01

Development and generalization of the method of reflections in problems of electrostatics and thermal conductivity of plane-layered media

© A.B. Petrin

Joint Institute for High Temperatures, Russian Academy of Sciences,
Moscow, Russia
e-mail: petrin@ihed.ras.ru, a_petrin@mail.ru

Received November 22, 2022
Revised January 14, 2023
Accepted January 16, 2023

The method of mirror reflections of electrostatics for a point charge located next to a plane-layered medium consisting of a single film on a dielectric half-space is formulated. The method is generalized to the case of an arbitrary system of charges. The proposed approach is applied to mathematically similar problems of electrostatics and stationary heat conduction of plane-layered media. In particular, the problems of finding distributions of the electrostatic potential around a conducting sphere, an ellipsoid of revolution and a drop-shaped body located near the dielectric film on the dielectric half-space. It is shown how to apply the results obtained for electrostatic problems to similar problems of finding the temperature distribution of uniformly heated bodies of the same geometry located near a heat-conducting film in a heat-conducting half-space.

Keywords: plane-layered medium, mirror reflection method, electrostatics, thermal conductivity.

DOI: 10.21883/TP.2023.03.55802.251-22

Introduction

In many modern technical applications, it becomes necessary to analyze electromagnetic fields and heat transfer in plane-layered structures. In problems of heat conduction in plane-layered media, the matrix method [1–4] is used, which, as a rule, is used to solve two-dimensional problems. Such matrix methods are successfully used in solving problems related to the emission and propagation of electromagnetic waves in plane layered media [5]. In the paper [6] an original option of the rigorous electromagnetic theory of the radiation of an elementary dipole located at the boundary or inside a plane-layered structure was proposed being a development of the papers [7,8]. In particular, in [6] a method of analytical solution simplification was demonstrated, which has a potentially important general theoretical value. The generalization of this method for the case of an arbitrary number of films in a plane-layered structure allowed to reduce the formulas for radiated fields to one-dimensional integrals, which significantly simplified the analysis of the problem and accelerated numerical calculations. So, in [9] developed mathematical approaches were applied to finding a three-dimensional fundamental solution of electrostatics (quasi-electrostatics) in flat-layered media, i.e. to finding the field of a point charge in plane-layered media. On the basis of the results obtained, a formulation of the generalized method of mirror reflections for a point charge located next to the plane-layered structure consisting of one film in a half-space was proposed. This result was generalized in the paper [9] for the case of an arbitrary charges distribution near plane-layered structure and an arbitrary number of films. However, in the generalized method of mirror reflections developed in [9], one has to perform double integration, which limits the computational speed.

In this paper, the generalized method of mirror reflections is modified in relation to an important particular problem — a point charge located next to the plane-layered medium consisting of a single film on a dielectric half-space. The new modified formulation of the reflection method makes it possible to exclude double integration, simplify and speed up calculations.

Below, based on the proposed method, the problems of finding the distribution of the electrostatic potential near a conducting sphere, an ellipsoid of revolution, and a drop-shaped body located near a dielectric film in a dielectric half-space are solved. Based on the analogy between electrostatics and stationary thermal conductivity, similar problems of finding the distribution of temperature fields are discussed.

1. Problem formulation. Electric field of point charge located inside a plane-layered structure

Consider the problem of finding the electrostatic field from point charge q located inside the plane-layered structure. Let, for generality, this charge be located inside a flat layered structure consisting of several films and two half-spaces surrounding the layered structure. For definiteness we will first assume that the charge is located in one of the films, and then we generalize this problem to the case when the charge is located at their boundary or in one of the half-spaces.

Let the total number of films be N_f, the thickness of m-th film be d_m, and the total thickness of the layered structure be $d_{tot} = \sum_{m=1}^{N_f} d_m$. The total number of boundaries between films will be denoted as $N = N_f + 1$. Let us number the regions of the space $j = 1, \dots, (N+1)$. Let us assume that the films have absolute permittivities equal to ε_j, and in front of and behind the layered structure there are homogeneous half-spaces with permittivities ε_1 and ε_{N+1}. Also denote by z_j-the coordinates N of the film boundaries along the axis Z — as follows: $z_1 = 0$, $z_j = \sum_{m=1}^{j-1}$ for $j = 2, \dots, N$.

The equations of electrostatics (or quasi-electrostatics) in the region with number j can be written in terms of the electric potential φ_j in the form: $\Delta\varphi_j = -\rho/\varepsilon_j$, where Δ — Laplace operator, ρ — volumetric charge density, ε_j — absolute permittivity of the j-th region. Solving the Laplace equations in each region, taking into account the boundary conditions, we find the electric field in all regions. Consider first the following auxiliary problem.

2. Electric field in layer free of charges

Let there be no extraneous charges between the boundaries z_{j-1} and z_j in the region with the number j. The permittivity of the medium in this film is ε_j.

The electric potential can be represented as a Fourier expansion:

$$\varphi_j(x, y, z) = (2\pi)^{-2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\xi x + i\eta y}\tilde{\varphi}(\xi, \eta, z)d\xi d\eta.$$

Let us substitute the potential in the form of a Fourier expansion into the in the Laplace equation, then in the region under consideration for $\hat{\varphi}_j$ we obtain equation

$$d^2\tilde{\varphi}_j/dz^2 - \gamma^2\tilde{\varphi}_j = 0, \tag{1}$$

where $\gamma = \sqrt{\xi^2 + \eta^2}$. The equations (1) for fixed values of ξ and η are ordinary differential equations with respect to the variable z. We write the general solution of equation (1) in the region $[z_{j-1}, z_j]$ in the form [9]:

$$\varphi_j(x, y, z) = (2\pi)^{-2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{\varphi}_j^{+} e^{-\gamma(z - z_{j-1})} e^{i(\xi x + \eta y)} d\xi d\eta$$

$$+ (2\pi)^{-2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{\varphi}_j^{-} e^{\gamma(z - z_j)} e^{i(\xi x + \eta y)} d\xi d\eta, \tag{2}$$

where $\hat{\varphi}_j^{+}$ and $\hat{\varphi}_j^{-}$ are the functions of only ξ and η. The first term (on the right from equal sign) in the formula (2) is the field from sources located to the left of the left boundary of the layer. The second term on the right in the formula (2) is the field from sources located to the right of the right boundary of the layer.

From (2) we write the Fourier transforms of the electric potential and the normal component of the electric field induction at the boundaries of the region j:

$$\begin{pmatrix}\tilde{\varphi}_j \\ \tilde{D}_{j,z}\end{pmatrix}\Bigg|_{z=z_{j-1}} = \mathbf{L}_j \times \hat{\mathscr{F}}_j, \quad \begin{pmatrix}\tilde{\varphi}_j \\ \tilde{D}_{j,z}\end{pmatrix}\Bigg|_{z=z_j} = \mathbf{R}_j \times \hat{\mathscr{F}}_j, \tag{3}$$

where the column vector $\hat{\mathscr{F}}_j = (\hat{\varphi}_j^{+}; \hat{\varphi}_j^{-})^T$ is introduced. Matrices $\mathbf{L}_j$ and $\mathbf{R}_j$ have the form

$$\mathbf{L}_j = \begin{pmatrix} 1 & e^{-\gamma d_{j-1}} \\ \varepsilon_j\gamma & -\varepsilon_j\gamma e^{-\gamma d_{j-1}} \end{pmatrix}, \quad \mathbf{R}_j = \begin{pmatrix} e^{-\gamma d_{j-1}} & 1 \\ \varepsilon_j\gamma e^{-\gamma d_{j-1}} & -\varepsilon_j\gamma \end{pmatrix}, \tag{4}$$

where $d_{j-1} = z_j - z_{j-1}$.

3. Electric field in multilayer structure free from external charges

Let us now consider a multilayer structure, inside which there are no extraneous charges. Let us consider the boundary $z = z_j$ between regions with numbers j and $j + 1$. Continuity of the tangential components of the electric field strengths $E_{j,x}$, $E_{j+1,x}$, $E_{j,y}$, $E_{j+1,y}$, and of normal electric induction components $D_{j,z}$ and $D_{j+1,z}$ on this boundary can be written in terms of the corresponding electric potentials φ_j and φ_{j+1} as follows:

$$\varphi_j\big|_{(x,y,z_j)} - \varphi_{j+1}\big|_{(x,y,z_j)} = 0,$$

$$\varepsilon_j \partial\varphi_j/\partial z\big|_{(x,y,z_j)} - \varepsilon_{j+1}\partial\varphi_{j+1}/\partial z\big|_{(x,y,z_j)} = 0,$$

where the electric potential φ_{j+1} in the region $j + 1$ is expressed by the formula (2), in which the indices $j \to j + 1$ are changed. Since the equations of electrostatics (quasi-statics) are linear equations, the boundary conditions must be satisfied for each term of the Fourier expansion,i.e. boundary conditions are satisfied for the Fourier transforms of the corresponding quantities:

$$\tilde{\varphi}_j\big|_{(\xi,\eta,z_j)} - \tilde{\varphi}_{j+1}\big|_{(\xi,\eta,z_j)} = 0,$$

$$\tilde{D}_{j,z}\big|_{(\xi,\eta,z_j)} - \tilde{D}_{j+1,z}\big|_{(\xi,\eta,z_j)} = 0. \tag{5}$$

Writing (5) using (3) and (4), we obtain a matrix equation on the boundary $z = z_j$:

$$\mathbf{R}_j \times \hat{\mathscr{F}}_j = \mathbf{L}_{j+1} \times \hat{\mathscr{F}}_{j+1}. \tag{6}$$

Equation (6) can be written for $j = 2, \dots, (N-1)$, where $(N + 1)$ — the total number of regions, N – the number of boundaries, i.e. for all boundaries except the first ($j = 1$) and last ($j = N$) boundaries, i.e. excluding the boundaries z_1 and $z_N = d_{tot} = \sum_{m=1}^{N-1} d_m$.

The general solution for the electric potential in the region $j = 1$, i.e. in the interval $(-\infty, z_1]$, we write it as

$$\varphi_j(x, y, z) = (2\pi)^{-2} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \left(\hat{\varphi}_1^+ e^{-\gamma(z-z_1)} + \hat{\varphi}_1^- e^{\gamma(z-z_1)}\right) \times e^{i(\xi x+\eta y)} d\xi d\eta. \quad (7)$$

Then, introducing the vector $\hat{\mathscr{F}}_1 = (\hat{\varphi}_1^+, \hat{\varphi}_1^-)^T$, we write the boundary condition (5) for $z = z_1$:

$$\begin{pmatrix} 1 & 1 \\ \varepsilon_1\gamma & -\varepsilon_1\gamma \end{pmatrix} \times \hat{\mathscr{F}}_1 = \mathbf{L}_2 \times \hat{\mathscr{F}}_2. \quad (8)$$

Similarly, the general solution for the potential in the region $j = N + 1$, i.e. in the interval $[z_N + \infty)$, we write as

$$\varphi_{N+1}(x, y, z) = (2\pi)^{-2} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \left(\hat{\varphi}_{N+1}^+ e^{-\gamma(z-z_N)} + \hat{\varphi}_{N+1}^- e^{\gamma(z-z_N)}\right) e^{i(\xi x+\eta y)} d\xi d\eta. \quad (9)$$

Then, introducing the vector $\hat{\mathscr{F}}_{N+1} = (\hat{\varphi}_{N+1}^+, \hat{\varphi}_{N+1}^-)^T$, we write the boundary condition (5) for $z = z_N$:

$$\mathbf{R}_N \times \hat{\mathscr{F}}_N = \begin{pmatrix} 1 & 1 \\ \varepsilon_{N+1}\gamma & -\varepsilon_{N+1}\gamma \end{pmatrix} \times \hat{\mathscr{F}}_{N+1}. \quad (10)$$

Equations (6), (8) and (10) allow us to relate the column vectors $\hat{\mathscr{F}}_1$ and $\hat{\mathscr{F}}_{N+1}$ in the first and last domains of the problem (i.e. in half-spaces) by the following matrix equation:

$$\hat{\mathscr{F}}_1 = \left(\mathbf{T}_1 \times \prod_{m=2}^{N} \mathbf{T}_m \times \mathbf{T}_{N+1}\right) \times \hat{\mathscr{F}}_{N+1},$$

where

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 1 \\ \varepsilon_1\gamma & -\varepsilon_1\gamma \end{pmatrix}^{-1}, \quad \mathbf{T}_m = \mathbf{L}_m \times (\mathbf{R}_m)^{-1},$$

$$\mathbf{T}_{N+1} = \begin{pmatrix} 1 & 1 \\ \varepsilon_{N+1}\gamma & -\varepsilon_{N+1}\gamma \end{pmatrix}. \quad (11)$$

It is easy to show that the column vectors in regions with arbitrary numbers S and P, where $1 \le S < P \le (N + 1)$ and $S + 2 \le P$, will be connected by formula

$$\hat{\mathscr{F}}_S = \mathbf{R}_S^{-1} \times \left(\prod_{m=S+1}^{P-1} \mathbf{T}_m\right) \times \mathbf{L}_P \times \hat{\mathscr{F}}_P. \quad (12)$$

4. Electric field in multilayer structure from point charge located in one of the films

Let there be the point charge q located in the point $(0, 0, z_q)$ in the region with number s. Let this charge be determined by the density distribution $\rho(x, y, z) = q\delta(x)\delta(y)\delta(z-z_q)$, where $\delta(x)$ — Dirac delta function. The Fourier transform of this distribution is given by the following expression:

$$\tilde{\rho}(\xi, \eta, z) = q \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \delta(x)\delta(y)\delta(z - z_q) e^{-i(\xi x+\eta y)} dx dy = q\delta(z - z_q).$$

Let this point charge be in an infinitely thin layer $(z_q-\Delta z/2, z_q + \Delta z/2)$. Then from the equations of electrostatics $\operatorname{rot}\mathbf{E} = 0$ and $\operatorname{div}\mathbf{D} = \rho$ for the Fourier transforms of the fields at $\Delta z \to 0$ we can write

$$i\eta\tilde{E}_{s,z} - \frac{\Delta\tilde{E}_{s,y}}{\Delta z} = 0, \quad \frac{\Delta\tilde{E}_{s,x}}{\Delta z} - i\xi\tilde{E}_{s,z} = 0, \quad i\xi\tilde{E}_{s,y} - i\eta\tilde{E}_{s,x} = 0,$$

$$i\xi\tilde{D}_{s,x} + i\eta\tilde{D}_{s,y} + \frac{\Delta\tilde{D}_{s,z}}{\Delta z} = q\delta(z - z_q).$$

From the equations obtained, we find the field increments in the layer

$$\Delta\tilde{E}_{s,y} = i\eta\tilde{E}_{s,z}\Delta z, \quad \Delta\tilde{E}_{s,x} = i\xi\tilde{E}_{s,z}\Delta z,$$

$$\Delta\tilde{D}_{s,z} = -(i\xi\tilde{D}_{s,x} + i\eta\tilde{D}_{s,y})\Delta z + q\delta(z - z_q)\Delta z.$$

Whence it follows that in the limit $\Delta z \to 0$, the jumps of the tangential components of the electric field strengths and the normal component of the electric field induction upon passing through an infinitely thin layer with a charge are equal to $\Delta\tilde{E}_{s,y} \to 0$, $\Delta\tilde{E}_{s,x} \to 0$, $\Delta\tilde{D}_{s,z} \to q$. In matrix form, these limit equations can be written in terms of the electric potential in an equivalent form as follows:

$$\begin{pmatrix} \tilde{\varphi}_s \\ \tilde{D}_{s,z} \end{pmatrix}\Bigg|_{z=z_q+0} - \begin{pmatrix} \tilde{\varphi}_s \\ \tilde{D}_{s,z} \end{pmatrix}\Bigg|_{z=z_q-0} = \begin{pmatrix} 0 \\ q \end{pmatrix}. \quad (13)$$

Let us now express the left side of the boundary condition (13) in terms of the column vectors $\hat{\mathscr{F}}_1$ and $\hat{\mathscr{F}}_{N+1}$ of half-spaces outside the plane-layered structure. To do this, let's divide the region with number s into two regions and denote them by indices l and r (left and right, if you look at Fig. 3). Let us introduce the column vectors $\hat{\mathscr{F}}_l$ and $\hat{\mathscr{F}}_r$ in these regions. Then the terms to the left of the equal sign in (13) can be expressed as

$$\begin{pmatrix} \tilde{\varphi}_s \\ \tilde{D}_{s,z} \end{pmatrix}\Bigg|_{z=z_d-0} = \mathbf{R}_l \times \hat{\mathscr{F}}_l \text{ and } \begin{pmatrix} \tilde{\varphi}_s \\ \tilde{D}_{s,z} \end{pmatrix}\Bigg|_{z=z_d+0} = \mathbf{L}_r \times \hat{\mathscr{F}}_r. \quad (14)$$

It follows from (6) and (12) that

$$\hat{\mathscr{F}}_l = \left(\prod_{m=1}^{s-1} \mathbf{T}_m\right)\mathbf{L}_l\hat{\mathscr{F}}_l, \text{ and } \hat{\mathscr{F}}_r = (\mathbf{R}_r)^{-1}\left(\prod_{m=s+1}^{N+1} \mathbf{T}_m\right)\hat{\mathscr{F}}_{N+1}, \tag{15}$$

where

$$\mathbf{L}_l = \begin{pmatrix} 1 & e^{-\gamma(z_q - z_{s-1})} \\ \varepsilon_s\gamma & -\varepsilon_s\gamma e^{-\gamma(z_q - z_{s-1})} \end{pmatrix},$$

$$\mathbf{R}_r = \begin{pmatrix} e^{-\gamma(z_s - z_q)} & 1 \\ \varepsilon_s\gamma e^{-\gamma(z_s - z_q)} & -\varepsilon_s\gamma \end{pmatrix}.$$

Substituting (15) into (14) and then the resulting expressions into (13), we obtain

$$\mathbf{H}_R \times \hat{\mathscr{F}}_{N+1} = \mathbf{H}_L \times \hat{\mathscr{F}}_1 + \mathbf{V}, \tag{16}$$

where $\mathbf{V} = (0, q)^T$ is a column vector characterizing the exciting action on the plane-layered structure of the point charge, and the matrices $\mathbf{H}_R$ and $\mathbf{H}_L$ characterize the response to external excitation of the layered structure to the right and left of the charge and are expressed as follows:

$$\mathbf{H}_R = \mathbf{T}_R\left(\prod_{m=s+1}^{N+1} \mathbf{T}_m\right), \ \mathbf{H}_L = \left(\left(\prod_{m=1}^{s-1} \mathbf{T}_m\right)\mathbf{T}_L\right)^{-1},$$

where the matrices $\mathbf{T}_m$ are defined by (11), and the matrices $\mathbf{T}_L$ and $\mathbf{T}_R$ — by the formulas $\mathbf{T}_L = \mathbf{L}_l(\mathbf{R}_l)^{-1}$, $\mathbf{T}_R = \mathbf{L}_r(\mathbf{R}_r)^{-1}$.

In the problem under consideration, the point charge (source of fields) is located exclusively inside the plane-layered structure. Therefore, $\hat{\mathscr{F}}_1$ and $\hat{\mathscr{F}}_{N+1}$ columns contain only the components that determine the waves coming from the flat-layered structure.

To obtain the remaining non-zero components $\hat{\mathscr{F}}_1$ and $\hat{\mathscr{F}}_{N+1}$, we split the matrices into $\mathbf{H}_R$ and $\mathbf{H}_L$ into elements H_{RA}, H_{RB}, H_{RC}, H_{RD} and H_{LA}, H_{LB}, H_{LC}, H_{LD}, then equation (16) takes the form

$$\begin{pmatrix} H_{RA} & H_{RB} \\ H_{RC} & H_{RD} \end{pmatrix}\begin{pmatrix} \hat{\varphi}^+_{N+1} \\ 0 \end{pmatrix} = \begin{pmatrix} H_{LA} & H_{LB} \\ H_{LC} & H_{LD} \end{pmatrix}\begin{pmatrix} 0 \\ \hat{\varphi}^-_1 \end{pmatrix} + \begin{pmatrix} 0 \\ q \end{pmatrix}. \tag{17}$$

Equation (17) can be represented by the following system of two matrix equations: $H_{RA}\hat{\varphi}^+_{N+1} = H_{LB}\hat{\varphi}^-_1$, $H_{RC}\hat{\varphi}^+_{N+1} = H_{LD}\hat{\varphi}^-_1 + q$. The resulting equations can be combined again into a single matrix 2×2 equation:

$$\begin{pmatrix} -H_{LB} & H_{RA} \\ -H_{LD} & H_{RC} \end{pmatrix}\hat{\mathscr{F}}_{out} = \begin{pmatrix} 0 \\ q \end{pmatrix}, \tag{18}$$

where the column vector $\hat{\mathscr{F}}_{out} = (\hat{\varphi}^-_1, \hat{\varphi}^+_{N+1})^T$ is introduced. Solving this equation, we find $\hat{\varphi}^-_1$ and $\hat{\varphi}^+_{N+1}$, which means that the fields decrease with distance from the flat-layered structure:

$$\hat{\varphi}^-_1 = qH_{RA}/(H_{RC}H_{LB} - H_{RA}H_{LD})$$

and

$$\hat{\varphi}^+_{N+1} = H_{LB}q/(H_{RC}H_{LB} - H_{RA}H_{LD}). \tag{19}$$

The field decreasing to the left in the half-space $j = 1$ is found by the formula

$$\varphi_1(x, y, z) = (2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{\varphi}^-_1 e^{\gamma(z - z_1)} e^{i(\xi x + \eta y)} d\xi d\eta, \tag{20}$$

and the field decreasing to the right in the half-space $j = N + 1$ by the formula

$$\varphi_{N+1}(x, y, z) = (2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{\varphi}^+_{N+1} e^{-\gamma(z - z_N)} e^{i(\xi x + \eta y)} d\xi d\eta. \tag{21}$$

Finally, if necessary, knowing $\hat{\mathscr{F}}_1$ and $\hat{\mathscr{F}}_{N+1}$, one can find the column vector of potentials in any internal region $\hat{\mathscr{F}}_j$, since they are uniquely determined by the boundary conditions. After that, the electric potential in any of these areas can be found using the formula (2). This way the fields will be defined throughout the space.

5. Electric field of a point charge located at some distance from the film boundary

Let us consider the problem of finding the electric potential from point charge q located in half-space with permittivity ε_f (Fig. 1). The charge is located at some distance from the film with permittivity ε_p deposited on the half-space with permittivity ε_d.

In the coordinate system of Fig. 1, the charge is located at a point with radius vector $\mathbf{r}_q = (0; 0; z_q)$ at distance $d = (z_b - z_q)$ along the axis Z from film h thick.

This problem can be considered as a problem of finding the electric potential from a point charge located on the surface of an auxiliary film of thickness $d = (z_b - z_q)$, and the dielectric constants of this auxiliary film and the half-space on the left are equal ε_f.

In this formulation, there are four regions. Let us introduce the following enumeration of regions: the index $j = 1$ corresponds to the half-space c ε_f, $j = 2$ — auxiliary film with $\varepsilon_2 = \varepsilon_f$ and thickness d, $j = 3$ — real film with $\varepsilon_3 = \varepsilon_p$ with thickness h, and the index $j = 4$ corresponds to the half-space with $\varepsilon_4 = \varepsilon_d$ (Fig. 1).

Then $N = 3$, $z_1 = z_q$, $z_2 = z_b$, $z_3 = h + z_b$, $\mathbf{H}_R = \mathbf{T}_2 \times \mathbf{T}_3 \times \mathbf{T}_4$, $\mathbf{H}_L = (\mathbf{T}_1)^{-1}$ and equation (16) becomes

$$(\mathbf{T}_2 \times \mathbf{T}_3 \times \mathbf{T}_4)\hat{\mathscr{F}}_4 = (\mathbf{T}_1)^{-1}\hat{\mathscr{F}}_1 + \mathbf{V}, \tag{22}$$

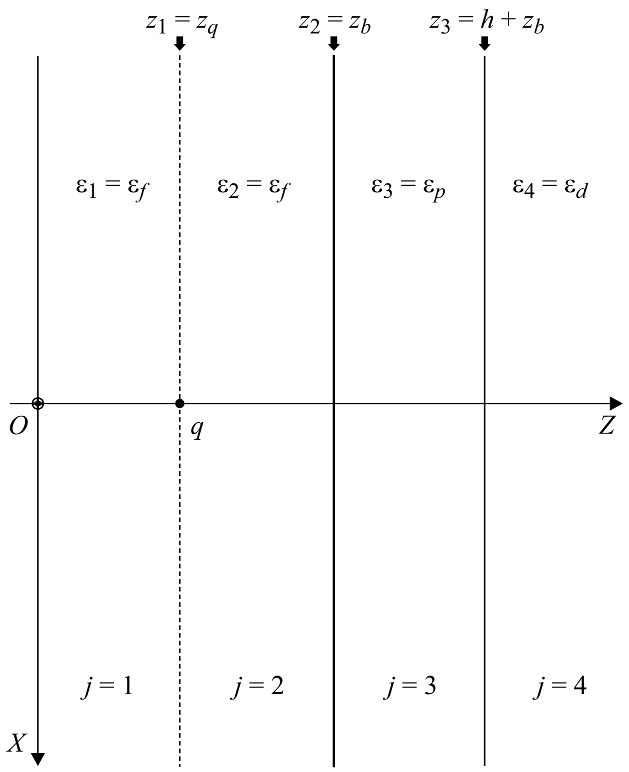

Figure 1. Point charge q located in free space with permittivity ε_f at a point with coordinate z_q at a distance of $(z_b - z_q)$ from film with thickness of h and permittivity ε_p located on the boundary of the half-space with permittivity ε_d.

where the matrices are expressed by the following formulas:

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 1 \\ \varepsilon_f \gamma & -\varepsilon_f \gamma \end{pmatrix}^{-1},$$

$$\mathbf{T}_2 = \mathbf{L}_2 (\mathbf{R}_2)^{-1} = \begin{pmatrix} 1 & e^{-\gamma d} \\ \varepsilon_f \gamma & -\varepsilon_f \gamma e^{-\gamma d} \end{pmatrix} \times \begin{pmatrix} e^{-\gamma d} & 1 \\ \varepsilon_f \gamma e^{-\gamma d} & -\varepsilon_f \gamma \end{pmatrix}^{-1},$$

$$\mathbf{T}_3 = \mathbf{L}_3 (\mathbf{R}_3)^{-1} = \begin{pmatrix} 1 & e^{-\gamma h} \\ \varepsilon_p \gamma & -\varepsilon_p \gamma e^{-\gamma h} \end{pmatrix} \times \begin{pmatrix} e^{-\gamma h} & 1 \\ \varepsilon_p \gamma e^{-\gamma h} & -\varepsilon_p \gamma \end{pmatrix}^{-1},$$

$$\mathbf{T}_4 = \begin{pmatrix} 1 & 1 \\ \varepsilon_d \gamma & -\varepsilon_d \gamma \end{pmatrix},$$

and the column vector of the point charge is equal to $\mathbf{V} = (0; q)^T$. Then

$$\mathbf{H}_L = (\mathbf{T}_1)^{-1} = \begin{pmatrix} 1 & 1 \\ \varepsilon_f \gamma & -\varepsilon_f \gamma \end{pmatrix},$$

$$\mathbf{H}_R = \begin{pmatrix} 1 & e^{-\gamma d} \\ \varepsilon_f \gamma & -\varepsilon_f \gamma e^{-\gamma d} \end{pmatrix} \begin{pmatrix} e^{-\gamma d} & 1 \\ \varepsilon_f \gamma e^{-\gamma d} & -\varepsilon_f \gamma \end{pmatrix}^{-1} \times \begin{pmatrix} 1 & e^{-\gamma h} \\ \varepsilon_p \gamma & -\varepsilon_p \gamma e^{-\gamma h} \end{pmatrix} \begin{pmatrix} e^{-\gamma h} & 1 \\ \varepsilon_p \gamma e^{-\gamma h} & -\varepsilon_p \gamma \end{pmatrix}^{-1} \times \begin{pmatrix} 1 & 1 \\ \varepsilon_d \gamma & -\varepsilon_d \gamma \end{pmatrix}.$$

Let us introduce the column vector $\hat{\mathscr{F}}_{out} = (\hat{\varphi}_1^-, \hat{\varphi}_4^+)^T$ then equation (18) for the given objective will take the form

$$\begin{pmatrix} -1 & H_{RA} \\ \varepsilon_f \gamma & H_{RC} \end{pmatrix} \begin{pmatrix} \hat{\varphi}_1^- \\ \hat{\varphi}_4^+ \end{pmatrix} = \begin{pmatrix} 0 \\ q \end{pmatrix}. \quad (23)$$

From (23) we have

$$\hat{\varphi}_1^- = q H_{RA} / (H_{RC} + \varepsilon_f \gamma H_{RA}) \text{ and } \hat{\varphi}_4^+ = q / (H_{RC} + \varepsilon_f \gamma H_{RA}). \quad (24)$$

We can explicitly express H_{RA} and H_{RC} in terms of hyperbolic sines and cosines

$$H_{RA} = \mathrm{ch}(\gamma d)\,\mathrm{ch}(\gamma h) + \frac{\varepsilon_p}{\varepsilon_f}\,\mathrm{sh}(\gamma d)\,\mathrm{sh}(\gamma h) + \frac{\varepsilon_d}{\varepsilon_p}\,\mathrm{ch}(\gamma d)\,\mathrm{sh}(\gamma h) + \frac{\varepsilon_d}{\varepsilon_f}\,\mathrm{sh}(\gamma d)\,\mathrm{ch}(\gamma h), \quad (25)$$

$$H_{RC} = \gamma \varepsilon_f \Big[\mathrm{sh}(\gamma d)\,\mathrm{ch}(\gamma h) + \frac{\varepsilon_p}{\varepsilon_f}\,\mathrm{ch}(\gamma d)\,\mathrm{sh}(\gamma h) + \frac{\varepsilon_d}{\varepsilon_p}\,\mathrm{sh}(\gamma d)\,\mathrm{sh}(\gamma h) + \frac{\varepsilon_d}{\varepsilon_f}\,\mathrm{ch}(\gamma d)\,\mathrm{ch}(\gamma h) \Big]. \quad (26)$$

Let us find the potential φ_1 in the half-space $(j = 1)$ for $z \le z_q$. Substituting the expressions (25) and (26) for H_{RA} and H_{RC} into (24) for $\hat{\varphi}_1^-$, after simple transformations we obtain the expression

$$\hat{\varphi}_1^- = q H_{RA} / (H_{RC} + \varepsilon_f \gamma H_{RA}) = \frac{q}{2\gamma \varepsilon_f} + \frac{q}{2\gamma \varepsilon_f}\, e^{-2\gamma d} \mathscr{R}(\gamma, h), \quad (27)$$

where

$$\mathscr{R}(\gamma, h) = \frac{\varepsilon_f - \varepsilon_p}{\varepsilon_p + \varepsilon_f} + \frac{4 \varepsilon_p \varepsilon_f}{(\varepsilon_p + \varepsilon_f)} \times \frac{(\varepsilon_p - \varepsilon_d)}{\left[(\varepsilon_p + \varepsilon_f)(\varepsilon_p + \varepsilon_d) e^{2\gamma h} + (\varepsilon_p - \varepsilon_d)(\varepsilon_f - \varepsilon_p) \right]}. \quad (28)$$

From the expression for the potential (20) we have

$$\varphi_1(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{q}{2\gamma \varepsilon_f}\, e^{\gamma (z - z_q)} + \frac{q}{2\gamma \varepsilon_f} \times e^{\gamma \left(z - (2 z_b - z_q) \right)}\, \mathscr{R}(\gamma, h) \right) e^{i(\xi x + \eta y)} d\xi d\eta. \quad (29)$$

Taking into account that $\gamma = \sqrt{\xi^2+\eta^2}$, we use the mathematical identity

$$\frac{q}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{e^{-\gamma|z-z_q|}}{2\varepsilon_f\gamma}e^{i\left(\xi(x-x_q)+\eta(y-y_q)\right)}d\xi d\eta$$

$$= \frac{q}{4\pi\varepsilon_f\sqrt{(x-x_q)^2+(y-y_q)^2+(z-z_q)^2}}. \quad (30)$$

Then (29) can be rewritten:

$$\varphi_1(x,y,z) = \frac{q}{4\pi\varepsilon_f\sqrt{x^2+y^2+(z-z_q)^2}}$$

$$+\frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}\frac{q}{4\pi\varepsilon_f\sqrt{x^2+y^2+\left(z-(2z_b-z_q)\right)^2}}$$

$$+\frac{q}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{2\varepsilon_p}{\gamma(\varepsilon_p+\varepsilon_f)}\right.$$

$$\left.\times\frac{(\varepsilon_p-\varepsilon_d)e^{\gamma\left(z-(2z_b-z_q)\right)}}{\left[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)\right]}\right)$$

$$\times e^{i(\xi x+\eta y)}d\xi d\eta. \quad (31)$$

Let us find the potential φ_4 in half-space ($j=4$) at $z \geq h+z_b$. Substituting expressions (25) and (26) for H_{RA} and H_{RC} into expression for $\hat{\varphi}_4^+$, we obtain

$$\hat{\varphi}_4^+ = \frac{q}{H_{RC}+\varepsilon_f\gamma H_{RA}}$$

$$= \frac{2q\varepsilon_q e^{-\gamma d}e^{\gamma h}}{\gamma\left[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)\right]}. \quad (32)$$

From the expression (21) substituting (32) we have

$$\varphi_4(x,y,z) = \frac{q}{(2\pi)^2}$$

$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{2\varepsilon_p e^{2\gamma h}}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}$$

$$\times\frac{e^{-\gamma(z-z_q)}}{\gamma}e^{i(\xi x+\eta y)}d\xi d\eta. \quad (33)$$

Let us now find the potential for $z_b \leq z \leq (h+z_b)$, i.e. in the layer $j=3$. Boundary conditions (6) on the plane $z=z_3=h+z_b$ can be written as

$$\begin{pmatrix} e^{-\gamma h} & 1 \\ \varepsilon_p\gamma e^{-\gamma h} & -\varepsilon_p\gamma \end{pmatrix}\begin{pmatrix}\hat{\varphi}_3^+ \\ \hat{\varphi}_3^-\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \varepsilon_d\gamma & -\varepsilon_d\gamma \end{pmatrix}\begin{pmatrix}\hat{\varphi}_4^+ \\ 0\end{pmatrix}.$$

Then, solving this equation and using (32), we obtain the solution

$$\hat{\varphi}_3^+ = \frac{\varepsilon_p+\varepsilon_d}{2\varepsilon_p}e^{\gamma h}\hat{\varphi}_4^+ = \frac{\varepsilon_p+\varepsilon_d}{2\varepsilon_p}$$

$$\times\frac{2q\varepsilon_p e^{-\gamma(d-2h)}}{\gamma\left[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)\right]},$$

$$\hat{\varphi}_3^- = \frac{\varepsilon_p-\varepsilon_d}{2\varepsilon_p}\hat{\varphi}_4^+ = \frac{\varepsilon_p-\varepsilon_d}{2\varepsilon_p}$$

$$\times\frac{2q\varepsilon_p e^{-\gamma(d-h)}}{\gamma\left[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)\right]}.$$

The total potential of the field inside the layer $j=3$ is expressed by the formula (2):

$$\varphi_3(x,y,z) = (2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\hat{\varphi}_3^+e^{-\gamma(z-z_b)}+\hat{\varphi}_3^-e^{\gamma(z-(z_b+h))}\right)$$

$$\times e^{i(\xi x+\eta y)}d\xi d\eta.$$

Substituting here the obtained expressions for $\hat{\varphi}_3^+$ and $\hat{\varphi}_3^-$, we obtain

$$\varphi_3(x,y,z) = \frac{q(\varepsilon_p+\varepsilon_d)}{(2\pi)^2}$$

$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{e^{-\gamma\left(z-(z_q+2h)\right)}}{\gamma\left[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)\right]}$$

$$\times e^{i(\xi x+\eta y)}d\xi d\eta + \frac{q(\varepsilon_p-\varepsilon_d)}{(2\pi)^2}$$

$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{e^{\gamma\left(z-(2z_b-z_q)\right)}}{\gamma\left[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)\right]}$$

$$\times e^{i(\xi x+\eta y)}d\xi d\eta. \quad (34)$$

Let us now find the potential for $z_q \leq z \leq z_b$, i.e. in the layer $j=2$. Boundary conditions (6) on the plane $z=z_2=z_b$ can be written as

$$\begin{pmatrix} e^{-\gamma d} & 1 \\ \varepsilon_f\gamma e^{-\gamma d} & -\varepsilon_f\gamma \end{pmatrix}\begin{pmatrix}\hat{\varphi}_2^+ \\ \hat{\varphi}_2^-\end{pmatrix} = \begin{pmatrix} 1 & e^{-\gamma h} \\ \varepsilon_p\gamma & -\varepsilon_p\gamma e^{-\gamma h} \end{pmatrix}\begin{pmatrix}\hat{\varphi}_3^+ \\ \hat{\varphi}_3^-\end{pmatrix}.$$

Then after simple calculations we get

$$\hat{\varphi}_2^+ = \frac{\varepsilon_f+\varepsilon_p}{2\varepsilon_f}e^{\gamma d}\hat{\varphi}_3^+ + \frac{\varepsilon_f-\varepsilon_p}{2\varepsilon_f}e^{\gamma(d-h)}\hat{\varphi}_3^- = \frac{q}{2\gamma\varepsilon_f}, \quad (35)$$

$$\hat{\varphi}_2^- = \frac{\varepsilon_f-\varepsilon_p}{2\varepsilon_f}\hat{\varphi}_3^+ + \frac{\varepsilon_f+\varepsilon_p}{2\varepsilon_f}e^{-\gamma h}\hat{\varphi}_3^- = \mathscr{R}(\gamma,h)\frac{q}{2\gamma\varepsilon_f}e^{-\gamma d}, \quad (36)$$

where $\mathscr{R}(\gamma,h)$, as before, is expressed by formula (28).

Then the total potential of the field inside the layer $j=2$ (see (2)):

$$\varphi_2(x,y,z) = (2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{q}{2\gamma\varepsilon_f}e^{-\gamma(z-z_q)}+\mathscr{R}(\gamma,h)\right.$$

$$\left.\times\frac{q}{2\gamma\varepsilon_f}e^{\gamma(z-(2z_b-z_q))}\right)e^{i(\xi x+\eta y)}d\xi d\eta.$$

Substituting here the expression for $\mathscr{R}(\gamma, h)$ and using identity (30), we obtain

$$\varphi_2(x, y, z) = \frac{q}{4\pi\varepsilon_f\sqrt{x^2 + y^2 + (z - z_q)^2}}$$

$$+ \frac{\varepsilon_f - \varepsilon_p}{\varepsilon_p + \varepsilon_f} \frac{q}{4\pi\varepsilon_f\sqrt{x^2 + y^2 + \big(z - (2z_b - z_q)\big)^2}}$$

$$+ \frac{q}{(2\pi)^2} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \Bigg(\frac{2\varepsilon_p}{\gamma(\varepsilon_p + \varepsilon_f)}$$

$$\times \frac{(\varepsilon_p - \varepsilon_d)e^{\gamma(z-(2z_b-z_q))}}{\big[(\varepsilon_p + \varepsilon_f)(\varepsilon_p + \varepsilon_d)e^{2\gamma h} + (\varepsilon_p - \varepsilon_d)(\varepsilon_f - \varepsilon_p)\big]} \Bigg)$$

$$\times e^{i(\xi x + \eta y)} d\xi d\eta. \quad (37)$$

Note that this is the same expression (31) as for $\varphi_1(x, y, z)$ in the half-space $j = 1$. The first term of this expression is the potential of point charge q located at the point $\mathbf{r}_q = (0, 0, z_q)$. The second term — the potential of a point charge of magnitude $q(\varepsilon_f - \varepsilon_p)/(\varepsilon_p + \varepsilon_f)$ in a medium with permittivity ε_f, which is mirrored relative to the $z = z_b$ plane, at the point $\mathbf{r}_{ref} = (0, 0, 2z_b - z_q)$. The third term — the potential of the charge distributed over the plane $z = z_b$ with some surface density (below it will be calculated explicitly).

To find φ_{ind} — the potential of the charges induced in the plane-layered structure under consideration, it is necessary in the region $z < z_b$ to subtract the potential of the initial charge from the total potential in this region. Then denoting $z_{ref} = 2z_b - z_q$ from (37) we will receive

$$\varphi_{ind}(x, y, z) = \frac{\varepsilon_f - \varepsilon_p}{\varepsilon_p + \varepsilon_f} \frac{q}{4\pi\varepsilon_f\sqrt{x^2 + y^2 + (z - z_{ref})^2}}$$

$$+ \frac{q\varepsilon_p(\varepsilon_p - \varepsilon_d)}{2\pi^2(\varepsilon_p + \varepsilon_f)}$$

$$\times \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \frac{e^{\gamma(z - z_{ref})}e^{i(\xi x + \eta y)}}{\gamma\big[(\varepsilon_p + \varepsilon_f)(\varepsilon_p + \varepsilon_d)e^{2\gamma h} + (\varepsilon_p - \varepsilon_d)(\varepsilon_f - \varepsilon_p)\big]} d\xi d\eta. \quad (38)$$

It can be seen from (38) that for $h \to +\infty$ the integral vanishes, and we obtain the well-known formula for the charge at the boundary of two half-spaces with the appropriate change of the permittivity symbols [10].

6. Generalization of the method of mirror reflections to the case of point charge located near plane-layered medium

For the first time, the method of mirror reflections for the case of point charge located near the plane-layered medium, as applied to the particular case of single film, was proposed in [11]. A further generalization of the method, including to multifilm structure, was given in [9]. Below we consider a new formulation of the reflection method.

If we introduce a function

$$U(x, y, z) = q/4\pi\varepsilon_f\sqrt{x^2 + y^2 + z^2}, \quad (39)$$

that determines the potential of the point charge q located at the origin of coordinates in space with permittivity ε_f, then expression (38) can be represented as

$$\varphi_{ind}(x, y, z) = \frac{\varepsilon_f - \varepsilon_p}{\varepsilon_p + \varepsilon_f} U(x, y, z - z_{ref}) + \frac{q}{(2\pi)^2}$$

$$\times \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \chi(\gamma, h) \frac{e^{\gamma(z - z_{ref})}e^{i(\xi x + \eta y)}}{2\varepsilon_f\gamma} d\xi d\eta, \quad (40)$$

where $z_{ref} = 2z_b - z_q$, and function

$$\chi(\gamma, h) = \mathscr{R}(\gamma, h) - (\varepsilon_f - \varepsilon_p)/(\varepsilon_p + \varepsilon_f)$$

is determined by formula

$$\chi(\gamma, h) = \frac{4\varepsilon_p\varepsilon_f}{(\varepsilon_p + \varepsilon_f)}$$

$$\times \frac{(\varepsilon_p - \varepsilon_d)}{\big[(\varepsilon_p + \varepsilon_f)(\varepsilon_p + \varepsilon_d)e^{2\gamma h} + (\varepsilon_p - \varepsilon_d)(\varepsilon_f - \varepsilon_p)\big]}. \quad (41)$$

For further generalizations, it is easy to obtain formulas of the form (31), (33), (34), (37) and (38) for the field potential of the point charge q at arbitrary point $\mathbf{r}_q = (x_q, y_q, z_q)$:

$$\varphi_1(x, y, z) = \varphi_2(x, y, z)$$

$$= \frac{q}{4\pi\varepsilon_f\sqrt{(x - x_q)^2 + (y - y_q)^2 + (z - z_q)^2}}$$

$$+ \frac{q(\varepsilon_f - \varepsilon_p)/(\varepsilon_p + \varepsilon_f)}{4\pi\varepsilon_f\sqrt{(x - x_q)^2 + (y - y_q)^2 + (z - (2z_b - z_q))^2}} + \frac{q}{(2\pi)^2}$$

$$\times \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \frac{\chi(\gamma, h)}{2\gamma\varepsilon_f} e^{\gamma(z - (2z_b - z_q))} e^{i(\xi(x - x_q) + \eta(y - y_q))} d\xi d\eta, \quad (42)$$

where $z_{ref} = 2z_b - z_q$, and $\chi(\gamma, h)$ is expressed by formula (41). In the region of the film ($j = 3$), the potential can be written as

$$\varphi_3(x,y,z)=\frac{2\varepsilon_f}{(\varepsilon_p+\varepsilon_f)}U(x-x_q,y-y_q,z-z_q)$$
$$-\frac{(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p}\frac{q}{(2\pi)^2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\chi(\gamma,h)$$
$$\times\frac{e^{-\gamma(z-z_q)}e^{i(\xi(x-x_q)+\eta(y-y_q))}}{2\gamma\varepsilon_f}d\xi d\eta+\frac{2\varepsilon_f(\varepsilon_p-\varepsilon_d)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}$$
$$\times U(x-x_q,y-y_q,z-[2(z_b+h)-z_q])$$
$$-\frac{(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p(\varepsilon_p+\varepsilon_d)}\frac{q}{(2\pi)^2}$$
$$\times\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\chi(\gamma,h)\frac{e^{\gamma(z-[2(z_b+h)-z_q])}e^{i(\xi(x-x_q)+\eta(y-y_q))}}{2\gamma\varepsilon_f}d\xi d\eta. \quad (43)$$

And then in the half-space behind the film $(j=4)$ for $\varphi_4(x,y,z)$ we obtain the expression

$$\varphi_4(x,y,z)=\frac{4\varepsilon_f\varepsilon_p U(x-x_q,y-y_q,z-z_q)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}$$
$$-\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_d)}\frac{q}{(2\pi)^2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{\chi(\gamma,h)}{2\gamma\varepsilon_f}\exp\big(-\gamma(z-z_q)\big)$$
$$\times\ \exp\Big(i\big(\xi(x-x_q)+\big(\eta(y-y_q)\big)\big)\Big)d\xi d\eta. \quad (44)$$

Let us represent the expression for $\chi(\gamma,h)$ in (41) by a series in powers $e^{-2\gamma h}$:

$$\chi(\gamma,h)=\frac{4\varepsilon_p+\varepsilon_f}{(\varepsilon_p+\varepsilon_f)}$$
$$\times\frac{(\varepsilon_p-\varepsilon_d)}{[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}$$
$$=\frac{ae^{-2\gamma h}}{1+be^{-2\gamma h}}=a\sum_{m=1}^{\infty}(-1)^{m-1}b^{m-1}e^{-2\gamma mh}, \quad (45)$$

where the following notations are introduced

$$a=\frac{4\varepsilon_p\varepsilon_f(\varepsilon_p-\varepsilon_d)}{(\varepsilon_p+\varepsilon_d)(\varepsilon_p+\varepsilon_f)^2},\ b=\frac{(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}. \quad (46)$$

Note that series (45) converges at $|b|<1$, i.e. at

$$-1<\frac{(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}<1.$$

Further discussion will assume that the permittivities of the problem satisfy this convergence condition. Obviously, for positive values of permittivities, the convergence condition is always satisfied. Then the expression (42) takes the form

$$\varphi_1(x,y,z)=U(x-x_q,y-y_q,z-z_q)+\frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}$$
$$\times U(x-x_q,y-y_q,z-z_{ref})+\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}$$
$$\times\frac{q}{(2\pi)^2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{e^{\gamma(z-z_{ref}-2mh)}e^{i(\xi(x-x_q)+\eta(y-y_q))}}{2\varepsilon_f\gamma}d\xi d\eta.$$

Taking into account the mathematical identity (30), we obtain

$$\varphi_1(x,y,z)=U(x-x_q,y-y_q,z-z_q)+\frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}$$
$$\times U(x-x_q,y-y_q,z-z_{ref})+\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}$$
$$\times U(x-x_q,y-y_q,z-z_{ref}-2mh). \quad (47)$$

Thus, the generalized method of mirror reflections can be formulated as follows: if the point charge q is located in half-space next to the film h thick located on the boundary of another half-space, then the potential in half-space, in which the charge is located, is infinite sum of the potentials of the following charges:

— the first term of the sum — is the potential of the initial point charge q (source of the field) located at the point z_q,

— the second term of the sum — this is the potential of the virtual charge $q(\varepsilon_f-\varepsilon_p)/(\varepsilon_p+\varepsilon_f)$ located at the point z_{ref}, mirrored with respect to the nearest film boundary $z=z_b$, and the remaining members of the sum is infinite sum of potentials of virtual charges located at the points $z=z_{ref}+2mh$,and the value $qa(-1)^{m-1}b^{m-1}$, where $m=1,2,3,\ldots,\infty$, and the values a and b are expressed in terms of the permittivities of the problem media using formulas (46).

Thus, in the expressions for the potential $\varphi_1(x,y,z)$, one can exclude double integration, as in [9], and replace it by summing a rather rapidly convergent series in virtual charges.

Similarly, substituting expression (45) into (43) and (44), we obtain

$$\varphi_3(x,y,z)=\frac{2\varepsilon_f}{(\varepsilon_p+\varepsilon_f)}U(x-x_q,y-y_q,z-z_q)-\frac{(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p}$$
$$\times\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}U\big(x-x_q,y-y_q,z-(z_q-2mh)\big)$$
$$\times\frac{2\varepsilon_f(\varepsilon_p-\varepsilon_d)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}U(x-x_q,y-y_q,z-[2(z_b+h)-z_q])$$
$$-\frac{(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p(\varepsilon_p+\varepsilon_d)}\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}$$
$$\times U\big(x-x_q,y-y_q,z-[2(z_b+(m+1)h)-z_q]\big), \quad (48)$$

$$\varphi_4(x,y,z)=\frac{4\varepsilon_f\varepsilon_p U(x-x_q,y-y_q,z-z_q)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}-\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_d)}$$

$$\times\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}U\big(x-x_q,y-y_q,z-(z_q-2mh)\big). \quad (49)$$

7. Generalization of mirror reflection method to the case of system of charges

Let us now generalize the obtained method of mirror reflections and find the potential of the total field $\Phi_{tot}(x,y,z)$ in the region $z<z_b$ in front of the film (Fig. 2) of an arbitrary compact system N_q of source charges q_k located at points with radius vectors $\mathbf{r}_{q,k}=(x_{q,k},y_{q,k},z_{q,k})$, where $k=1,2,\ldots,N_q$.

If there were no plane-layered structure, then the potential of this system of charges-sources would be represented by the formula

$$\Phi_s(x,y,z)=\sum_{k=1}^{N_q}q_k\Big/4\pi\varepsilon_f$$

$$\times\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+(z-z_{q,k})^2}. \quad (50)$$

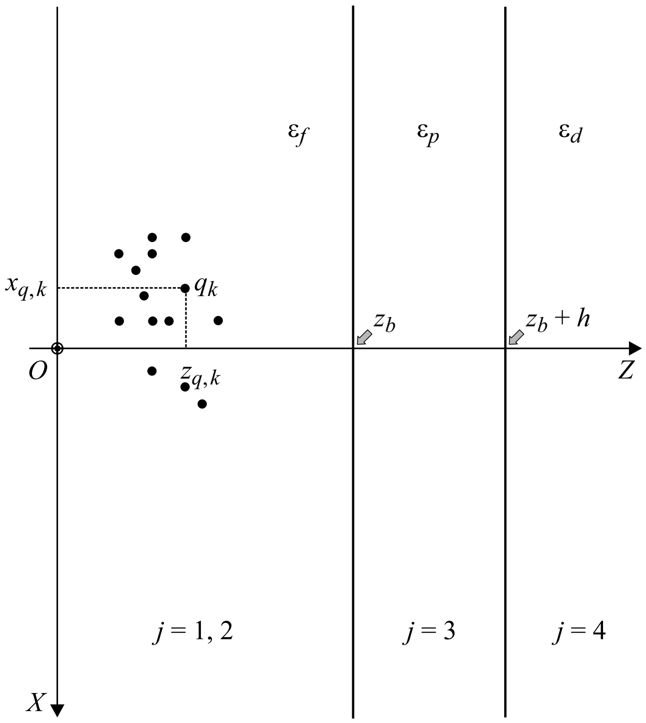

Figure 2. System of point charges q_k for film located on the boundary of half-space.

Summing expressions (47) for each charge q_k over all N_q charges of the system and noticing that

$$\sum_{k=1}^{N_q}q_k\Big/4\pi\varepsilon_f$$

$$\times\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+\big(z-(2z_b+2mh-z_{q,k})\big)^2}$$

$$=\sum_{k=1}^{N_q}q_k\Big/4\pi\varepsilon_f$$

$$\times\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+\big((2z_b+2mh-z)-z_{q,k}\big)^2}$$

$$=\Phi_s(x,y,2z_b+2mh-z),$$

we obtain a generalization of the method of mirror reflections in electrostatics for arbitrary charge distributions for the total potential of the system of charges in the half-space in front of the film in the form

$$\Phi_{tot}(x,y,z)=\Phi_s(x,y,z)+\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_f)}\Phi_s(x,y,2z_b-z)$$

$$+\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}\Phi_s(x,y,2z_b+2mh-z). \quad (51)$$

It follows from the derivation of formula (51) that the potential of induced charges Φ_{tot}, which is generated by the potential Φ_s of distributions of charge-sources of the field, is represented by the formula

$$\Phi_{ind}(x,y,z)=\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_f)}\Phi_s(x,y,2z_b-z)$$

$$+\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}\Phi_s(x,y,2z_b+2mh-z). \quad (52)$$

Similarly, considering that

$$\sum_{k=1}^{N_q}q_k\Big/4\pi\varepsilon_f\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+\big(z-(z_{q,k}-2mh)\big)^2}$$

$$=\sum_{k=1}^{N_q}q_k\Big/4\pi\varepsilon_f\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+\big((z+2mh)-z_{q,k}\big)^2}$$

$$=\Phi_s(x,y,z+2mh),$$

$$\sum_{k=1}^{N_q}q_k\Big/4\pi\varepsilon_f$$

$$\times\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+\big(z-[2(z_b+(m+1)h)-z_{q,k}]\big)^2}$$

$$=\sum_{k=1}^{N_q}q_k\Big/4\pi\varepsilon_f$$

$$\times\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+\big([2(z_b+(m+1)h)-z]-z_{q,k}\big)^2}$$

$$=\Phi_s\big(x,y,2(z_b+(m+1)h)-z\big),$$

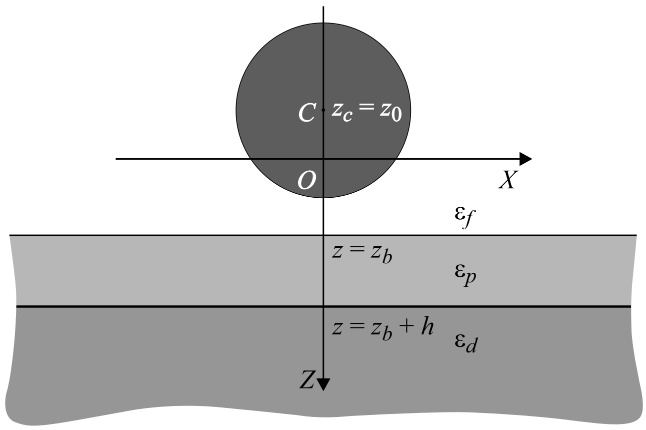

Figure 3. Metal sphere at film located on the boundary of half-space. Geometry of the problem.

we obtain, taking into account (48), for the potential Φ_3 of the system of charges in the film at $z_b \le z \le (z_b + h)$:

$$\Phi_3(x,y,z)=\frac{2\varepsilon_f\Phi_s(x,y,z)}{(\varepsilon_p+\varepsilon_f)}-\frac{(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p}\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}$$
$$\times\Phi_s(x,y,z+2mh)+\frac{2\varepsilon_f(\varepsilon_p-\varepsilon_d)\Phi_s(x,y,2(z_b+h)-z)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}$$
$$-\frac{(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p(\varepsilon_p+\varepsilon_d)}\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}$$
$$\times\Phi_s\big(x,y,2(z_b+(m+1)h)-z\big). \tag{53}$$

Taking into account (49), the charges potential Φ_4 in the half-space at $z \ge (z_b + h)$ $(j = 4)$ is equal to

$$\Phi_4(x,y,z)=\frac{4\varepsilon_f\varepsilon_p}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}\Phi_s(x,y,z)$$
$$-\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_d)}\sum_{m=1}^{\infty}a(-1)^{m-1}b^{m-1}\Phi_s(x,y,z+2mh). \tag{54}$$

Note that generalization of the method of mirror reflections was obtained earlier in the paper [9] in a different formulation, which was successfully applied to the problem of nanofocusing of a surface plasmon wave at the top of a metal tip. The proposed method of this paper avoids double integration and has an advantage during numerical implementation.

8. Potential distribution around metal body located near flat boundary of film on dielectric half-space

Let us consider the metal body, for certainty a sphere with a radius R is considered first. The surface of the sphere is represented by the formula $x^2 + y^2 + (zz_0^2) = R^2$, where R — the radius of the sphere, and the coordinates of the center are $x_c = 0$, $y_c = 0$ and $z_c = z_0$ (Fig. 3). Let a film of thickness h with boundaries $z = z_b$ and $z = (z_b + h)$ be located near the sphere. The permittivities of the external homogeneous medium, the film, and the semi-infinite medium behind the film are denoted by ε_f, ε_p and ε_d and ε_d, respectively (in the general case of quasistatics — these are complex quantities).

Let us consider the distribution of the electric potential, which will be established in space. As it is known, the electrostatic field potential Φ satisfies the Laplace equation $\Delta\Phi = 0$. Moreover, at the boundary of the metal sphere the potential U_s will be constant. Then the boundary conditions can be written as:

$$\text{on tip surface: } \quad \Phi = U_s, \tag{55}$$

$$\text{on film boundary } z=z_b:\ \varepsilon_p E_{3,n} = \varepsilon_f E_{f,n} \text{ and } E_{3,\tau} = E_{f,\tau}, \tag{56}$$

$$\text{on film boundary } z=z_b+h:\ \varepsilon_d E_{4,n}=\varepsilon_p E_{3,n} \text{ and } E_{4,\tau}=E_{3,\tau}. \tag{57}$$

The problem under consideration has axial symmetry with respect to the axis Z. Therefore, the solution of the Laplace equation will have the same symmetry. Let the potential of charges located on the equipotential metallic sphere in space with permeability ε_f be described by the function $\Phi_s(x,y,z)$. Then the total potential $\Phi_{tot}(x,y,z)$ in the region filled with dielectric with ε_f can be expressed in terms of $\Phi_s(x,y,z)$ by formula (51), and in the film and in the dielectric space behind it — by formulas (53) and (54). In this case, the boundary conditions (56) and (57) will be satisfied automatically. Thus, the problem of determining the potential in the entire space is to find the potential $\Phi_s(x,y,z)$ such that the total potential $\Phi_{tot}(x,y,z)$ satisfies the boundary condition (55). This can be done by expanding the potential $\Phi_s(x,y,z)$ in suitable harmonic functions, and the expansion coefficients can be determined from the condition (55) for $\Phi_{tot}(x,y,z)$ on the equipotential boundary of metal.

Bearing in mind the generality of the further presentation, we pass to dimensionless coordinates: $\tilde{x} = x/R$, $\tilde{y} = y/R$, $\tilde{z} = z/R$, where R — radius of sphere. The Laplace equation in dimensionless coordinates will not change. In addition, we normalize the potential to its value U_s on the surface of the sphere, then we can pass from the dimensional to the dimensionless potential $\tilde{\Phi}(\tilde{x},\tilde{y},\tilde{z}) = \Phi/U_s$ in dimensionless coordinates. Then the boundary condition (55) on the sphere can be written in the form $\tilde{\Phi}_{tot} = 1$.

In the considered axisymmetric case, to fulfill the boundary condition on the surface of the sphere, it is sufficient to satisfy its on the line of intersection of the tip surface with any plane of symmetry passing through the Z axis. As such a plane we choose the plane $(\tilde{x},\tilde{z})$ at $\tilde{y} = 0$. More specifically, it suffices to satisfy the boundary condition $\tilde{\Phi}_{tot}(\tilde{x},0,\tilde{z}) = 1$ only on the boundary of the intersection of the half-plane $\tilde{y} = 0$ at $\tilde{x} \ge 0$ and the surface of the sphere. In dimensionless coordinates, this will be the curve $\tilde{x}^2 + (\tilde{z} - \tilde{z}_0) = 1$ at $\tilde{y} = 0$ and $\tilde{x} \geq 0$.

We will look for the solution, assuming that the potential outside the sphere $\tilde{\Phi}_s$ has the form

$$\tilde{\Phi}_s = \sum_{j=1}^{N} A_j P_{j-1}(\cos\theta)/\tilde{r}^j, \tag{58}$$

where A_j — constant expansion coefficients, $P_j(\cos\theta)$ — Legendre polynomials of degree j, angle θ is measured from the axis Z from the sphere center, $\tilde{r}$ — dimensionless radius-vector of the observation point drawn from the center of the sphere. In the coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$, expression (58) can be represented as

$$\tilde{\Phi}_s(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{j=1}^{N} A_j \mathscr{F}_j(\tilde{x}, \tilde{y}, \tilde{z}), \tag{59}$$

where

$$\mathscr{F}_j(\tilde{x}, \tilde{y}, \tilde{z}) = P_{j-1}\Big((\tilde{z} - \tilde{z}_0) \Big/ \sqrt{\tilde{x}^2 + \tilde{y}^2 + (\tilde{z} - \tilde{z}_0)^2}\Big) \Big/ \Big(\sqrt{\tilde{x}^2 + \tilde{y}^2 + (\tilde{z} - \tilde{z}_0)^2}\Big)^j.$$

Note that the choice of functional dependences (58) from the general solution of the Laplace equation is determined by the fact that the field potential outside the sphere must tend to zero with increase in distance from its surface and be finite. Then, the potential of induced charges $\tilde{\Phi}_{ind}(\tilde{x}, \tilde{y}, \tilde{z})$ can be represented, taking into account (52), in the form

$$\tilde{\Phi}_{ind}(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{j=1}^{N} A_j \mathscr{P}_j(\tilde{x}, \tilde{y}, \tilde{z}), \tag{60}$$

where

$$\mathscr{P}_j(\tilde{x}, \tilde{y}, \tilde{z}) = \big((\varepsilon_f - \varepsilon_p)/(\varepsilon_p + \varepsilon_f)\big)\mathscr{F}_j(\tilde{x}, \tilde{y}, 2\tilde{z}_b - \tilde{z}) + \sum_{m=1}^{\infty} a(-1)^{m-1} b^{m-1} \mathscr{F}_j(\tilde{x}, \tilde{y}, 2\tilde{z}_b + 2m\tilde{h} - \tilde{z}).$$

Then we obtain the total potential in the medium with ε_f (between the sphere and the film):

$$\tilde{\Phi}_{tot}(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{m=1}^{\infty} A_j\big(\mathscr{F}_j(\tilde{x}, \tilde{y}, \tilde{z}) + \mathscr{P}_j(\tilde{x}, \tilde{y}, \tilde{z})\big). \tag{61}$$

The boundary condition on the sphere $\tilde{\Phi}_{tot}(\tilde{x}, \tilde{y}, \tilde{z}) = 1$ was satisfied in this paper approximately, by the collocation method [12]. These equations were written at uniformly distributed N points of the semicircle $\tilde{x}^2 + (\tilde{z} - \tilde{z}_0)^2 = 1$ at $\tilde{y} = 0$ and $\tilde{x} > 0$ on the surface of the sphere, and N linear algebraic equations with N unknown coefficients A_j were obtained. As a result of solving the obtained system A_j were found, and by the formulas (61) of the potential distribution in the region between the sphere and the front surface of the film.

Taking into account (59), the potential distribution in the film and behind it, taking into account (53) and 54), was determined by the formulas

$$\tilde{\Phi}_3(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{j=1}^{N} A_j\bigg(\frac{2\varepsilon_f}{(\varepsilon_p + \varepsilon_f)}\mathscr{F}_j(\tilde{x}, \tilde{y}, \tilde{z}) - \frac{(\varepsilon_f - \varepsilon_p)}{2\varepsilon_p} \times \sum_{m=1}^{\infty} a(-1)^{m-1} b^{m-1} \mathscr{F}_j(\tilde{x}, \tilde{y}, \tilde{z} + 2m\tilde{h}) \times \frac{2\varepsilon_f(\varepsilon_p - \varepsilon_d)\mathscr{F}_j(\tilde{x}, \tilde{y}, 2(\tilde{z}_b + \tilde{h}) - \tilde{z})}{(\varepsilon_p + \varepsilon_f)(\varepsilon_p + \varepsilon_d)} - \frac{(\varepsilon_p - \varepsilon_d)(\varepsilon_f - \varepsilon_p)}{2\varepsilon_p(\varepsilon_p + \varepsilon_d)} \times \sum_{m=1}^{\infty} a(-1)^{m-1} b^{m-1} \mathscr{F}_j\big(\tilde{x}, \tilde{y}, 2(\tilde{z}_b + (m+1)\tilde{h}) - \tilde{z}\big)\bigg), \tag{62}$$

$$\tilde{\Phi}_4(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{j=1}^{N} A_j\bigg(\frac{4\varepsilon_f\varepsilon_p}{(\varepsilon_p + \varepsilon_f)(\varepsilon_p + \varepsilon_d)}\mathscr{F}_j(\tilde{x}, \tilde{y}, \tilde{z}) - \frac{(\varepsilon_f - \varepsilon_p)}{(\varepsilon_p + \varepsilon_d)} \sum_{m=1}^{\infty} a(-1)^{m-1} b^{m-1} \mathscr{F}_j(\tilde{x}, \tilde{y}, \tilde{z} + 2m\tilde{h})\bigg). \tag{63}$$

As a result, formulas (61), (62) and (63) solve the problem of finding the normalized potential in normalized coordinates outside the sphere and in the entire layered structure using known values A_j.

The obtained solutions for the normalized potential have an important property: they depend on the ratios of permittivities, i.e. if all permittivities are increased by k times, then the distribution of the normalized potential will not change.

9. Potential distribution around some bodies located near the film on half-space

First, as an example of the above theory application we performed numerical calculations of the normalized potential distribution of the charged metal sphere near plane-layered structure of single film. The sphere with center at the point $(\tilde{x}_0, \tilde{y}_0, \tilde{z}_0) = (0, 0, -0.5)$ was in vacuum with $\varepsilon_f = 1$ near with film with thickness equal to $\tilde{h} = 1$ (in units normalized to the radius of curvature of the sphere) with a permittivity $\varepsilon_p = 2$. The half-space behind the film had permittivity $\varepsilon_d = 4$. The film boundaries were determined by the equations $\tilde{z} = \tilde{z}_b = 1$ and $\tilde{z} = \tilde{z}_b + \tilde{h} = 2$ (Fig. 3). The shortest distance from the sphere to the film was $\Delta\tilde{z} = 0.5$.

Fig. 4, a shows the normalized potential distribution in the $(\tilde{x}, \tilde{z})$ plane for the indicated parameters. One can see increase in the distance between the equipotentials in the transition from free space to the dielectric film and further to the dielectric half-space. This is due to the natural screening

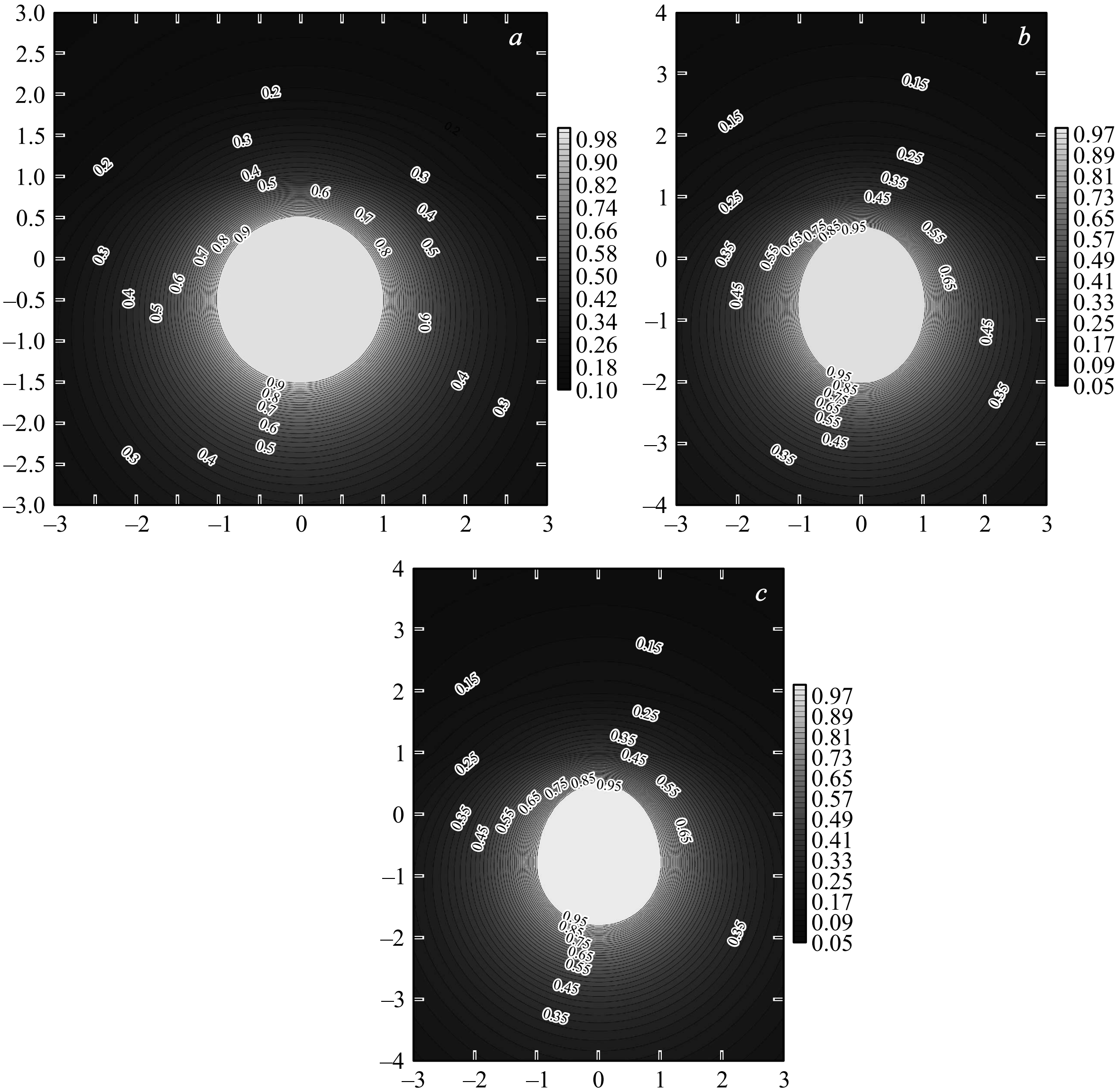

Figure 4. Distributions of the normalized potential around metal charged bodies: sphere (a), ellipsoid of revolution elongated along Z (b) and drop-shaped body of revolution (c) in normalized coordinates. Permittivities: $\varepsilon_f = 1$, $\varepsilon_p = 2$, $\varepsilon_d = 4$. The film thickness is $\tilde{h} = 1$, its boundaries are $\tilde{z} = 1$ and $\tilde{z} = 2$. The distance between the bodies and the film is $\Delta\tilde{z} = 0.5$.

of the electric field in the dielectric, which occurs due to the induced charges. The greater the dielectric constant is, the stronger the shielding is. An important property of the resulting distribution is that its characteristic size is approximately equal to the size of the sphere.

The question arises: can the obtained method be used to solve other charged metal bodies? It turned out that it is possible. Thus, the problem was considered of finding the normalized potential of metal charged ellipsoid near the same plane-layered structure of one film as in the previous example. The elongated ellipsoid with the axis of symmetry along the axis Z was considered. Normalization of coordinates was carried out with respect to the minor semi-axis in a plane perpendicular to the axis. The normalized length of the major semi-axis was 1.25. The center of the ellipse was at the point $(\tilde{x}_0, \tilde{y}_0, \tilde{z}_0) = (0, 0, -0.75)$. The shortest distance from the ellipse to the film was $\Delta\tilde{z} = 0.5$. Fig. 4, b shows the normalized potential distribution in the $(\tilde{x}, \tilde{z})$ plane. The expansion $\tilde{\Phi}_s$ was carried out in the same functions as in the previous case (59).

Similarly, the problem of finding the normalized potential of metal charged body of drop-type shape was considered, this body is semi-ellipsoid superimposed on hemisphere. As in the previous example, the body was located near the same plane-layered structure of one film. Normalization of coordinates was carried out with respect to the radius of the hemisphere. The normalized length of the hemisphere major semi-axis was 1.3. The center of the hemisphere curvature was at the point $(\tilde{x}_0, \tilde{y}_0, \tilde{z}_0) = (0, 0, -0.8)$. The shortest distance from the body to the film was $\Delta\tilde{z} = 0.5$. Fig. 4, c shows the normalized potential distribution in the

$(\tilde{x}, \tilde{z})$ plane. The expansion $\tilde{\Phi}_s$ was carried out in the same functions as in the previous case (59).

Note that as the ellipsoid eccentricity increases, the convergence of the numerical method decreases; for large eccentricities the potential should be expanded into a series in solutions of the Laplace equation in the elliptic system of coordinates of rotation [13]. The best approximation problem $\tilde{\Phi}_s$ requires additional study.

10. Application of results obtained to problems of thermal conductivity of plane-layered media

As it is known, there is an analogy between the problems of electrostatics and stationary problems of thermal conductivity [14]. Not only the corresponding equations are similar, but also the boundary conditions [15,16]. If, in the problems discussed in the previous Sections, simultaneously the following are replaced: 1) potential φ — by the temperature $(T - T_0)$, where T_0 — temperature at infinity; 2) permittivity ε — by thermal conductivity K; 3) electric induction vector $\mathbf{D} = \varepsilon\mathbf{E} = -\varepsilon\nabla\varphi$ — by heat flux density $\mathbf{h} = -K\nabla T$; 4) point charge q — by the heat release power of point source Q_h, then similar problems can be solved by the same method.

For example, to solve the problem of temperature distribution around the body heated to a constant temperature T_m, one can first solve the corresponding electrostatic problem of potential distribution in the vicinity of the body with constant surface potential. The normalized solution of the electrostatic problem will also be the solution of the similar normalized problem of thermal conductivity in normalized coordinates (with the specified change of values and notations). The result of calculating the temperature distribution (more precisely, the temperature rise over the external temperature T_0, expressed in units $(T_m - T_0)$)in the problems of heated sphere, ellipsoid, and drop-type body will be identical to the distributions shown in Fig. 4.

The normalized temperature distribution, just as in the case of the distribution of normalized electrostatic potentials has the following property. The distribution depends on the ratios of the thermal conductivity coefficients of the regions. That is, if we increase all the coefficients by k times, then the normalized temperature distribution will not change in the normalized coordinates. Thus, Fig. 4 shows the temperature distributions for the thermal conductivity coefficients $K_f = \text{k}$, $K_p = 2\text{k}$, $K_d = 4\text{k}$, where k is arbitrary number.

Especially note that the analogy between the electrostatic and thermal conductivity problems does not mean that the electric potential distribution in vacuum is similar to the temperature distribution in vacuum. In the considered case, the analogue of vacuum is a medium with some finite thermal conductivity coefficient K_f, and the coefficients K_p and K_d correlate with K_f in the same way as the permittivities of similar electrostatic problem.

Conclusion

In this paper, a new formulation of the generalized mirror reflection method for single film located on half-space is proposed, it eliminates the double integration used in the previous formulation of the method [9].

The method application for finding electrostatic fields from symmetric bodies of revolution located next to one film is demonstrated. The applicability of the proposed theoretical method to similar problems of stationary thermal conductivity is shown. This was done using the example of solving the problem of finding the temperature field around a uniformly heated body located near thermal conducting film and half-space.

Conflict of interest

The author declares that he has no conflict of interest.

References

[1] H.N. Pollack. J. Geophys. Research, **70** (22), 5645 (1965). DOI: 10.1029/JZ070i022p05645

[2] J.G. Negi, R.N. Singh. Pure Appl. Geophys., **73** (1), 143 (1969). DOI: 10.1007/BF00875132

[3] J.G. Negi, R.N. Singh. Pure Appl. Geophys., **69** (1), 110 (1968). DOI: 10.1007/BF00874909

[4] S.J. Matysiak, D.M. Perkowski. Heat Mass Transfer, **50** (1), 75 (2014). DOI: 10.1007/s00231- 013-1225-9

[5] W.C. Chew. *Waves and Fields in Inhomogeneous Media* (IEEE Press, NY., 1995)

[6] A.B. Petrin. ZhETF, **159** (1), 35 (2021). (in Russian). DOI: 10.31857/S004445102101003X

[7] R. King, G. Smith. *Antenny v material'nykh sredakh:* V 2-kh kn. Per s angl. (Mir, M., 1984) (in Russian).

[8] R.W.P. King. IEEE Trans. Microwave Theory Tech., **36** (6), 1080 (1988). DOI: 10.1109/22.3635

[9] A.B. Petrin. Opt. i spektr., **130** (9), 1436 (2022) (in Russian). DOI: 10.21883/OS.2022.09.53308.3636-22

[10] D.V. Sivukhin, *Obshchiy kurs fiziki. Ucheb. posobie: Dlya vuzov. V 5 t. T. III. Elektrichestvo* (Fizmatlit, Izd-vo MFTI, M., 2004), 4-e izd., stereotip. (in Russian)

[11] A.B. Petrin. Uspekhi prikladnoi fiziki, **4** (4), 326 (2016).(in Russian).

[12] *Chislennye metody teorii difrakcii (Matematika. Novoe v zarubezhnoy nauke. Vyp. 29):* Sb. statej. Per. s angl. (Mir, M., 1982) (in Russian).

[13] A. Ango. Matematika dlya elektro- i radioinzhenerov (Nauka, M., 1967).(in Russian).

[14] R. Feynman, R. Leighton., M. Sands. *Feynmanovskie lektsii po fizike. (in Russian) T. 5. Elektrichestvo i magnetizm* (Mir, M., 1965) (in Russian).

[15] L.D. Landau, E.M. Lifshits. *Teoreticheskaya fizika. T. VI. Gidrodinamika* (Nauka, M., 1986) (in Russian)

[16] E.M. Kartashov. *Analiticheskie metody v teorii teploprovodnosti tverdykh tel* (Vysshaya shkola, M., 2001) (in Russian)

Translated by I.Mazurov