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Elastic waves in media with bimodular nonlinearity taking into account the effects of reflection from shock fronts

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A theoretical study of the propagation of longitudinal strong low-frequency and weak high-frequency elastic waves in non-dispersing solids with a bimodular nonlinearity is carried out, taking into account the effects of reflection from the shock fronts of the wave. Expressions are obtained for the waveform, as well as for the amplitudes, frequencies, and phases of the harmonic components of the perturbation reflected from the discontinuities of the nonlinear wave. Numerical and graphical analysis of the obtained solutions is carried out. It is noted that the experimental study of the effects of wave reflection from discontinuities can be used to determine the nonlinear parameter of the bimodular solids.

Keywords: longitudinal elastic waves, bimodular nonlinearity, shock waves, reflection from a discontinuity.

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Introduction

The theory of nonlinear wave processes (NWPs) in homogeneous non-dispersing media with quadratic elastic nonlinearity is fully developed [1–3]. Nonlinear propagation of plane longitudinal waves in such media (without linear dissipation) is described by the equation of simple waves [1–3]. The solutions of this equation are valid up to the formation of a physically unrealizable ambiguity in the wave profile — „overlap"; to eliminate it, a shock front — gap is introduced into the wave profile, connecting the wave parameters on its profile [1–3]. After the formation of the „overlap" (and shock front), the wave ceases to be simple, while a wave perturbation occurs in the medium associated with the reflection of the continuous part of the wave running into the gap, and propagating in the opposite direction — to the emitter [1–5]. Certainly, the amplitude of the reflected from the gap is small and for its description it is necessary to take into account the following: the cubic term in the equation of state of the medium. To compare the magnitude of nonlinear effects arising from the propagation of an initially harmonic wave (IHW) in homogeneous media, we note that at the beginning, with a small distortion of the wave, the amplitude of its second harmonic is proportional to the square of the amplitude of the IHW, at the stage of a sawtooth shock wave, the amplitude of its second harmonic is proportional to the first degree of the amplitude of the shock wave, and the amplitude of the reflected from the gap the perturbation is proportional to the third degree of the shock wave amplitude [2,4,5]. Despite the relative smallness of the reflection effects, the characteristics of the harmonic components of the reflected perturbation (amplitude, frequency and phase) depend on the nonlinear properties of the medium, which can be used to determine its nonlinearity parameter.

The effects of the formation of „overlap" (and gap) in shock waves, as well as the occurrence of perturbation reflected from gaps, also occur in microinhomogeneous solid-state media containing various microdefects (cracks, grains, dislocations, etc.) [6] and having non-analytical, in particular, bimodular nonlinearity [7,8]. In such „non-analytical" media, the laws of the NWPs differ from similar laws for media with power-law — quadratic and cubic (analytical) nonlinearity. The identification of these differences is necessary for the development of the theory of NWPs in microinhomogeneous media with analytical nonlinearity, as well as for the classification of such media and the creation of effective methods for nonlinear acoustic diagnostics of defects in their structure.

Quite a lot of studies have been devoted to the study of nonlinear propagation of longitudinal elastic waves in bimodular solids without dispersion (and without taking into account reflection from gaps) [8–17]. In such media, the nonlinear propagation mode occurs only for heteropolar waves, and unipolar single pulses propagate linearly, with constant but different velocities depending on their polarity. In the first approximation (i.e., without taking into account reflection from gaps), the propagation of IHW in an ideal (without linear dissipation) bimodular medium occurs in such a way that at each period of the wave, a symmetrical „overlap"is formed in its profile, as well as in a quadratic medium, eliminated by the introduction of a gap — symmetric shock front [14]. As a result, the amplitudes of the higher (multiple) harmonics of such a shock wave are proportional to the first degree of the amplitude of the IHW, while due to nonlinear absorption at the gap, the wave completely attenuates at a finite distance. Considering the effects of wave reflection from gaps (second approximation), the NWPs dynamics in an environment with a multi-modulus nonlinearity will differ slightly from the above, which does not take into account reflections. To identify such differences, it is necessary to conduct appropriate theoretical studies.

In this paper, a theoretical study is carried out of the propagation of longitudinal strong low-frequency (LF) and weak high-frequency (HF) elastic waves in non-dispersing media with bimodular nonlinearity, taking into account the effects of wave reflection from its shock fronts.

1. Basic equations

The equation of state of an ideal (without linear dissipation) bimodular solid for longitudinal stresses σ and strains has the form [7,14]

$$\sigma(\varepsilon) = \left\{ \begin{array}{ll} E_1\varepsilon, & \varepsilon \geq 0 \\ E_2\varepsilon, & \varepsilon \leq 0 \end{array} \right| = E[\varepsilon - \gamma|\varepsilon|], \quad (1)$$

where $E_{1,2}$ — elastic modulus of the medium under its tension and compression, $E = \frac{E_1+E_2}{2}$, $\gamma = \frac{E_1-E_2}{E_1+E_2}$, $E_{1,2} = E(1 \pm \gamma)$, $\gamma \ll 1$, $\varepsilon \ll 1$. For solids with cracks $E_2 > E_1$, but for other microinhomogeneous materials it may be conversely $E_2 < E_1$.

When performing the inequality $|\varepsilon| \ll |\gamma| \ll 1$, it is possible not to take into account the geometric nonlinearity of the equations of motion in comparison with the physical (or material) nonlinearity of the equation of state. In this approximation, the equations of elasticity theory in Lagrangian and Eulerian forms coincide [14].

From the equation of motion $\rho_0 U_{tt} = \sigma_x(\varepsilon)$ [18] and the equation of state (1) we obtain (in two equivalent forms) a nonlinear wave equation for the longitudinal (along the axis x) strain $\varepsilon(t, x) = \partial U(t, x)/\partial x$:

$$\varepsilon_{tt} - C_0^2\varepsilon_{xx} = -\gamma C_0^2[|\varepsilon|]_{xx}, \;\; \varepsilon_{tt} - C_0^2[(1 - \gamma sign\varepsilon)\varepsilon]_{xx} = 0, \quad (2)$$

where $U = U(t, x)$ — displacement, ρ_0 — undisturbed density of the medium, $C_0 = (E/\rho_0)^{1/2}$. From the equations (2) it follows that their nonlinearity is determined not by the amplitude of strain, but by the sign of deformation (or its polarity). As a result, positive ($\varepsilon > 0$) and negative ($\varepsilon < 0$) pulse perturbations propagate at constant velocities C_1 and C_2, with $C_{1,2} = (E_{1,2}/\rho_0)^{1/2} = (E/\rho)^{1/2}(1 \pm \gamma) = C_0(1 \pm \gamma)$. (For certainty, we will assume that $\gamma > 0$, while $C_1 > C_2$.)

The boundary condition is defined as the sum of two harmonic (low and high frequency) oscillations with frequencies Ω and ω:

$$\varepsilon(t, x = 0) = \epsilon_0 \sin\Omega t + \epsilon_1 \sin(\omega t + \varphi), \quad (3)$$

where $\epsilon_0 \gg \epsilon_1$, $\Omega \ll \omega$, $\varphi = \text{const}$.

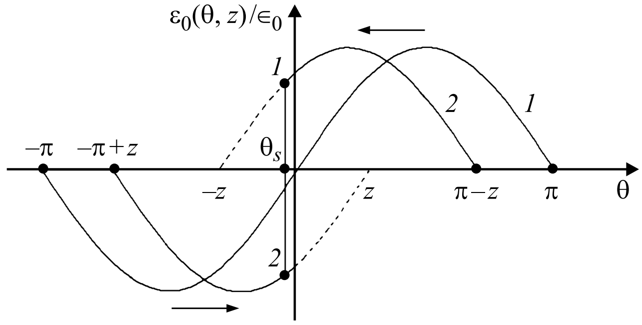

Figure 1. Evolution of IHW: *1* — form of oscillation at $z = 0$, *2* — waveprofile at a distance of $z < \pi$.

When waves with a frequency of Ω (or ω) are separately excited, a low frequency (or HF) deformation wave [14] will propagate in a bimodular medium:

$$\varepsilon_0(\theta, z) = \epsilon_0 \sin\left\{\Omega\left[t - \frac{x}{C_0\sqrt{1 + \gamma sign\varepsilon_0}}\right]\right\}$$
$$\approx \epsilon_0 \sin\left\{\Omega\left[t - \frac{x}{C(\gamma)} + \frac{\gamma x}{2C_0} sign\varepsilon_0\right]\right\}$$
$$= \epsilon_0 \sin\left[\Omega\tau + \frac{\gamma Kx}{2} sign\varepsilon_0\right] = \epsilon_0 \sin[\theta + z sign\varepsilon_0], \quad (4)$$

$$\varepsilon_1(\theta, z) = \epsilon_1 \sin\left\{\omega\left[t - \frac{x}{C_0\sqrt{1 + \gamma sign\varepsilon_0}}\right] + \varphi\right\}$$
$$\approx \epsilon_1 \sin[n(\theta + z sign\varepsilon_1) + \varphi], \quad (5)$$

where $\theta = \Omega\tau$, $\tau = t - \frac{x}{C(\gamma)}$, $C(\gamma) = \frac{2C_0}{(1+\gamma)^{-1/2}+(1-\gamma)^{-1/2}}$, $z = \frac{\gamma kx}{2} \geq 0$, $K = \frac{\Omega}{C_0}$, $\omega = n\Omega$, $n = \omega/\Omega \gg 1$.

Respectively, for the velocities $V_0(t, x) = \partial V_0(t, x)/\partial t$ and $V_1(t, x) = \partial V_1(t, x)/\partial t$ LF and HF waves have the following expressions:

$$V_0(\theta, z) = -C_0\epsilon_0 \sin[\theta - z signV_0], \quad (6)$$

$$V_1(\theta, z) = -C_0\epsilon_1 \sin[n(\theta - z signV_1) + \varphi]. \quad (7)$$

Solutions (4) and (5) describe the evolution of the LF (or HF) wave profile in a coordinate system moving at a speed of $C(\gamma)$. Figure 1 shows the evolution of the LF wave $\varepsilon_0 = \varepsilon_0(\theta, z)$ at a distance of $z < \pi$; the arrows show the directions of movement of the positive and negative half-periods of the wave. From expressions (4) and Figure 1 it follows that the distortion of the initially harmonic wave in bimodular medium occurs in such a way that at each period of the wave (at an arbitrarily small distance z) in its profile (at $|\theta| \leq z$) a „overlap“is formed, eliminated by introducing a gap — of the shock front at $\theta = \theta_S$. (When $\gamma < 0$, ambiguity occurs on the interval $|\pi - \theta| \leq z$.) In the first approximation by a small parameter γ, $C(\gamma) = C_0$ and there is no reflection from the gap [14], $\theta_S = 0$ for the low frequency wave, and $\theta_S = -\varphi/n$ for an RF wave, while the symmetric shock front relative to the medium moves at a speed of $C_S = C_0$, and $C_2 < C_S < C_1$. In the second approximation by γ, when $C(\gamma) = \frac{C_0}{1+3\gamma^2/8}$, there is a reflection from the gap, and its position is $\theta_S(z)$ shifts somewhat relative to the previous $\theta_S = 0$ (or $\theta_S = -\varphi/n$), while the asymmetric shock front will move at a speed of $C_S \neq C_0$, $C_2 < C_S < C_1$.

With the joint excitation of low-frequency and high-frequency waves, taking into account the reflection from the gap of the continuous part of the wave and the formation of a weak reflected perturbation $e_2(\tau_2)$, expressions for deformation waves $\varepsilon(\theta, z)$ and the velocities $V(\theta, z)$ have the form

$$\varepsilon(\theta, z) = \epsilon_0 \sin[\theta + z sign\varepsilon] + \epsilon_1 \sin[n(\theta + z sign\varepsilon) + \varphi] + e_2(\tau_2), \quad (8)$$

$$V(\theta, z) = -C_0\epsilon_0 \sin[\theta - z signV] - C_0\epsilon_1 \sin[n(\theta - z signV) + \varphi] + C_1 e_2(\tau_2), \quad (9)$$

where $\theta = \Omega\left[t - \frac{x}{C_0}\left(1 + \frac{3\gamma^2}{8}\right)\right]$, $\tau_2 = t + \frac{x}{C_1} \approx t + \frac{2z}{\gamma\Omega}$, i.e. the perturbation reflected from the gap $e_2(\tau_2)$ propagates at the rate of $C_1 > C_S > C_2$, while $e_2(\tau_2)$ there is only after the gap, where $\varepsilon(\theta, z) > 0$, and before the gap it is not. Here, the position of the gap $\widetilde{\theta}_S(z)$ in the shock wave $\varepsilon(\theta, z)$ will be slightly shifted by the action of a weak RF wave relative to the previous position $\theta_S(z)$ when $\epsilon 1 = 0$.

From the expressions (8), (9) we get the values $\varepsilon_1(\widetilde{\theta}_S, z)$, $\varepsilon_2(\widetilde{\theta}_S, z)$, $V_1(\widetilde{\theta}_S, z)$, $V_2(\widetilde{\theta}_S, z)$ and $e_2(\widetilde{\theta}_S, z)$ at the gap $\widetilde{\theta}_S = \widetilde{\theta}_S(z)$:

$$\varepsilon_1(\widetilde{\theta}_S, z) = \epsilon_0 \sin[\widetilde{\theta}_S + z] + \epsilon_1 \sin[n(\widetilde{\theta}_S + z) + \varphi] + e_2(\widetilde{\theta}_S, z) > 0,$$

$$\varepsilon_2(\widetilde{\theta}_S, z) = \epsilon_0 \sin[\widetilde{\theta}_S - z] + \epsilon_1 \sin[n(\widetilde{\theta}_S - z) + \varphi] < 0,$$

$$V_1(\widetilde{\theta}_S, z) = -C_0\epsilon_0 \sin[\widetilde{\theta}_S + z] - C_0\epsilon_1 \sin[\widetilde{\theta}_S + z + \varphi] + C_1 e_2(\widetilde{\theta}_S, z),$$

$$V_2(\widetilde{\theta}_S, z) = -C_0\epsilon_0 \sin[\widetilde{\theta}_S - z] - C_0\epsilon_1 \sin[\widetilde{\theta}_S - z + \varphi]. \quad (10)$$

The velocity $C_S(\widetilde{\theta}_S, z)$ and the position of the gap $\widetilde{\theta}_S(z)$, as well as the parameters of the perturbation reflected from it $e_2(\widetilde{\theta}_S, z)$ we define from the equations of elasticity theory for one-dimensional longitudinal waves [18]:

$$\rho_0 \frac{\partial V}{\partial t} = \frac{\partial \sigma(\varepsilon)}{\partial x}, \quad (11)$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial V}{\partial t} = 0, \quad (12)$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial V}{\partial x}, \quad (13)$$

where ρ — the perturbed density of the medium, $V(t, x) = \partial U(t, x)/\partial t$ — the velocity of the particles of the medium. Writing down the solution of equations (11)−(13) in the form of a stationary wave, i.e. assuming $V, \sigma, \varepsilon, \rho \propto F(\xi = x - C_S t)$ and integrating by ξ from $-\infty$ to $+\infty$, we get the boundary conditions on the gap

$$\rho_0 C_S(V_1 - V_2) = -(\sigma_1 - \sigma_2), \quad (14)$$

$$C_S(\rho_1 - \rho_2) = \rho_0(V_1 - V_2), \quad (15)$$

$$C_S(\varepsilon_1 - \varepsilon_2) = -(V_1 - V_2), \quad (16)$$

where the indices 1 and 2 correspond to the values of the shock wave parameters for and before the gap.

From the equations (15), (16) we obtain $\rho_1 - \rho_2 = -\rho_0(\varepsilon_1 - \varepsilon_2)$.

From expressions (10), (14), (16) find $C_S(\widetilde{\theta}_S, z)$:

$$C_S(\widetilde{\theta}_S, z) = \sqrt{\frac{\sigma_1 - \sigma_2}{\rho_0(\varepsilon_1 - \varepsilon_2)}} = \sqrt{\frac{E_1\varepsilon_1 - E_2\varepsilon_2}{\rho_0(\varepsilon_1 - \varepsilon_2)}} = C_0\sqrt{\frac{(1+\gamma)\varepsilon_1 - (1-\gamma)\varepsilon_2}{\varepsilon_1 - \varepsilon_2}}$$
$$\approx C_0\left(1 + \frac{\gamma}{2}\frac{\operatorname{tg}\widetilde{\theta}_S}{\operatorname{tg} z} + \frac{\gamma\epsilon_1}{2\epsilon_0 \cos\widetilde{\theta}_S \sin z} \times \left(\sin(n\widetilde{\theta}_S + \varphi)\cos nz - \frac{\operatorname{tg}\widetilde{\theta}_S}{\operatorname{tg} z}\cos(n\widetilde{\theta}_S + \varphi)\sin nz\right)\right), \quad (17)$$

where

$$\left|\frac{\gamma}{2}\frac{\operatorname{tg}\widetilde{\theta}_S}{\operatorname{tg} z} + \frac{\gamma\epsilon_1}{2\epsilon_0\cos\widetilde{\theta}_S \sin z}\left(\sin(n\widetilde{\theta}_S + \varphi)\cos nz - \frac{\operatorname{tg}\widetilde{\theta}_S}{\operatorname{tg} z}\cos(n\widetilde{\theta}_S + \varphi)\sin nz\right)\right| \ll 1.$$

Subtracting (16) from (14), we get the value of the strain $e_2(\widetilde{\theta}_S, z)$ at the gap $\widetilde{\theta}_S(z)$:

$$e_2(\widetilde{\theta}_S, z) \approx -\frac{\gamma}{2}\left[\epsilon_0 \sin\widetilde{\theta}_S \cos z + \epsilon_1 \sin(n\widetilde{\theta}_S + \varphi)\cos nz\right],$$
$$|e_2(\widetilde{\theta}_S, z)| \ll \epsilon_0. \quad (18)$$

To determine the position of $\widetilde{\theta}_S(z)$ the gap we will differentiate the expression $\widetilde{\theta}_S = \Omega\left[\widetilde{t}_S - \frac{x}{C_0}\left(1 + \frac{3\gamma^2}{8}\right)\right]$ by $z = \frac{\gamma K x}{2}$ and assuming that $\frac{d\widetilde{t}_S}{dx} = C_S^{-1}(\widetilde{\theta}_S, z)$,

we get the equation

$$\frac{d\widetilde{\theta}_S}{dz} = -\frac{3\gamma}{4} - \frac{\operatorname{tg}\widetilde{\theta}_S}{\operatorname{tg} z} - \frac{\epsilon_1}{\epsilon_0\cos\widetilde{\theta}_S \sin z}\left(\sin(n\widetilde{\theta}_S + \varphi)\cos nz - \frac{\operatorname{tg}\widetilde{\theta}_S}{\operatorname{tg} z}\cos(n\widetilde{\theta}_S + \varphi)\sin nz\right). \quad (19)$$

2. Propagation and reflection from gaps of a strong LF wave

Let us first consider the propagation and reflection from the gaps of one strong low-frequency wave, assuming that $\epsilon_1 = 0$. In this case $\widetilde{\theta}_S(z) = \theta_S(z)$ and from equation (19) we have

$$\frac{d\widetilde{\theta}_S}{dz} = -\frac{3\gamma}{4} - \frac{\operatorname{tg}\theta_S}{\operatorname{tg} z}. \tag{20}$$

From this equation it can be seen that the position of the gap θ_S in a bimodular medium for IHW does not depend on its amplitude. Approximate solution of equation (20) with boundary condition $\theta_S(z = 0) = 0$ has the form

$$\theta_S(z) \approx -\frac{3\gamma}{4} \operatorname{tg}\frac{z}{2} \le 0, \quad |\theta_S(z)| \ll 1, \tag{21}$$

where $z \le z_0 < \pi$, z_0 —
the distance (defined below) at which a unipolar pulse is formed at each IHW period.

It can be seen from the expression (21) that when $\gamma > 0$, the gap moves towards negative θ, so that the duration of the negative half-period in the wave decreases faster than the positive one. As a result, at some distance z_0, the negative half-period of the wave should disappear, and the remaining small part of the positive half-period will propagate at a linear velocity C_1. (At $\gamma < 0$, the gap formed near $\theta = \pi$ also moves to the left, but the positive half-period disappears, and the remaining part of the negative half-period will propagate with a linear velocity of C_2).

Let us show that the solution (21) is valid up to the formation (at $z = z_0$) of a unipolar positive pulse wave at each period of the initial LF wave. To do this, determine the position of the gap $\theta_S(z = z_0) = \theta_0$ at the moment of formation of a unipolar pulse, when the gap connects to the trailing edge of the negative half-period (point $-\pi + z_0$ in Figure 1). From this condition we obtain an equation for determining the distance z_0 of the formation of a unipolar pulse

$$-\pi + z_0 = -\frac{3\gamma}{2} \operatorname{tg}\frac{z_0}{2} = \theta_0. \tag{22}$$

Solving this equation by the perturbation technique, we obtain

$$z_0 \approx \pi - \sqrt{\frac{3\gamma}{2}} < \pi, \quad \theta_0 \approx -\sqrt{\frac{3\gamma}{2}}. \tag{23}$$

Substituting (23) into (21), it is easy to make sure that the solution (21) is valid everywhere $(0 \le z \le z_0)$, up to the formation of a unipolar pulse, the shape of which is shown in Figure 2; its amplitude and duration are $\varepsilon_0(z \ge z_0) = \sqrt{6\gamma}\epsilon_0 \ll \epsilon_0$ and $2\theta_0 \approx \sqrt{6\gamma} \ll 2\pi$. At $z \ge z_0$, the shape of the unipolar pulse does not change:

$$\varepsilon_0(\theta_1, z \ge z_0) = \epsilon_0 \sin\theta_1 \ge 0, \ |\theta_1| \le \theta_0 \approx \sqrt{3\gamma/2}, \tag{24}$$

where $\theta_1 = \Omega\tau_1$, $\tau_1 = t - x/C_1$.

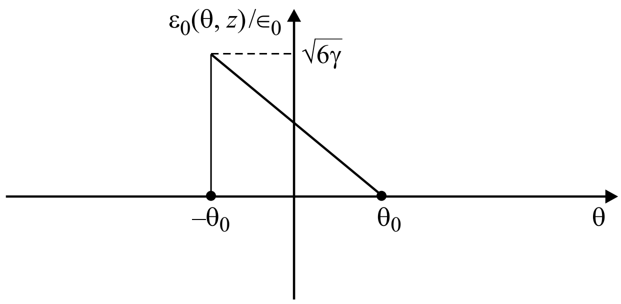

Figure 2. The form of a unipolar pulse at a distance of $z \ge z_0$.

Further, it follows from equations (15), (19) that the amplitude of the $e_{2,0}(\theta_S, z)$ of the perturbation reflected from the gap $\theta_S(z)$ is determined by the expression

$$e_{2,0}(\theta_S, z) \approx -\frac{\gamma}{2}\epsilon_0 \sin\theta_S(z)\cos z$$

$$\approx \frac{3\gamma^2\epsilon_0}{8} \operatorname{tg}\frac{z}{2} \cdot \cos z \ll \epsilon_0. \tag{25}$$

Let us decompose the periodic function (4) at $\theta_S(z) \approx -\frac{3\gamma}{2} \operatorname{tg}\frac{z}{2} \le 0$ (Figure 1) in Fourier series:

$$\varepsilon_0(\theta, z) = \epsilon_0 \cdot \begin{cases} 0, & -\pi \le \theta \le -\pi + z, \\ \sin(\theta - z) \le 0, & -\pi + z \le \theta \le \theta_S(z), \\ \sin(\theta + z) \ge 0, & \theta_S(z) \le \theta \le \pi - z, \\ 0, & \pi - z \le \theta \le \pi, \end{cases} \tag{26}$$

$$\varepsilon_0(\theta, z) = \epsilon_0\left(\frac{a_0(z)}{2} + \sum_{m=1}^{\infty} a_m(z)\cos m\theta + b_m(z)\sin m\theta\right), \tag{27}$$

where $a_0(z)$, $a_m(z)$ and $b_m(z)$ — Fourier series coefficients,

$$a_0(z) = -\frac{2\sin\theta_S(z)\sin z}{\pi} \approx \frac{3\gamma}{2\pi}\sin^2\frac{z}{2},$$

$$a_1(z) = -\frac{2\theta_S(z) + \sin 2\theta_S(z)}{2\pi}\sin z \approx \frac{3\gamma}{2\pi}\operatorname{tg}\frac{z}{2}\sin z$$

$$= \frac{3\gamma}{2\pi}\sin^2\frac{z}{2},$$

$$b_1(z) = \frac{(\pi - z)\cos z + \cos^2\theta_S(z)\sin z}{\pi}$$

$$\approx \frac{(\pi - z)\cos z + \sin z}{\pi},$$

$$a_m(z) = \frac{2[\sin\theta_S(z)\cos m\theta_S(z) - m\sin m\theta_S(z)\cos\theta_S(z)]}{\pi(m^2 - 1)},$$

$$b_m(z) = \frac{2\{\cos[(m-1)\theta_S(z)] + (m-1)\cos m\theta_S(z)\cos\theta_S(z)\}\sin z + (-1)^m\sin mz}{\pi(m^2 - 1)}.$$

When the low frequency wave $\varepsilon_0(\theta, z)$ is reflected from the gap $\theta_S(z)$ located at a distance of z $(0 \le z \le z_0)$, the reflected perturbation $e_{2,0}(\tau_2, z)$ will be defined by the expression

$$e_{2,0}(\tau_2, z) = \frac{3\gamma^2\epsilon_0}{8}\ \mathrm{tg}\ \frac{z}{2} \cdot \cos z \cdot \left(\frac{a_0(z)}{2} + \sum_{m=1}^{\infty} a_m(z)\right.$$

$$\left.\times \cos[m\Omega_2(z)\tau_2(t, z)] + b_m(z) \sin[m\Omega_2(z)\tau_2(t, z)]\right), \tag{28}$$

where is the frequency $\Omega_2(z)$ of the fundamental harmonic reflected from the gap $\theta_S(z)$ of the perturbation $e_{2,0}(\tau_2, z)$due to the double Doppler effect [1] has the form

$$\Omega_2(z) = \Omega\left(\frac{1-(C_S/C_1)}{1+(C_S/C_1)}\right) \approx \frac{\gamma\Omega}{4}\left(1+\frac{\gamma}{8}\right.$$

$$\left.\times\left(1 - 3\,\mathrm{tg}^2\ \frac{z}{2}\right)\right) \ll \Omega, \tag{29}$$

$$\tau_2(t, z) = t + x/C_1 \approx t + \frac{2z}{\gamma\Omega},$$

$$\Omega_2(z)\tau_2(t, z) = \Phi_1(t, z) + \Phi_2(t, z),$$

$$\Phi_1(t, z) = \frac{\gamma\Omega t}{4}\ \left(1 + \frac{\gamma}{8}\left(1 - 3\,\mathrm{tg}^2\ \frac{z}{2}\right)\right),$$

$$\Phi_2(t, z) = \frac{z}{2}\ \left(1 + \frac{\gamma}{8}\left(1 - 3\,\mathrm{tg}^2\ \frac{z}{2}\right)\right).$$

It follows from the expression (29) that the frequency $\Omega_2(z)$ of the perturbation reflected from the gap depends on the coordinate z the gap, while $\Omega_2(z=0) = \frac{\gamma\Omega}{4}\left(1 + \frac{\gamma}{8}\right)$, $\Omega_2(z = \frac{\pi}{2}) = \frac{\gamma\Omega}{4}\left(1 - \frac{\gamma}{4}\right)$ $\Omega_2(z = z_0) = \frac{\gamma^2\Omega}{32}$.

After simple transformations from the expression (28) we get

$$e_{2,0}(\tau_2, z) = \frac{3\gamma^2\epsilon_0}{8}\ \mathrm{tg}\,\frac{z}{2}\ \cos z\ \left(\frac{a_0(z)}{2}\right.$$

$$\left.+ \sum_{m=1}^{\infty} c_m(z)\cos m\Phi_1(t, z) - d_m(z)\sin m\Phi_1 t, z)\right), \tag{30}$$

where $c_m(z) = a_m(z)\cos m\Phi_2(z) + b_m(z)\sin m\Phi_2(z)$, $d_m(z) = a_m(z)\sin m\Phi_2(z) - b_m(z)\cos m\Phi_2(z)$.

Assuming in the equations (21), (24), (29), (30) $z = z_j \approx j\,\frac{\gamma K\Lambda}{2} \approx j\pi\gamma \le z_0 = \frac{\gamma K x_0}{2} \approx \pi - \sqrt{\frac{3\gamma}{2}}$ and summing reflections from all gaps $\theta_S(z_j) \approx -\frac{3\gamma}{4}\ \mathrm{tg}\,\frac{z_j}{4} = -\frac{3\gamma}{4}\ \mathrm{tg}\,\frac{j\pi\gamma}{2}$, we find

$$e_{2,0}(\tau_2, z) \approx \frac{3\gamma^2\epsilon_0}{8} \cdot \sum_{j=1}^{N} \mathrm{tg}\,\frac{z_j}{2} \cdot \cos z_j \cdot \left(\frac{a_0(z_j)}{2}\right.$$

$$\left.+ \sum_{m=1}^{\infty} c_m(z_j)\cos m\Phi_1(t, z_j) - d_m(z_j)\sin m\Phi_1(t, z_j)\right), \tag{31}$$

where

$$\Omega_2(z_j) = \frac{\gamma\Omega}{4}\left(1 + \frac{\gamma}{8}\left(1 - 3\,\mathrm{tg}^2\ \frac{j\pi\gamma}{2}\right)\right) \ll \Omega,$$

$$\tau_2(z_j) \approx t + \frac{2z_j}{\gamma\Omega}, \quad j = 0, 1, 2, \ldots,$$

$$x_0 = \frac{2z_0}{\gamma K} = \frac{\Lambda}{\gamma}\left(1 - \frac{1}{\pi}\sqrt{\frac{3\gamma}{2}}\right), \quad \Lambda = \frac{2\pi}{K},$$

$$j \le N = \left[\frac{x_0}{\Lambda}\right] \approx \left[\frac{1}{\gamma} - \frac{1}{\pi}\sqrt{\frac{3}{2\gamma}}\right] \gg 1,$$

$$\Phi_1(t, z_j) = \frac{\gamma\Omega t}{4}\ \left(1 + \frac{\gamma}{8}\left(1 - 3\,\mathrm{tg}^2\ \frac{z_j}{2}\right)\right),$$

$$\Phi_2(z_j) = \frac{z}{2}\ \left(1 + \frac{\gamma}{8}\left(1 - 3\,\mathrm{tg}^2\ \frac{z_j}{2}\right)\right).$$

Since $\pi\gamma \ll 1$, the sum of j in (31), it can be replaced by an integral over $y = z_j = j\pi\gamma$:

$$e_{2,0}(\tau_2) \approx \frac{3\gamma\epsilon_0}{8\pi} \cdot \int_0^{\pi-\sqrt{3\gamma/2}} \mathrm{tg}\,\frac{y}{2} \cdot \cos y \cdot \left(\frac{a_0(y)}{2}\right.$$

$$\left.+ \sum_{m=1}^{\infty} c_m(y)\cos m\Phi_1(t, y) - d_m(y)\sin m\Phi_1(t, y)\right)dy, \tag{32}$$

where

$$c_m(y) = a_m(y)\cos m\Phi_2(y) + b_m(y)\sin m\Phi_2(y),$$

$$d_m(y) = a_m(y)\sin m\Phi_2(y) - b_m(y)\cos m\Phi_2(y),$$

$$\Omega_2(y) = \frac{\gamma\Omega}{4}\ \left(1 + \frac{\gamma}{8}\left(1 - 3\,\mathrm{tg}^2\ \frac{y}{2}\right)\right), \quad \tau_2(y) \approx t + \frac{2y}{\gamma\Omega},$$

$$\Omega_2(y)\tau_2(y) = \left(\frac{\gamma\Omega t}{4} + \frac{y}{2}\right)\left(1 + \frac{\gamma}{8}\left(1 - 3\,\mathrm{tg}^2\ \frac{y}{2}\right)\right)$$

$$= \Phi_1(t, y) + \Phi_2(y),$$

$$\Phi_1(t, y) = \frac{\gamma\Omega t}{4}\ \left(1 + \frac{\gamma}{8}\left(1 - 3\,\mathrm{tg}^2\ \frac{y}{2}\right)\right),$$

$$\Phi_2(y) = \frac{y}{2}\ \left(1 + \frac{\gamma}{8}\left(1 - 3\,\mathrm{tg}^2\ \frac{y}{2}\right)\right).$$

It can be seen from the expression (32) that the reflected perturbation is $e_{2,0}(\tau_2)$ contains a constant component and a set of low-frequency harmonics with frequencies $m\Omega_2 \approx m(\gamma\Omega/4) \ll \Omega$, whose amplitudes decrease with increasing numbers m. Generally speaking, calculating them is quite a difficult task. However, it is easy to obtain an expression for the constant component of the strain $e_{2,0}(\tau_2)$:

$$\langle e_{2,0}(\tau_2)\rangle \approx \frac{3\gamma\epsilon_0}{16\pi} \cdot \int_0^{\pi-\sqrt{3\gamma/2}} a_0(y) \cdot \mathrm{tg}\,\frac{y}{2} \cdot \cos y\, dy$$

$$= \frac{9\gamma^2\epsilon_0}{16\pi^2}\left(1 + \ln\sqrt{\frac{3\gamma}{2}}\right) < 0.$$

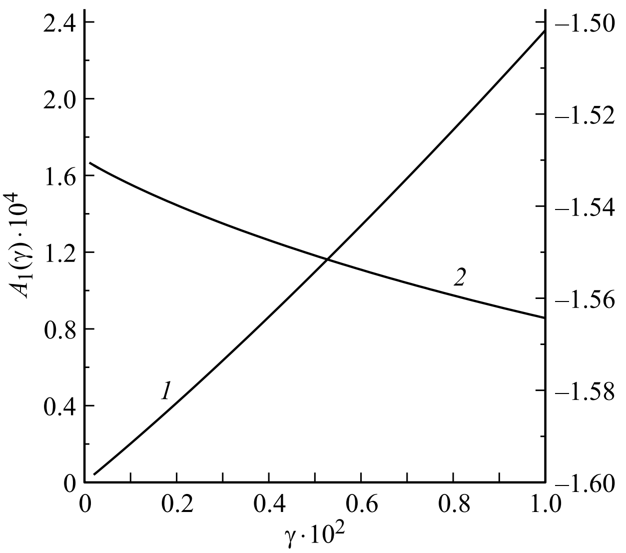

Figure 3. Dependences: *1* — $A_1 = A_1(\gamma)$, *2* — $\psi_1 = \psi_1(\gamma)$.

It is also possible to obtain approximate expressions for the normalized by ϵ_0 amplitudes $A_m(\gamma)$ of harmonic components of the wave $e_{2,0}(\tau_2)$ with frequencies $m\Omega_2 \approx m\gamma\Omega/4$ and their phases $\psi_m(\gamma)$, assuming in (32) $\Omega_2 \approx \frac{\gamma\Omega}{4}$, $\Phi_1(t, y) \approx \frac{\gamma\Omega t}{4}$, $\Phi_2(y) \approx \frac{y}{2}$:

$$e_{2,0}(\tau_2) \approx \frac{3\gamma\epsilon_0}{8\pi} \cdot \int\limits_0^{\pi-\sqrt{3\gamma/2}} \operatorname{tg}\frac{y}{2} \cdot \cos y \cdot \left(\frac{a_0(y)}{2} + \sum_{m=1}^{\infty} c_m(y)[m\Omega_2 t] - d_m(y)\sin[m\Omega_2 t]\right) dy,$$

$$A_m(\gamma) = \sqrt{A_{mS}^2(\gamma) + A_{mC}^2(\gamma)}, \quad \operatorname{tg}\psi_m(\gamma) = \frac{A_{mS}(\gamma)}{A_{mC}(\gamma)}, \quad (33)$$

where

$$A_{mC}(\gamma) = \frac{3\gamma}{8\pi} \cdot \int\limits_0^{\pi-\sqrt{3\gamma/2}} \operatorname{tg}\frac{y}{2} \cdot \cos y \cdot c_m(y) dy,$$

$$A_{mS}(\gamma) = \frac{3\gamma}{8\pi} \cdot \int\limits_0^{\pi-\sqrt{3\gamma/2}} \operatorname{tg}\frac{y}{2} \cdot \cos y \cdot d_m(y) dy,$$

$$c_m(y) = a_m(y)\cos\frac{my}{2} + b_m(y)\cos\frac{my}{2},$$

$$d_m(y) = a_m(y)\sin\frac{my}{2} - b_m(y)\cos\frac{my}{2}.$$

Figure 3 shows the dependencies $A_1 = A_1(\gamma)$ and $\psi_1 = \psi_1(\gamma)$, from which it follows that in the range $10^{-4} \le \gamma \le 10^{-2}$ the dependence of A_1 on γ is close to linear — $A_1(\gamma) \propto \gamma$, $A_1(\gamma) \ll 1$, while $\psi_1(\gamma)$ weakly depends on $\gamma - \psi_1(\gamma) \approx -\pi/2$.

3. Joint propagation of strong LF and weak HF waves

As already noted, with the joint propagation of strong low-frequency and weak HF waves, the position of the gap $\widetilde{\theta}_S(z)$ in the wave $\varepsilon = \varepsilon(\theta, z)$ will be slightly shifted by the weak HF wave relative to the previous position $\widetilde{\theta}_S(z) \approx -\frac{3\gamma}{4}\operatorname{tg}\frac{z}{2} \le 0$ when $\epsilon_1 = 0$. Solving equation (19) by the perturbation technique, i.e. assuming that $\widetilde{\theta}_S(z) = \theta_S(z) + \psi_S(z)$, $|\psi_S(z)| \ll |\theta_S(z)|$, we get the equation for $\psi_S(z)$:

$$\frac{d\psi_S}{dz} = -\frac{\psi_S}{\operatorname{tg} z} - \frac{\epsilon_1}{\epsilon_0 \sin z}\left(\sin(n\theta_S + \varphi)\cos nz + \frac{3\gamma}{8}\left(1 - \operatorname{tg}^2\frac{z}{2}\right)\cos(n\theta_S + \varphi)\sin nz\right). \quad (34)$$

A solution of the equation (34) is as follows

$$\psi_S(z) \approx -\frac{\epsilon_1}{n\epsilon_0}\frac{\sin nz}{\sin z}\sin[n\theta_S(z) + \varphi]. \quad (35)$$

In this case $\widetilde{\theta}_S(z) = \theta_S(z) - \frac{\epsilon_1}{n\epsilon_0}\frac{\sin nz}{\sin z}\sin[n\theta_S(z) + \varphi]$ From the expression (18) we find:

$$e_2(\widetilde{\theta}_S, z) \approx -\frac{\gamma}{2}\epsilon_0 \sin\theta_S(z)\cos z - \frac{\gamma}{2}\epsilon_1\left(1 - \frac{\operatorname{tg} nz}{n\operatorname{tg} z}\right) \times \sin[n\theta_S(z) + \varphi]\cos nz. \quad (36)$$

It can be seen from the expression (36) that a weak HF wave does not affect the amplitude of a strong low-frequency wave reflected from the gap, and a strong low-frequency wave affects the amplitude of a reflected weak HF wave (via $\theta_S(z) \approx \frac{3\gamma}{4}\operatorname{tg}\frac{z}{2}$), at

$$e_{2,1}(\widetilde{\theta}_S, z) \approx -\frac{\gamma}{2}\epsilon_1\left(1 - \frac{\operatorname{tg} nz}{n\operatorname{tg} z}\right)\sin[n\theta_S(z) + \varphi]\cos nz. \quad (37)$$

When the wave is reflected $\varepsilon_1(\theta, z)$ from all gaps $\widetilde{\theta}_S(z_j)$ located at a distance of $0 \le z_j \le z_0$, reflected perturbation $e_{2,1}(\tau_2)$ will be defined by the expression

$$e_{2,1}(\tau_2) \approx -\frac{\gamma\epsilon_1}{2}\sum_{j=1}^{N}\left(1 - \frac{\operatorname{tg} nz_j}{n\operatorname{tg} z_j}\right)\sin[n\theta_S(z_j) + \varphi] \times \cos nz_j \cdot \sin[\omega_2(z_j)\tau_2(z_j) + \varphi]$$
$$= -\frac{\gamma\epsilon_1}{2}\sum_{j=1}^{N}\left(1 - \frac{\operatorname{tg} nz_j}{n\operatorname{tg} z_j}\right)\sin[n\theta_S(z_j) + \varphi]\cos nz_j \times \left\{\cos[\Psi_2(z_j) + \varphi]\sin\Psi_1(t, z_j) + \sin[\Psi_2(z_j) + \varphi] \times \cos\Psi_1(t, z_j)\right\}, \quad (38)$$

where

$$\omega_2(z_j) = \frac{1-(C_S/C_1)}{1+(C_S/C_1)}\omega$$

$$\approx \frac{\gamma\omega}{4}\left(1+\frac{\gamma}{8}\left(1-3\,\mathrm{tg}^2\,\frac{z_j}{2}\right)\right) \gg \Omega_2(z_j),$$

$$\omega_2(z=0) = \frac{\gamma\omega}{4}\left(1+\frac{\gamma}{8}\right),$$

$$\omega_2(z=\pi/2) = \frac{\gamma\omega}{4}\left(1-\frac{\gamma}{4}\right),$$

$$\omega_2(z=z_0) = \frac{\gamma^2\omega}{32}, \quad \tau_2(z_j) = t + x/C_1 \approx t + \frac{2z_j}{\gamma\Omega},$$

$$\omega_2(z_j)\tau_2(z_j) \approx \left(\frac{\gamma\omega t}{4}+\frac{nz_j}{2}\right)\left(1+\frac{\gamma}{8}\left(1-3\,\mathrm{tg}^2\,\frac{z_j}{2}\right)\right)$$

$$= \Psi_1(t,z_j) + \Psi_2(t,z_j),$$

$$\Psi_1(t,z_j) = \frac{\gamma\omega t}{4}\left(1+\frac{\gamma}{8}\left(1-3\,\mathrm{tg}^2\,\frac{z_j}{2}\right)\right),$$

$$\Psi_2(z_j) = \frac{nz_j}{2}\left(1+\frac{\gamma}{8}\left(1-3\,\mathrm{tg}^2\,\frac{z_j}{2}\right)\right).$$

Replacing the amount by j in (38) by the integral of $y = z_j = j\pi\gamma$, we get

$$e_{2,1}(\tau_2) = -\frac{\epsilon_1}{2\pi}\cdot\int_0^{\pi-\sqrt{3\gamma/2}}\left(1-\frac{\mathrm{tg}\,ny}{n\,\mathrm{tg}\,y}\right)\cdot\sin[n\theta_S(y)+\varphi]$$
$$\times\cos ny\cdot\{\cos[\Psi_2(y)+\varphi]\sin\Psi_1(t,y)$$
$$+\sin[\Psi_2(y)+\varphi]\cos\Psi_1(t,y)\}dy, \qquad (39)$$

where

$$\Psi_1(t,y) = \frac{\gamma\omega t}{4}\left(1+\frac{\gamma}{8}\left(1-3\,\mathrm{tg}^2\,\frac{y}{2}\right)\right),$$

$$\Psi_2(y) = \frac{ny}{2}\left(1+\frac{\gamma}{8}\left(1-3\,\mathrm{tg}^2\,\frac{y}{2}\right)\right).$$

Here you can also get approximate expressions for the normalized by ϵ_1 amplitude $B(\gamma,\varphi) = |e_{2,1}(\tau_2)/\epsilon_1|$of the wave $e_{2,1}(\tau_2)$ with a frequency of $\omega_2 \approx \gamma\omega/4$ and its phase $\psi(\gamma,\varphi)$, assuming in (39) $\omega_2(y) \approx \frac{\gamma\omega}{4}$, $\Psi_1(t,y) \approx \frac{\gamma\omega t}{4}$, $\Phi_2(y) \approx \frac{ny}{2}$:

$$e_{2,1}(\tau_2) \approx -\frac{\epsilon_1}{2\pi}\cdot\int_0^{\pi-\sqrt{3\gamma/2}}\left(1-\frac{\mathrm{tg}\,ny}{n\,\mathrm{tg}\,y}\right)\cdot\sin[n\theta_S(y)+\varphi]$$
$$\times\cos ny\cdot\left(\cos\left[\frac{ny}{2}+\varphi\right]\sin\frac{\gamma\omega t}{4}+\sin\left[\frac{ny}{2}+\varphi\right]\right.$$
$$\left.\times\cos\frac{\gamma\omega t}{4}\right)dy, \quad |e_{2,1}(\tau_2)| \ll \epsilon_1. \qquad (40)$$

$$B(\gamma,\varphi) = \sqrt{B_C^2(\gamma,\varphi)+B_S^2(\gamma,\varphi)},$$

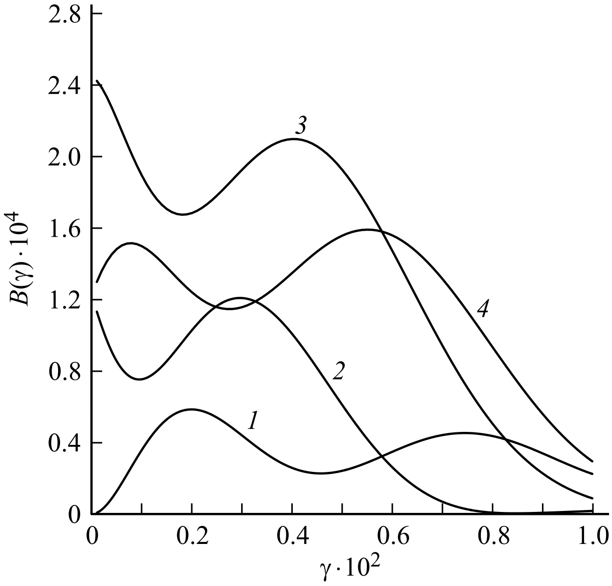

Figure 4. Dependencies $B = B(\gamma,\varphi)$ from γ at $n = 32$ and various values φ: *1* — 0, *2* — $\pi/4; 5\pi/4$, *3* — $\pi/3; 3\pi/2$, *4* — $3\pi/4; 7\pi/4$.

$$\mathrm{tg}\,\phi(\gamma,\varphi) = \frac{B_S(\gamma,\varphi)}{B_C(\gamma,\varphi)},$$

where

$$B_C(\gamma,\varphi) = -\frac{1}{2\pi}\cdot\int_0^{\pi-\sqrt{3\gamma/2}}\left(1-\frac{\mathrm{tg}\,ny}{n\,\mathrm{tg}\,y}\right)\cdot\sin[n\theta_S(y)+\varphi]$$
$$\times\cos ny\cdot\sin\left[\frac{ny}{2}+\varphi\right]dy,$$

$$B_S(\gamma,\varphi) = -\frac{1}{2\pi}\cdot\int_0^{\pi-\sqrt{3\gamma/2}}\left(1-\frac{\mathrm{tg}\,ny}{n\,\mathrm{tg}\,y}\right)\cdot\sin[n\theta_S(y)+\varphi]$$
$$\times\cos ny\cdot\cos\left[\frac{ny}{2}+\varphi\right]dy.$$

From the expressions (40) it can be seen that the characteristics of the wave reflected from the gaps $e_{2,1}(\tau_2)$ depend not only on the parameter γ of the nonlinearity of the medium, but also on the phase of the HF wave φ and on the value of $n = (\omega/\Omega) \gg 1$. Figures 4 and 5 show the dependencies $B(\gamma,\varphi)$ and $\phi(\gamma,\varphi)$ from the parameter γ for different φ and $n = 32$. From these figures it can be seen that these dependencies are non-monotonic and have a complex form. Figure 6 shows the dependencies $B(\gamma,\varphi)$ and $\phi(\gamma,\varphi)$ from the phase φ, with $\gamma = 10^{-3}$ and $n = 32$. These dependencies are periodic with a period equal to π, while the value of $B(\gamma,\varphi)$ within this period varies very noticeably — almost 10 times. Figure 7 shows the dependencies of $B(\gamma,\varphi)$ and $\phi(\gamma,\varphi)$ from n with $\gamma = 10^{-3}$ and $\varphi = \pi/2$. There are rather complex, non-monotonic and quasi-periodic dependencies

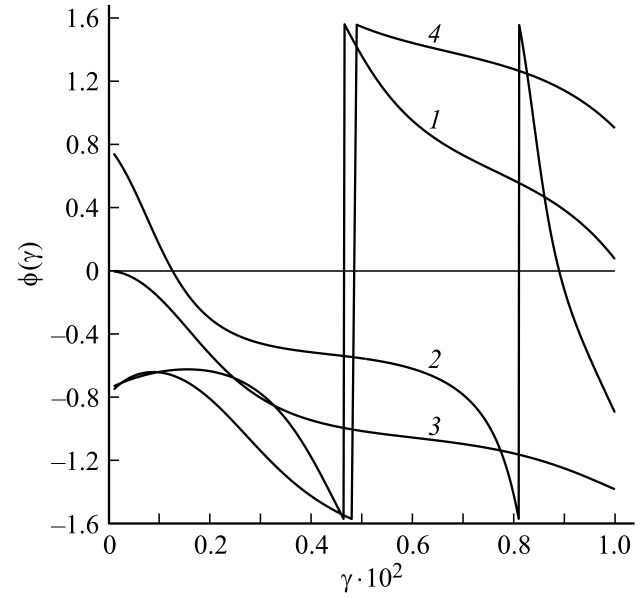

Figure 5. Dependencies $\phi(\gamma, \varphi)$ from γ at $n = 32$ and various values of φ: *1* — 0, *2* — $\pi/4; 5\pi/4$, *3* — $\pi/3; 3\pi/2$, *4* — $3\pi/4; 7\pi/4$.

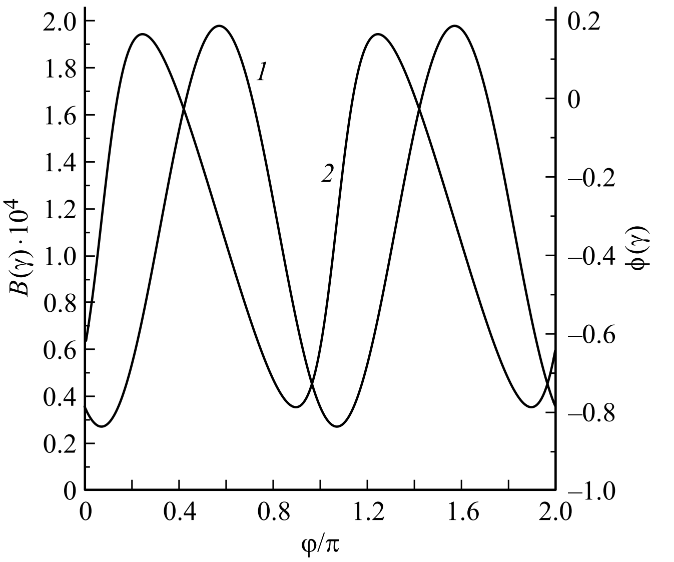

Figure 6. Dependencies: $B = B(\gamma, \varphi)$ (*1*), $\phi = \phi(\gamma, \varphi)$ (*2*) from the phase φ with $\gamma = 10^{-3}$ and $n = 32$.

$B(\gamma, \varphi)$ and $\phi(\gamma, \varphi)$from n related to the interference of wave perturbations arising from the reflection of a weak HF wave from many shock wave gaps (8), while $B(\gamma, \varphi)$ decreases with the growth of n (on average, of course).

Finally, we note that for $z \geq z_0 \approx \pi - \sqrt{3\gamma/2}$, i.e. after the formation of a unipolar pulse wave at each period, the wave propagating in the positive direction of the x axis will be a periodic sequence of unipolar pulses (24) and HF wave

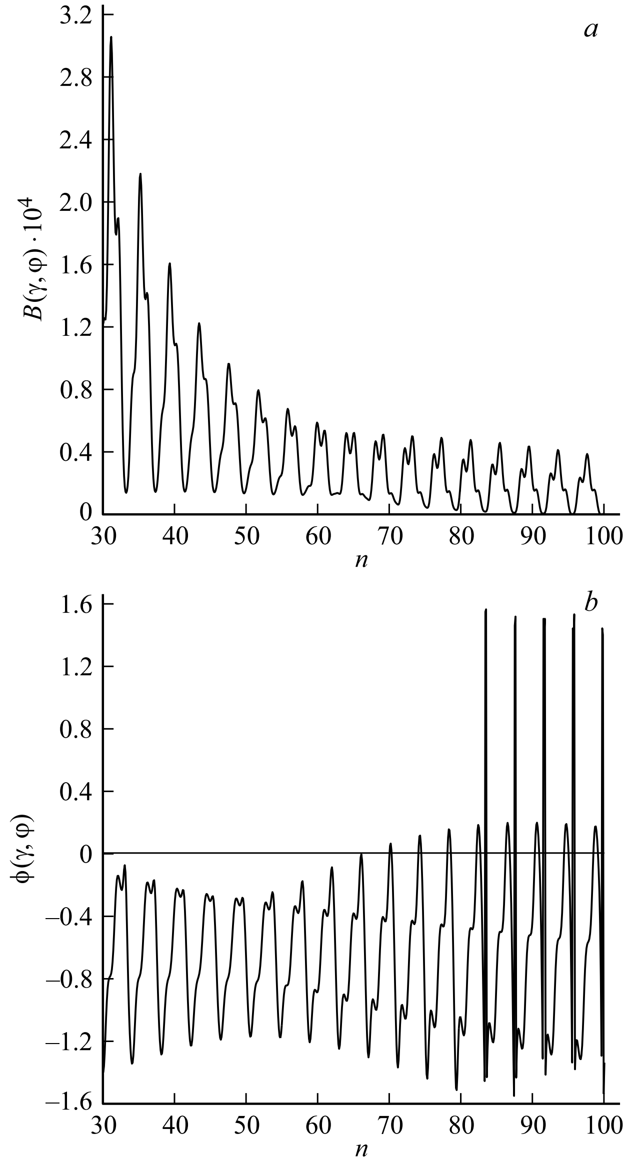

Figure 7. Dependencies $B(\gamma, \varphi)$ (*a*) and $\phi(\gamma, \varphi)$ (*b*) from n at $\gamma = 10^{-3}$ and $\varphi = \pi/2$.

pulses $\varepsilon_1(\theta, z)$ propagating at a speed of C_1:

$$\varepsilon(\theta_1, z \geq z_0) \approx \gamma[\epsilon_0 \sin\theta_1 + \epsilon_1 \sin(n\theta_1 + \varphi)]$$

$$\times \sum_{p=1}^{\infty} [h(\theta_1 + \theta_0) - 2\pi p) - [h(\theta_1 - \theta_0) - 2\pi p)] > 0,$$

where $\theta_1 = \Omega\tau_1$, $\tau_1 = t - x/C_1$, $\theta_0 = \sqrt{3\gamma/2}$, $h(\theta_1)$ — Heaviside function, $\epsilon_1 \ll \sqrt{6\gamma}\epsilon_0$.

Conclusion

In this paper, a theoretical study of the propagation of longitudinal initially harmonic strong low-frequency and weak high-frequency elastic waves in an ideal non-dispersing medium with bimodular nonlinearity, taking into account the effects of reflection from shock wave fronts, is carried out. Expressions are obtained for the nonlinear wave, as well as for the amplitudes, frequencies and phases of the harmonic components of the perturbation reflected from the gaps of the shock wave. It is shown that the nonlinear mode of propagation of a strong low-frequency harmonic wave initially takes place only up to a finite distance $z < z_0 \approx \pi - \sqrt{3\gamma/2}$, after which a weak unipolar pulse with an amplitude of ε_0 is formed at each period of the low-frequency wave $(z \geq z_0) = \sqrt{6\gamma}\epsilon_0 \ll \epsilon_0$ and duration $2\theta_0 \approx \sqrt{6\gamma} \ll 2\pi$, which propagates at a speed of C_1 without changing the form. With the combined propagation of LF and HF waves, at $z \geq z_0$, the shape of the nonlinear wave $\varepsilon(\theta, z)$ is a periodic sequence of unipolar LF and HF pulses propagating at a speed of C_1. It is shown that the amplitudes, frequencies and phases of the harmonic components of the strong low-frequency wave reflected from the gaps are determined by the parameter of the bimodular nonlinearity of the medium, and the same characteristics of the reflected weak high-frequency wave also depend on its initial phase and on the value of $n = (\omega/\Omega) \gg 1$. Thus, the results of the study of the effects of propagation of longitudinal LF and HF waves and their reflection from gaps can be used to determine the nonlinear properties of bimodular media when conducting appropriate experiments. The results obtained are of interest for the development of the theory of wave processes in media with non-analytical nonlinearity; they can also be used to create nonlinear techniques of acoustic diagnostics of microinhomogeneous solids and materials containing cracks.

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Conflict of interest

The author declares that he has no conflict of interest.

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