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Dynamics of coupled quasiperiodic generator and Rössler system

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The interaction of a system with quasi-periodic autonomous dynamics and a chaotic system (Rössler system) is considered. The behavior of Lyapunov exponents is studied to identify possible types of system dynamics: chaos with additional zero Lyapunov exponents, three-frequency and two-frequency quasi-periodic regimes, periodic oscillations and the mode of oscillation death.

Keywords: quasi-periodic oscillations, dynamical chaos, Rössler system, Lyapunov exponents.

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The interaction of nonlinear oscillation systems is of prime scientific and technological importance. The synchronization of periodic and chaotic oscillations is already understood well [1,2]. That said, studies touching on novel physical features of synchronization of quasi-periodic oscillations have been initiated in [3–6] and subsequent publications. The construction of autonomous low-dimensional radiophysical systems with quasi-periodic dynamics (quasi-periodic oscillators) was essential here. Issues of forced synchronization of quasi-periodic oscillators, mutual synchronization of such oscillators, etc., were examined next [3–10]. This research area also overlaps with the subject of bifurcations of invariant tori, which continues to attract interest [11–13]. Synchronization theory commonly deals with coupling of same-type systems (e.g., of van der Pol oscillators with each other, of Rössler systems with each other, etc.). However, the case of coupling of different-type systems is also of interest. The problem of interaction of a subsystem with quasi-periodic autonomous dynamics and a chaotic oscillator arises in this context. These issues are the ones examined in the present study. A quasi-periodic system [7,9] serves as the first subsystem, and a chaotic Rössler oscillator [1,2] was chosen to be the second one.

Let us write down the equations for an oscillator [7,9] and a Rössler system coupled to it:

$$\begin{aligned} \dot{x}_1 &= y_1, \\ \dot{y}_1 &= (\lambda + z_1 + x_1^2 - \beta x_1^4) y_1 - \omega_0^2 x_1 - \mu(\dot{x}_1 - \dot{x}_2), \\ \dot{z}_1 &= b(\varepsilon - z_1) - k y_1^2, \\ \dot{x}_2 &= -y_2 - z_2, \\ \dot{y}_2 &= x_2 + p y_2 + \mu(y_1 - y_2), \\ \dot{z}_2 &= q + (x_2 - r) z_2. \end{aligned} \tag{1}$$

Here, x_1, y_1, and z_1 are variables of the quasi-periodic oscillator; x_2, y_2, and z_2 are variables of the Rössler oscillator; ω_0 is the natural frequency of the oscillator; and μ is the coupling parameter. The coupling was introduced so as to be dissipative both for the first subsystem, which was constructed as a generalization of a van der Pol oscillator, and for the second one. Parameters were chosen in accordance with [7,9] ($\varepsilon = 4$, $b = 1$, $k = 0.02$, $\beta = 1/18$, $\lambda = -1$) and [1] ($p = 0.15$, $q = 0.4$, $r = 8.5$). Quasi-periodic oscillations are feasible in this case within a certain range of values of ω_0 for the first subsystem, while the Rössler oscillator with the indicated parameter set features chaotic dynamics.

Let us discuss the dynamics of system (1). Figure 1 shows the plots of its Lyapunov exponents for coupling $0 \leqslant \mu \leqslant 0.25$. Parameter ω_0 was set to 2π, which corresponds to quasi-periodic dynamics in the isolated first subsystem. Note that one exponent is positive at zero coupling ($\Lambda_1 > 0$), since chaos is observed in the isolated Rössler system. One exponent remains zero at any coupling, which is mandatory for flow systems. The plots in Fig. 1 reveal three characteristic regions. In the case of strong coupling, two the largest exponents are zero ($\Lambda_1 = \Lambda_2 = 0$), while the other ones are negative. This spectrum of exponents corresponds to a double-frequency quasi-periodic regime (two-frequency invariant torus in the phase space).

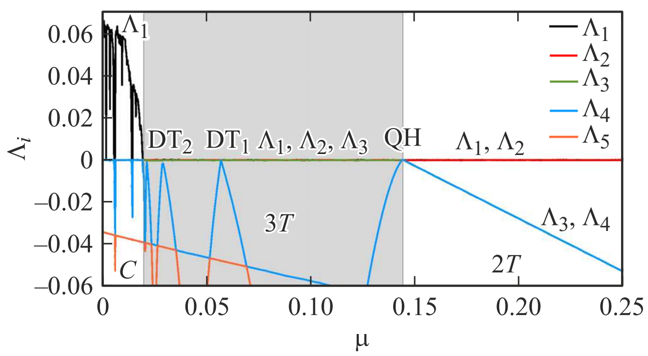

Figure 1. Dependences of Lyapunov exponents of coupled systems on the coupling strength at $\omega_0 = 2\pi$. A color version of the figure is provided in the online version of the paper.

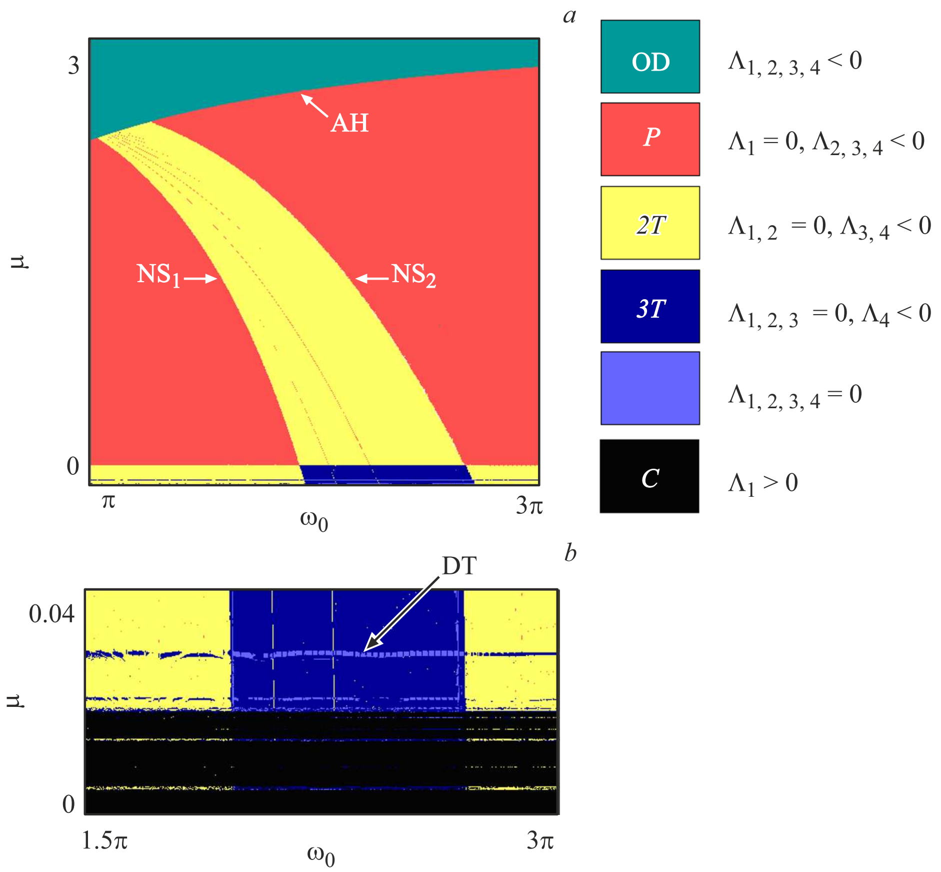

Figure 2. Lyapunov exponents chart of coupled systems (a) and an enlarged fragment of it (b). A color version of the figure is provided in the online version of the paper.

As the coupling grows weaker, exponents Λ_3, Λ_4 remain equal, increase, and become zero at the boundary of region $2T$. From this point onward, exponent Λ_3 remains zero, while Λ_4 again assumes a negative value. In accordance with [11], a quasi-periodic Hopf bifurcation occurs: a stable three-frequency torus $3T$ arises softly from a two-frequency one. Three Lyapunov exponents are equal to zero inside the $3T$ region ($\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$), while the rest are negative. As the coupling grows weaker, several successive bifurcations of doubling of three-frequency tori (DT) occur, wherein the first exponent turns to zero at the bifurcation point, but remains negative in its neighborhood [11]. The largest exponent becomes positive ($\Lambda_1 > 0$) in region C; thus, this is a mostly chaotic region. Very narrow windows of three-frequency tori are also possible. In the case of chaos, all three exponents Λ_2, Λ_3, and Λ_4 turn to zero. Note that the possibility of chaos with an additional zero Lyapunov exponent (specifically, resulting from disruption of doubling two-frequency tori) was discussed in [14–17]. In the present case, two additional zero exponents are observed instead of a single one. The presence of an additional zero Lyapunov exponent is debatable [16]. In view of this, the authors of [17] speak of a „very close to zero“ Lyapunov exponent, which may be indistinguishable from zero in numerical calculations. Let us now perform two-parameter analysis including the region of stronger coupling. Figure 2, a presents the Lyapunov exponents chart on the „natural frequency of the quasi-periodic oscillator–coupling strength of subsystems“ (ω_0, μ) plane, and an enlarged fragment of this chart is shown in Fig. 2, b. The regime type and the invariant torus dimension are determined from the spectrum of Lyapunov exponents and the number of zero exponents, respectively. The color code legend is shown on the right (a color version of the figure is provided in the online version of the paper).

This figure visualizes the regions of periodic (P) regimes, two-frequency ($2T$) and three-frequency ($3T$) tori, and chaos (C). The region of a stable equilibrium state of the system is also observed. This is a manifestation of the OD (oscillation death) effect [1] wherein strong dissipative coupling suppresses oscillations of both oscillators. Narrow windows of three-frequency tori are discernible in the chaos region in Fig. 2, b, and doubling lines DT are seen in the region of three-frequency tori.

The boundary between the OD region and the region of periodic regimes in Fig. 2, a is the Andronov–Hopf (AH) bifurcation line, and the region of two-frequency regimes is bounded from two sides by lines of Neimark–Sacker (NS) bifurcations of limit cycles. These bifurcations were identified using the XPPAUT numerical package. A small

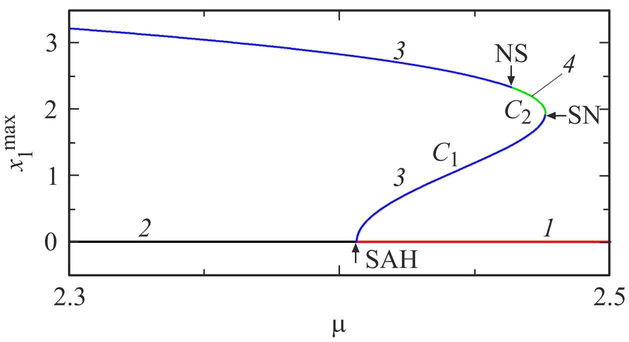

Figure 3. One-parameter bifurcation diagram. Lines *1* and *2* denote stable and unstable equilibrium states, and curves *3* and *4* represent saddle and stable limit cycles. The maximum value of variable x_1 is shown.

part of the OD region in Fig. 2,*a* also borders visually on the region of two-frequency tori. Bifurcation analysis was performed to clarify this issue. Figure 3 presents the corresponding one-parameter bifurcation diagram for $\omega_0 = 1.2\pi$. As the coupling grows stronger, unstable equilibrium at the subcritical Andronov–Hopf (SAH) bifurcation point becomes stable, and saddle limit cycle $C1$ arises from it. This cycle, in turn, merges with stable cycle $C2$ at the saddle-node bifurcation point (SN). Cycle $C2$ undergoes a Neimark–Sacker bifurcation at point NS, and a stable torus arises softly from it as the coupling grows weaker. Thus, a certain transition region is established between the SAH and SN points in a narrow coupling parameter interval of $2.4 \leqslant \mu \leqslant 2.476$, and bistability becomes possible (specifically, a stable equilibrium may exist alongside a stable torus). Any oscillations to the right of the SN point are suppressed by the coupling. Note that the authors of [9,18,19] discussed bifurcation at the OD region boundary resulting in the emergence of a two-frequency torus attractor and two saddle limit cycles from the equilibrium state in coupled oscillators with identical control parameters. This bifurcation got disrupted when a parameter mismatch was introduced. In the present case, the bifurcation scenario is different: the systems are definitely „non-identical“, and the bifurcation structure is stable against certain parameter variations.

Thus, the oscillation death regime is observed in the interaction of a system with quasi-periodic autonomous dynamics and a chaotic system under strong coupling. As the coupling becomes weaker, periodic self-oscillations and a two-frequency torus emerge. A three-frequency torus, which undergoes several doubling bifurcations, arises from this two-frequency one as a result of a quasi-periodic Hopf bifurcation. The three-frequency torus then gets disrupted, and chaos with two additional zero Lyapunov exponents emerges. The Lyapunov exponents chart provides an opportunity to localize different types of regimes on the parameter plane. A new scenario of transition from the oscillation death regime to quasi-periodicity occurring in a certain parameter region in coupled systems was characterized.

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Conflict of interest

The authors declare that they have no conflict of interest.

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