k(t)\hat{y}_m^s(z,t)dt}{\sqrt{\int_0^T \Omega_k^2(t)dt}}, \quad (33)$$

$$\hat{Q}_l(t) = \frac{1}{\sqrt{L}}\int_0^L \hat{q}_l^s(z,t)dz, \quad (34)$$

$$\hat{P}_l(t) = \frac{1}{\sqrt{L}}\int_0^L \hat{p}_l^s(z,t)dz. \quad (35)$$

The input-output relations for the new quadrature operators, the evolution of which was carried out by the Hamiltonian (28), can be written as follows:

$$\hat{X}_m^{out} = \hat{X}_m^{in} + \tilde{\chi}_{m-k}\hat{P}_{m-k}^{in}, \quad (36)$$

$$\hat{Q}_{m-k}^{out} = \hat{Q}_{m-k}^{in} + \tilde{\chi}_{m-k}\hat{Y}_m^{in}, \quad (37)$$

$$\hat{Y}_m^{out} = \hat{Y}_m^{in}, \quad (38)$$

$$\hat{P}_{m-k}^{out} = \hat{P}_{m-k}^{in}. \quad (39)$$

Here the superscripts in and out are entered according to the following rule: $\hat{X}_m^{in} = \hat{X}_m(z=0)$, $\hat{X}_m^{out} = \hat{X}_m^s(z=L)$, $\hat{Y}_m^{in} = \hat{Y}_m(z=0)$, $\hat{Y}_m^{out} = \hat{Y}_m(z=L)$, $\hat{Q}_{m-k}^{in} = \hat{Q}_{m-k}(t=0)$, $\hat{Q}_{m-k}^{out} = \hat{Q}_{m-k}(t=T)$, $\hat{P}_{m-k}^{in} = \hat{P}_{m-k}(t=0)$ and $\hat{P}_{m-k}^{out} = \hat{P}_{m-k}(t=T)$.

The dimensionless coupling constant for the QND interaction is defined as follows:

$$\tilde{\chi}_{m-k} = \frac{2\sqrt{2}g\sqrt{N}\chi_{m-k}}{\sqrt{S_m^{(s)}S_{m-k}^{(s)}}}\sqrt{\int_0^T \frac{\Omega_k^2(0,t)}{\Delta^2}dt}. \quad (40)$$

Here, as we mentioned earlier, the integral of the square of the Rabi frequency determines the effective integral interaction time and depends on the duration of the driving field pulse T.

In this section, the Hamiltonian of the interaction of multimode light with an OAM and an atomic ensemble with a complex structure of the driving field was analyzed. The features of modes with OAM allowed to formulate selection rules and show that the interaction is selective in OAM. In the basis of sum and difference modes, as well as when using a driving field with a symmetric spectrum along the OAM projection, the Hamiltonian describes the QND interaction of atomic and field modes. In this case, the indices of the interacting modes differ by the value of the OAM of the driving field. In the next section, we will review the resulting operator transformation in terms of the interaction of two qubits, and also pay attention to different ways of encoding qubits for different values of the OAM of the driving field.

3. Two-qubit closed subsystems and their evolution

3.1. Interaction with $k = 0$

For the purposes of this subsection, we will set the moment of the driving field equal to 0. For better clarity of the evolution of physical atomic-field states and logical states provided by the QND interaction of modes with the OAM, it is convenient to first write the input-output relations for field and atomic modes in the form of the Bogoliubov transformation:

$$\begin{pmatrix}\hat{\mathbf{A}}^\dagger\\ \hat{\mathbf{B}}^\dagger\end{pmatrix}^{out} = \begin{pmatrix}\mathcal{I} & \mathcal{S}\\ \mathcal{S} & \mathcal{I}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{A}}^\dagger\\ \hat{\mathbf{B}}^\dagger\end{pmatrix}^{in} + \begin{pmatrix}\mathbb{O} & \mathcal{S}^*\\ \mathcal{S}^* & \mathbb{O}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{A}}\\ \hat{\mathbf{B}}\end{pmatrix}^{in}. \quad (41)$$

Here, for ease of notation, the following designations are introduced: $\hat{\mathbf{A}} = \begin{pmatrix}\hat{A}_0 & \hat{A}_1 & \hat{A}_2 & \cdots\end{pmatrix}^T$, $\hat{\mathbf{B}} = \begin{pmatrix}\hat{B}_0 & \hat{B}_1 & \hat{B}_2 & \cdots\end{pmatrix}^T$. The $\hat{A}_i, \hat{B}_j$ operators are specified by quadrature components defined in (32) –(35), as $\hat{A}_i = \hat{X}_i + i\hat{Y}_i$, $\hat{B}_i = \hat{Q}_i + i\hat{P}_i$, the lower index, as before, is responsible for OAM modes, $\mathcal{I}, \mathbb{O}$ — identity and zero matrices. The matrix $\mathcal{S}$ for a driving field with OAM equal to 0 is diagonal and is defined as follows:

$$\mathcal{S}_{k,j} = \frac{-i\tilde{\chi}_{k-1}}{2}\delta_{k,j}, \quad \{k,j\} \in [1,\infty). \quad (42)$$

Since in the future the logical states of qubits will be encoded by means of the physical states of atomic or field systems with a unit excitation in a state superposition in the OAM, it will be convenient to rewrite the input-output relations (42) only for creation operators:

$$\begin{pmatrix}\hat{\mathbf{A}}^\dagger\\ \hat{\mathbf{B}}^\dagger\end{pmatrix}^{in} = \begin{pmatrix}\mathcal{I} & \mathcal{S}^*\\ \mathcal{S}^* & \mathcal{I}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{A}}^\dagger\\ \hat{\mathbf{B}}^\dagger\end{pmatrix}^{out} + F(\hat{\mathbf{A}}^{out}, \hat{\mathbf{B}}^{out}). \quad (43)$$

Here $F(\hat{\mathbf{A}}^{out}, \hat{\mathbf{B}}^{out})$ — some matrix function only from the annihilation operators at the output of the circuit, ensuring the preservation of commutation relations.

To construct a two-qubit transformation, it is required to select a basis of physical states of the field and atoms, encoding a logical basis in a two-qubit space. Let us select states with a single excitation as basic states in the physical state space:

$$|1\rangle_{m,L} \equiv \hat{A}_m^\dagger|0\rangle_{m,L}, \quad |1\rangle_{k,A} \equiv \hat{B}_k^\dagger|0\rangle_{k,A}. \quad (44)$$

The indices m, k are responsible for the OAM value, L, A indicate the field or atomic system, the superscripts in are

omitted. We can distinguish several closed subsystems: in the case under consideration, interaction occurs only between physical states with the same moment. If we take into account a finite number of states with different OAMs due to experimental difficulties in generating large OAM values, then, by limiting the maximum moment of a quantum state to a certain number K, we can identify closed two-qubit subsystems on the set of physical states $K/2$, defining the states of the qubits as follows:

$$|0\rangle_1^j \equiv |1\rangle_{2(j-1),L}, \quad |1\rangle_1^j \equiv |1\rangle_{2j-1,L}, \tag{45}$$

$$|0\rangle_2^j \equiv |1\rangle_{2(j-1),A}, \quad |1\rangle_2^j \equiv |1\rangle_{2j-1,A}. \tag{46}$$

The $j \in [1, K/2]$ index is responsible for the number of the two-qubit subsystem; index 1 or 2 numbers the qubits within one subsystem. Thus, we assumed that the logical state $|0\rangle_1^1$ of the first qubit of the first subsystem corresponds to a physical state with one photon in a mode with an OAM equal to 0, and the same state of the second qubit corresponds to one excitation in an atomic mode with an OAM equal to 0. The $|1\rangle_1^1$ state can be associated with excitation in a field mode with an OAM equal to 1, and so on. I.e., for every pair of qubits there are two excitations – one in light modes and one in atomic modes. The logical states of the second two-qubit subsystem in the notation used will be encoded through excitations in atomic and field modes with OAM 2 and 3.

Thus, the system under consideration contains many two-qubit subsystems, each of which evolves independently of the others. All coupling constants depend on the same Rabi frequency of Ω_0 and only the overlap integrals of $\chi_{m,k,m-k}$ can be varied (we indicate all three indices here and in the figures for better readability). As was shown in [18], it is possible to control the overlap integrals through the geometry of the fields: by varying the ratio of the waist width of the classical and quantum fields at the entrance to the atomic cell (we use the distance between the waists in units of Rayleigh length as a parameter), we can provide different values of overlap integrals for modes with different numbers. From the graph in Fig. 2 it is clear that with significant shifts in the waist of the classical field relative to the quantum one, i.e. when the waist radius of the driving beam is much larger than the waist radius of the quantum beam (the limit of a plane wave), all overlap integrals tend to unity. Thus, we can consider a system of $K/2$ two-qubit subsystems, where all subsystems interact with the same constants.

3.2. Interaction with $k = 1$

If the driving field has an OAM equal to 1, the interaction picture becomes significantly more complicated in comparison with the situation discussed in the previous subsection. It can be noted that identifying closed two-qubit subsystems is no longer a trivial task, since the states do not interact in pairs. The continuum of field states with even OAM interacts with the continuum of atomic states with

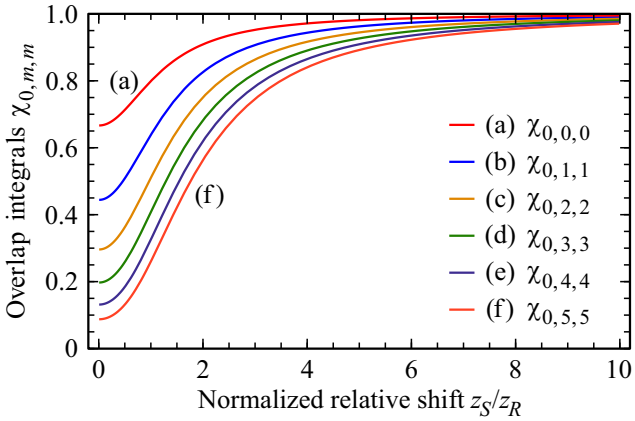

Figure 2. Dependence of the overlap integrals $\chi_{0,m,m}$ on the distance between the control and quantum field waists in units of Rayleigh length. It is easy to notice that all constants asymptotically approach unity at large distances between the waists, i.e. in the limit when the control field is a plane wave.

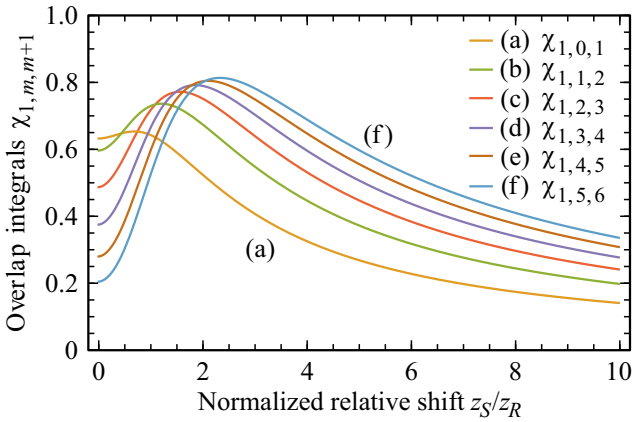

Figure 3. Graph of the dependence of the overlap integrals $\chi_{1,m,m+1}$ on the geometric parameter of the ratio of the waists of the driving and quantum fields. Unlike the $\chi_{0,m,m}$ integrals, in this case it is impossible to specify the value of the geometric parameter at which all integrals become equal in value.

odd numbers and vice versa, while the interaction constants for different states also differ (Fig. 3).

The question arises: how can such a basis of physical states of the system constructed that it becomes possible to identify non-interacting subsystems?

The input-output relations for birth operators are as follows:

$$\begin{pmatrix} \hat{\mathbf{A}}^\dagger \\ \hat{\mathbf{B}}^\dagger \end{pmatrix}^{in} = \begin{pmatrix} \mathscr{J} & \mathscr{H}^* \\ \mathscr{H}^* & \mathscr{J} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{A}}^\dagger \\ \hat{\mathbf{B}}^\dagger \end{pmatrix}^{out} + F(\hat{\mathbf{A}}^{out}, \hat{\mathbf{B}}^{out}) =$$

$$= \mathscr{M} \begin{pmatrix} \hat{\mathbf{A}}^\dagger \\ \hat{\mathbf{B}}^\dagger \end{pmatrix}^{out} + F(\hat{\mathbf{A}}^{out}, \hat{\mathbf{B}}^{out}). \tag{47}$$

Here $F(\hat{\mathbf{A}}^{out}, \hat{\mathbf{B}}^{out})$ — as before, denotes some matrix function only from the annihilation operators at the output

of the circuit, ensuring the preservation of commutation relations. The $\mathscr{H}$ matrix is defined through the following matrix elements:

$$\mathscr{H}_{k,j} = -i\frac{\tilde{\chi}_k\delta_{k+1,j} + \tilde{\chi}_{k-1}\delta_{k-1,j}}{2}, \quad \{k, j\} \in [1, \infty). \quad (48)$$

The $\mathscr{H}$ matrix here contains interaction constants and, unlike $\mathcal{S}$, is not diagonal: non-zero elements are located on two diagonals parallel to the main one. To identify closed subsystems, we turn to the properties of the $\mathscr{M}$ matrix. Let us define the basis of eigenvectors of the matrix $\mathscr{M}$, diagonalizing this matrix:

$$\mathscr{M}_{i,j} = \sum_n \lambda_n m_{n,i} m_{n,j}, \quad (49)$$

where λ_n — eigenvalues depending on all interaction constants $\tilde{\chi}_k$, $m_{n,i}$ — ith element of nth eigenvector of matrix $\mathscr{M}$. Knowing the elements of the eigenvectors, we define a set of eigenoperators $\hat{\mathscr{E}}_n^\dagger$ of the input-output transformation (47) as follows:

$$\hat{\mathscr{E}}_n^\dagger = \sum_{i=1}^{K} m_{n,i}\hat{A}_{i-1}^\dagger + \sum_{i=K+1}^{2K} m_{n,i}\hat{B}_{i-1}^\dagger. \quad (50)$$

The indices in and out are omitted hereinafter and will be indicated only where required. For convenience of notation, we also limit the dimension of the matrix $\mathscr{M}$ to a certain number $2K$, but for now we consider $2K$ to generally tend to infinity.

The $\hat{\mathscr{E}}_n^\dagger$ operators are native to the input-output transformation given by (47), that is, after the transformation they are simply multiplied by the eigenvalues of λ_n. Analysis shows that the $\hat{\mathscr{E}}_n^\dagger$ operators have an interesting property: in a linear combination (50) there will simultaneously be either only field operators with even indices and atomic operators with odd indices, or vice versa - field operators with odd indices and atomic operators with even indices. Meanwhile, in the spectrum of the matrix $\mathscr{M}$ of dimension $2K$ there are only $K/2$ eigenvalues that are different from each other in absolute value. Let us watch which $\hat{\mathscr{E}}_n^\dagger$ operators correspond to eigenvalues equal in modulus

1. A pair of equal eigenvalues λ_1, λ_2 will correspond to a pair of eigenoperators $\hat{\mathscr{E}}_1^\dagger, \hat{\mathscr{E}}_2^\dagger$ such that:

$$\begin{cases} m_{1,s} = m_{2,s+K} \\ m_{1,s+K} = m_{2,s} \end{cases} \quad \forall s \in (1, K/2] \quad \Leftrightarrow \quad \lambda_1 = \lambda_2. \quad (51)$$

That is, if some operator $\hat{\mathscr{E}}_1^\dagger$ includes only field operators with odd OAM with coefficients $m_{1,s}$ and atomic operators with even OAM with coefficients $m_{1,s+K}$, then the operator $\hat{\mathscr{E}}_2^\dagger$ will include only odd atomic operators with weights $m_{1,s}$ and even field operators with weights $m_{1,s+K}$.

2. A pair of complex conjugate eigenvalues λ_1, λ_3 will correspond to a pair of eigenoperators $\hat{\mathscr{E}}_1^\dagger, \hat{\mathscr{E}}_3^\dagger$ such that:

$$m_{1,s} = -m_{3,s} \ \ \forall s \in (1, K] \quad \Leftrightarrow \quad \lambda_1^* = \lambda_3. \quad (52)$$

That is, for each proper operator $\hat{\mathscr{E}}_1^\dagger$ there is an operator $\hat{\mathscr{E}}_3^\dagger$ such that the linear combination $\frac{\hat{\mathscr{E}}_1^\dagger + \hat{\mathscr{E}}_3^\dagger}{\sqrt{2}}$ is expressed only through atomic operators, and $\frac{\hat{\mathscr{E}}_1^\dagger - \hat{\mathscr{E}}_3^\dagger}{\sqrt{2}}$ depends only on field operators.

The matrix $\mathscr{M}$, written for combinations of eigenoperators $\frac{\hat{\mathscr{E}}_i^\dagger \pm \hat{\mathscr{E}}_{i+2}^\dagger}{\sqrt{2}}$, $\frac{\hat{\mathscr{E}}_{i+1}^\dagger \pm \hat{\mathscr{E}}_{i+3}^\dagger}{\sqrt{2}}$, is block-diagonal with block $\mathscr{U}_i$ with dimension 4×4:

$$\mathscr{M} = \begin{pmatrix} \mathscr{U}_1 & \cdots & \mathbb{O} \\ \vdots & \ddots & \vdots \\ \mathbb{O} & \cdots & \mathscr{U}_{2K} \end{pmatrix}, \quad (53)$$

$$\mathscr{U}_i = \begin{pmatrix} 1 & \mathrm{Im}[\lambda_i] & 0 & 0 \\ \mathrm{Im}[\lambda_i] & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{Im}[\lambda_i] \\ 0 & 0 & \mathrm{Im}[\lambda_i] & 1 \end{pmatrix}, \quad (54)$$

$$i = 1, 5, 9, \ldots, 2K - 3.$$

This division of the spectrum of the $\mathscr{M}$ matrix into tetrads of eigenvalues ($\lambda_i = \lambda_{i+1} = \lambda_{i+2}^* = \lambda_{i+3}^*$) allows us to determine groups of physical states that form a closed system with respect to the evolution through QND interaction described by one $\mathscr{U}_i$ block. For example, the first 4 states corresponding to the largest eigenvalue λ_1 can be written as follows:

$$|1\rangle_{A,o} \equiv \frac{1}{\sqrt{2}}(\hat{\mathscr{E}}_1^\dagger + \hat{\mathscr{E}}_3^\dagger)|0\rangle_{A,o}, \quad (55)$$

$$|1\rangle_{L,e} \equiv \frac{1}{\sqrt{2}}(\hat{\mathscr{E}}_1^\dagger - \hat{\mathscr{E}}_3^\dagger)|0\rangle_{L,e}, \quad (56)$$

$$|1\rangle_{A,e} \equiv \frac{1}{\sqrt{2}}(\hat{\mathscr{E}}_2^\dagger + \hat{\mathscr{E}}_4^\dagger)|0\rangle_{A,e}, \quad (57)$$

$$|1\rangle_{L,o} \equiv \frac{1}{\sqrt{2}}(\hat{\mathscr{E}}_2^\dagger - \hat{\mathscr{E}}_4^\dagger)|0\rangle_{L,o}. \quad (58)$$

The subscripts o, e are shorthand for *odd* and *even*. The state defined by equality (55) is the state of the field with one excitation, distributed over a superposition of modes with an even OAM. The probability amplitude to detect a photon in a mode with a specific OAM, equal to, for example, 4, is determined through the fifth element of the first eigenvector of the $\mathscr{M}$ matrix as $\sqrt{2}m_{1,5}$ (50). The remaining states can be described similarly.

As in the previous subsection, we can distinguish closed two-qubit subsystems in the space of physical states $K/2$, defining the states of the qubits as follows:

$$|0\rangle_1^j \equiv |1\rangle_{L,o}^{(4j-3)}, \quad |1\rangle_1^j \equiv |1\rangle_{A,e}^{(4j-3)}, \quad (59)$$

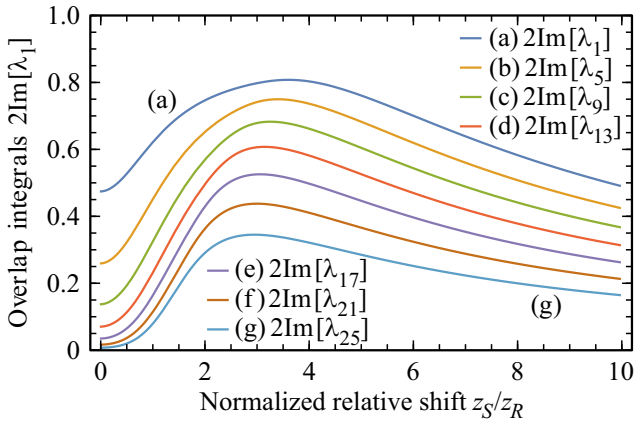

Figure 4. Graph of the dependence of the values of $2\mathrm{Im}[\lambda_i]$ for the first 7 different eigenvalues on the geometric parameter of the ratio of the waists of the driving and quantum fields.

$$|0\rangle_2^j \equiv |1\rangle_{L,e}^{(4j-3)}, \quad |1\rangle_2^j \equiv |1\rangle_{A,o}^{(4j-3)}. \tag{60}$$

The $j \in [1, K/2]$ index is responsible for the number of the two-qubit subsystem; index 1 or 2 numbers the qubits within one subsystem. Figure 4 shows the numerical calculation of the values of $2\mathrm{Im}[\lambda_i]$ (effective interaction constants between qubit states) for the first 7 different eigenvalues. As part of the calculation, we took for simplicity $\tilde{\chi}_m = \chi_m$, i.e. put in expression (40) the entire combination of parameters not related to the overlap integrals χ_m equal to one, keeping in mind, however, that the constants $\tilde{\chi}_m$ can be varied by changing the integral time of interaction, i.e. graph in Fig. 4 can be scaled and the required value of the interaction constants can be achieved by choosing the integral time. The quantities $2\mathrm{Im}[\lambda_i]$ are functions of the overlap integrals and, therefore, depend on the geometric parameter of the ratio of the waists of the control and quantum fields. It is important to note that in the case under consideration it is no longer possible to ensure the same interaction constants for all two-qubit subsystems.

4. Conclusion

In this paper, we reviewed the quantum non-demolition interaction of a multimode quantum field, possessing orbital angular momentum, with an atomic ensemble in the presence of a classical field, also possessing OAM. In the system under consideration, it is possible to write a multimode Hamiltonian of the QND interaction, with the indices of the interacting modes differing by the value of the OAM of the driving field. The interaction constants include the overlap integrals of transverse spatial mode profiles, which allows the constants to be controlled by varying the field geometry.

When trying to consider the evolution of an atomic field system in the language of discrete variables, the question arises of encoding the logical states of qubits through the physical states of the system. The analysis showed that different modes of interaction (determined by the OAM of the driving field) require specific coding. Thus, during evolution with the driving field Ω_0 , atomic and field modes with the same OAM interact, and coding can be carried out directly, identifying states with a unit excitation in a mode with a certain OAM with the logical state of the qubit. The system breaks up into closed two-qubit subsystems, and in the limit of a plane wave of the control field, the interaction constants for all two-qubit systems become equal. This allows us to talk about the parallel evolution of many systems and gives hope in the future to construct a protocol for parallel multi-qubit logical operations.

However, when considering the mode with Ω_1, identifying interacting subsystems turns out to be not so simple, and effectively interacting states turn out to be linear combinations of states with a certain OAM. This procedure guarantees the equality of interaction constants within each two-qubit system. The interaction constants for different subsystems turn out to be different for any value of the geometric parameter of the ratio of the waists of the classical and quantum fields, which can make it difficult to carry out many parallel two-qubit operations. In the future, we plan to develop protocols for various logical two-qubit operations carried out as part of the system under consideration, as well as to attempt to construct transformations in discrete variables for high-dimensional objects (qudits).

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Conflict of interest

The authors declare that they have no conflict of interest.

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