Technical Physics, 2022, Vol. 67, No. 14

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Two coupled quasiperiodic generators excited by external force

© A.P. Kuznetsov,[1] Yu.V. Sedova,[1] N.V. Stankevich[1,2]

[1] Saratov Branch, Kotel'nikov Institute of Radio Engineering and Electronics, Russian Academy of Sciences, 410019 Saratov, Russia
[2] National Research University Higher School of Economics, 603155 Nizhny Novgorod, Russia
e-mail: sedovayv@yandex.ru

Received May 17, 2021
Revised June 7, 2021
Accepted June 8, 2021

A system of two dissipatively coupled generators, which can exhibit autonomous quasiperiodic oscillations, excited by a harmonic signal, is studied. Lyapunov charts are presented that reveal the regimes of invariant tori of different dimensions and chaos. Phase portraits in stroboscopic section and double Poincare section are presented. The coexistence of different regimes, in particular, the bifurcations of invariant tori, is discussed.

Keywords: quasiperiodic oscillations, coupled generators, Lyapunov charts, invariant torus, Poincare section, quasiperiodic bifurcation.

DOI: 10.21883/TP.2022.14.55214.145-21

Introduction

It is known several examples of autonomous oscillators capable of exhibiting quasi-periodic oscillations. This is one of Chua's circuits [1], the modified Anishchenko-Astakhov oscillator [2–7], the family of autonomous oscillators with minimal phase space dimension [8]. Quasi-periodic oscillations occupy a kind of intermediate position between periodic and chaotic oscillations, but the dynamics of autonomous systems with quasi-periodicity has been little studied. At the same time, it is of fundamental interest. Two coupled quasi-periodic oscillators are studied in [9]. They demonstrate a sufficient number of interesting phenomena, including the effect of „oscillator death", synchronized quasi-periodicity, etc. It is of interest to study the excitation of such a system by an external periodic signal, i.e. study of a kind of control over the dynamics of coupled quasi-periodic oscillators. As we will show, the structure obtained is quite complicated, it is interesting to compare it with the problem of excitation of two coupled van der Pol oscillators by harmonic signal [10–12].

1. Chart of Lyapunov exponents

In accordance with [9], the system of coupled quasi-periodic oscillators is described by equations in which an external action is added:

$$\ddot{x}_1 - (\lambda_1 + z_1 + x_1^2 - \beta x_1^4)\dot{x}_1 + \omega_0^2 x_1 + M_c(\dot{x}_1 - \dot{x}_2) = a\cos\omega t,$$

$$\dot{z}_1 = b(\varepsilon - z_1) - k\dot{x}_1^2,$$

$$\ddot{x}_2 - (\lambda_2 + z_2 + x_2^2 - \beta x_2^4)\dot{x}_2 + (\omega_0 + \Delta)^2 x_2 + M_c(\dot{x}_2 - \dot{x}_1) = 0,$$

$$\dot{z}_2 = b(\varepsilon - z_2) - k\dot{x}_2^2, \qquad (1)$$

where x_1, $y_1 = \dot{x}_1$, z_1 are variables characterizing the first oscillator, x_2, $y_2 = \dot{x}_2$, z_2 are second oscillator variables, Δ is frequency mismatch of oscillators, M_C is dissipative coupling coefficient, ω is frequency of external signal, a is its amplitude. Next, we will use the parameter values $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\lambda = -1$, $\beta = 1/18$, $\omega_0 = 2\pi$.

Figure 1, a shows the Lyapunov chart of the system (1) without external influence (i.e., at $a = 0$) on the parameter plane, the frequency mismatch Δ — the value of the coupling M_C [9]. Different colors show the types of modes, the color palette is presented next to the picture. Modes of „oscillation death" OD, synchronous periodic modes P, broadband quasi-periodicity mode BQ were visualized. In Fig. 1 the designations T_n correspond to the n-frequency torus. Note that for a flow system, one of the Lyapunov exponents Λ is always equal to zero, and we will discard it. Thus, the case $\Lambda_1 = 0$ corresponds to the two-frequency torus T_2. An increase in the dimension of the torus by one corresponds to an increase by one in the number of zero Lyapunov exponents. For example, a three-frequency torus T_3 corresponds to the condition $\Lambda_1 = \Lambda_2 = 0$, and so on. Chaos C and hyperchaos CH modes correspond to one and two positive Lyapunov exponents: $\Lambda_1 > 0$ and $\Lambda_1 >, \Lambda_2 > 0$.

Along the periphery of Fig. 1, maps of the excitable system (1) are shown on the plane: frequency ω — amplitude of external signal a at selected points of the autonomous system for some characteristic modes: periodic mode (b), two-frequency torus (c), three-frequency torus (d), four-frequency torus (e) and chaotic mode (f).

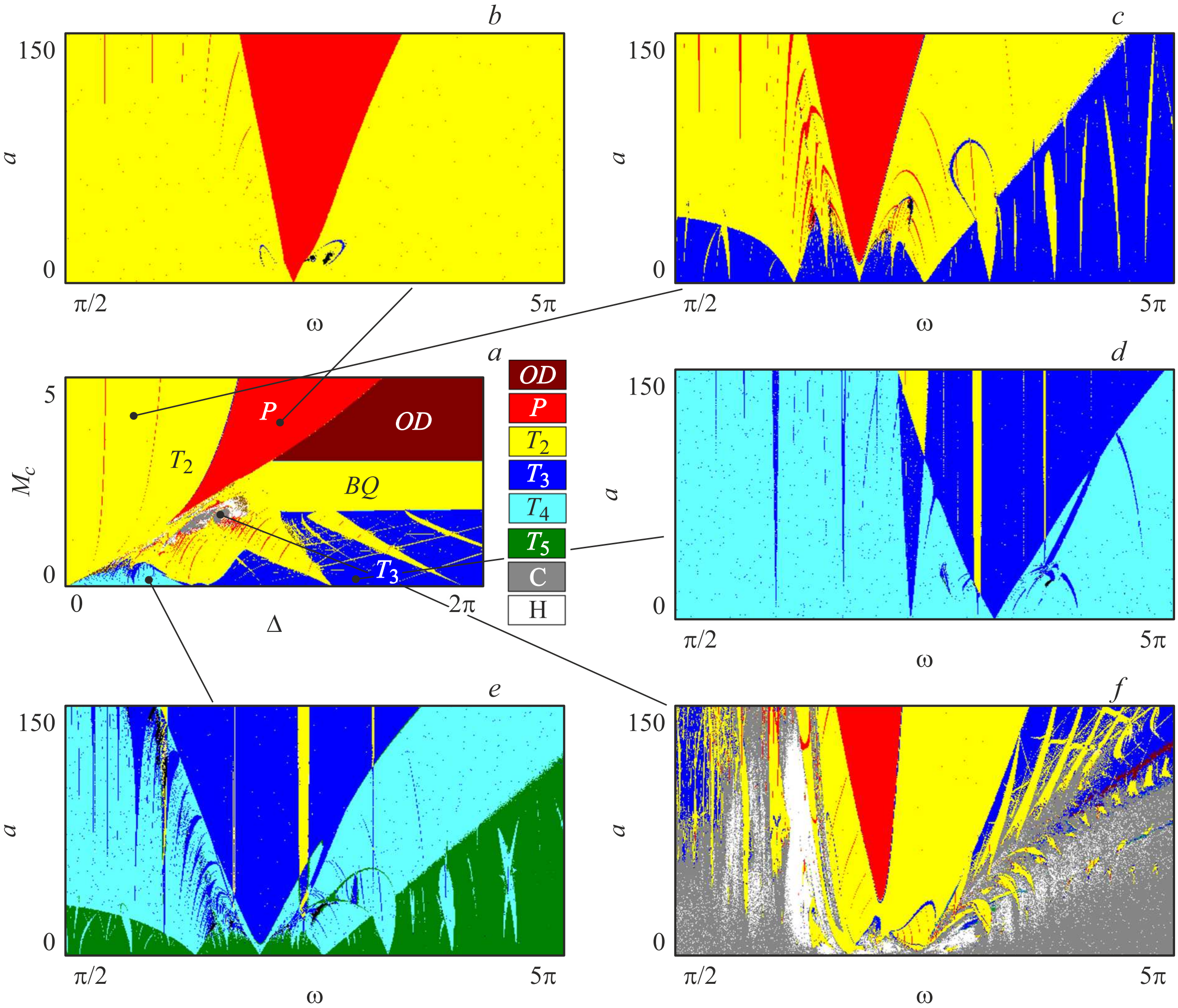

Figure 1. Lyapunov exponent charts of system (1): a — system without external influence, b–f — system with external signal. Parameter values: b are $\Delta = 3.14$, $M_c = 4$; c — $\Delta = 0.9$, $M_c = 4$; d — $\Delta = 4.3$, $M_c = 0.2$; e — $\Delta = 0.75$, $M_c = 0.05$; f — $\Delta = 2.58$, $M_c = 1.8$.

2. Illustration of dynamics and discussion of results

Fig. 1, b represents the case of the periodic mode of the autonomous model. As can be seen, in the excitable system, a periodic regime is observed in the form of Arnold tongue with a tip on the frequency axis, immersed in the region of two-frequency quasi-periodicity. This is traditional for the excitable model with limit cycle [13–15]. At the same time, very narrow tongues of synchronization on subharmonics and small regions of chaotic dynamics are observed.

Fig. 1, c corresponds to the case of a two-frequency torus T_2 in the autonomous system. The parameter values are chosen in such a way that the synchronous quasi-periodicity mode is realized in the autonomous mode, when the phases of the oscillators are mutually locked, but the system as a whole demonstrates quasi-periodicity [9]. At small values of the signal amplitude a, three-frequency tori are observed in Fig. 1, c. System of rather narrow tongues of two-frequency modes, which are resonant with respect to three-frequency tori, is built into the region of their existence. At large values of the signal amplitude, wide regions of two-frequency tori arise. Such structure is quite natural — the external signal locks one of the two frequencies of the autonomous model. At the same time, a fairly large region of periodic modes arises. Its difference from Fig. 1, b is the presence of a threshold in terms of the impact amplitude. Thus, the external signal in this case locks both frequencies of the autonomous system. Similar effect is also observed in the system of two coupled van der Pol [12] oscillators. When acting on coupled oscillators in the beatings regime, with the sufficient amplitude of signal and a certain frequency mismatch, the external signal locks not only the directly excitable oscillator, but also the second one (in this respect, it is interesting to compare Fig. 1, c and Fig. 1, c from [12]).

Fig. 2 shows examples of characteristic phase portraits in the non-autonomous system (1) in the stroboscopic section

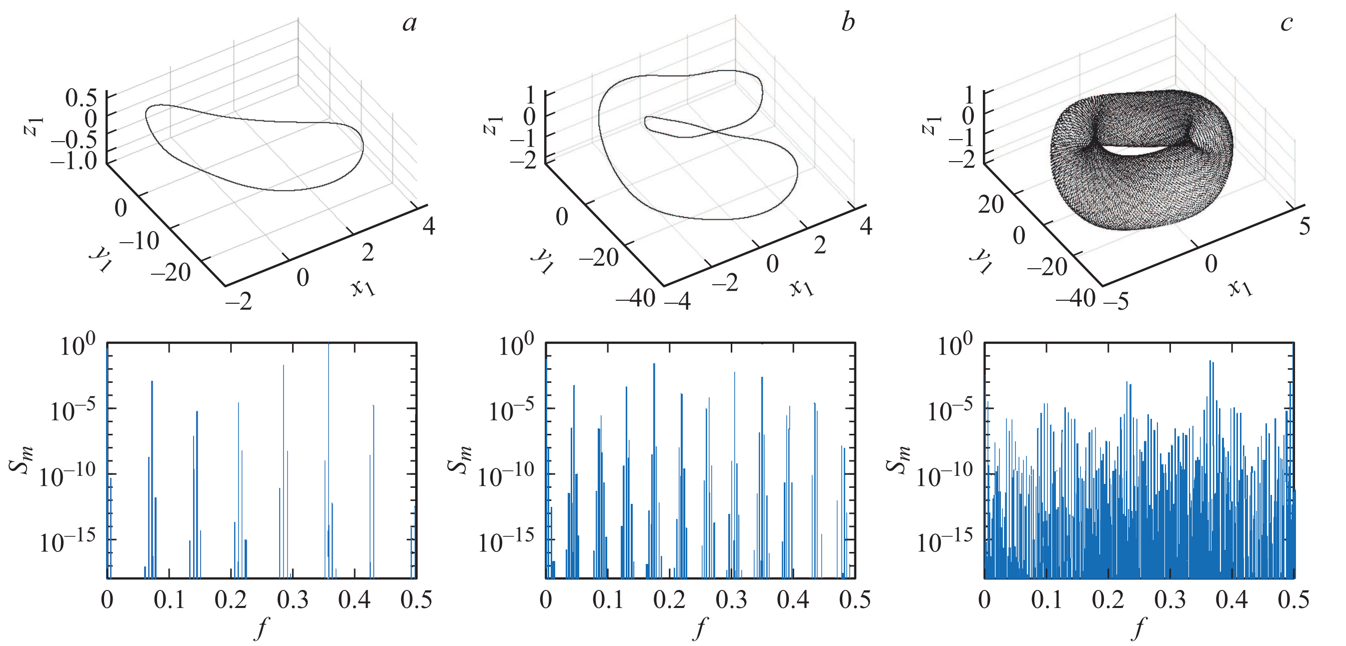

Figure 2. Projections of phase portraits (upper row) and corresponding Fourier spectra (lower row) at various points of the chart 1*c*; f — normalized frequency, S_m — spectral intensity expressed in dB. Parameter values: a — $\omega = 3.25\pi$, $a = 100$; b — $\omega = 3.25\pi$, $a = 60$; c — $\omega = 4.26\pi$, $a = 30$.

(i.e, after the period of external action). One can see the closed invariant curve corresponding to the two-frequency torus (Fig. 2, *a*), the doubling torus of this type (Fig. 2, *b*), as well as the three-frequency torus (Fig. 2, *c*). The bottom row shows the Fourier spectra for these modes. It is worth noting the enrichment of the spectra when a three-frequency torus appears.

Figure 3 shows graphs of the three largest Lyapunov exponents inside one of the tongues of two-frequency regimes along the line $\omega = \text{const}$ (*a*) and in the „transverse" direction $a = \text{const}$ (*b*). In the first case, we can note the doubling bifurcation of the two-frequency torus DT. It corresponds to the point when one of the exponents vanishes $\Lambda_1 = 0$, while the exponent Λ_2 remains negative, except for the bifurcation point, when it turns out to be zero. In the second case, we fix the bifurcation of two-frequency torus SNT when the value of the Lyapunov exponent Λ_2 becomes negative- from zero-value. Such bifurcation is a saddle-node bifurcation of two-frequency torus [16] with partial frequency locking, and the pair of two-frequency tori is born on the surface of the three-frequency torus: stable torus and saddle torus. At the exit from the region of two-frequency quasi-periodicity, the merging of stable and unstable tori is observed, which leads to the appearance of three-frequency torus. Thus, the side boundaries of the tongues of two-frequency tori in the chart in Fig. 1, *b* are the lines of the saddle-node bifurcations of the tori.

Fig. 1, *d* and e show the Lyapunov charts for the case of three- and four-frequency tori in the autonomous (without signal) system. In Fig. 1, *d* one can see the tongue of three-frequency torus immersed in the region of four-frequency tori. The side boundaries of the tongue are the lines of saddle-node bifurcations of the three-frequency torus. Inside the tongue, vertical windows of two-frequency tori are visible, the boundaries of which are also saddle-node bifurcations. Inside the region of four-frequency tori, narrow bands of three-frequency tori are observed, corresponding to partial synchronization tongues at higher harmonics.

In Fig. 1, *e* in the case of a four-frequency torus in the autonomous system in the presence of an external signal, the tongue of three-frequency tori is retained, but it has a small threshold on the signal amplitude. It is surrounded by a region of four-frequency quasi-periodicity, in which a large number of tongues of three-frequency tori are embedded. In turn, at a small amplitude of signal, the region of five-frequency tori arises, in which tongues of four-frequency tori are embedded, leaned on the frequency axis of the signal. The side boundaries of four-frequency tori are the saddle-node bifurcations of these tori.

In Fig. 4, the top line shows the phase portraits of three-, four-, and five-frequency tori for selected points of the chart of Fig. 1, e. It should be emphasized that visually they are practically indistinguishable. To single out three-frequency tori, one should construct a double Poincaré section. Note that the usual Poincaré section for the system under external harmonic signal is a set of discrete points obtained as a result of stroboscopic section. In order to plot a double section, it is necessary to take into account only those points from the mentioned discrete set that fall into some thin layer of the phase space, for example, in this case, the condition $|y_1| \le 0.01$ was used. The result of a double section (i.e., stroboscopic section and section by plane $y_1 = 0$) of the

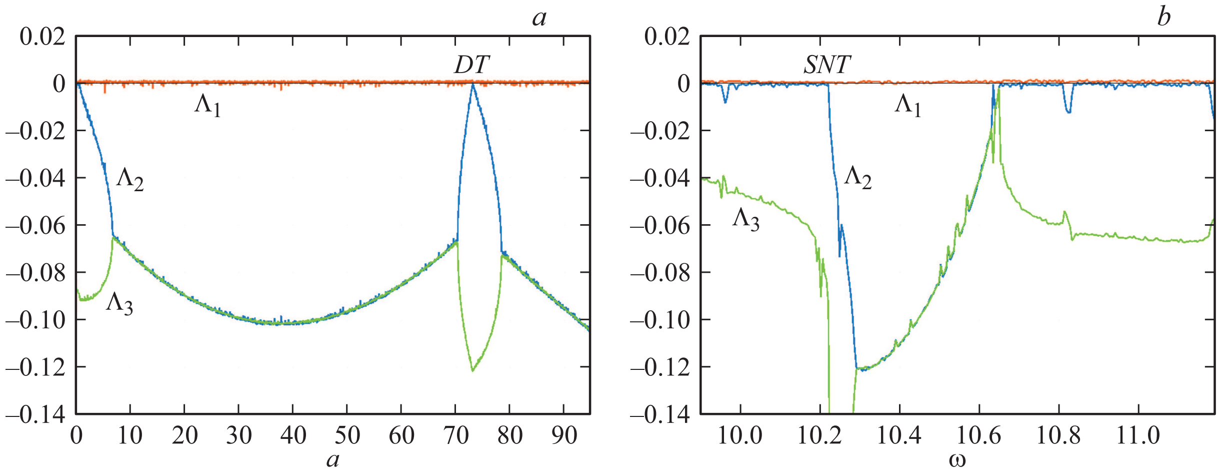

Figure 3. Graphs of the three largest Lyapunov exponents of the system (1) along the lines $\omega = 10.47$ (a) and $a = 23$ (b) in Fig. 1, c.

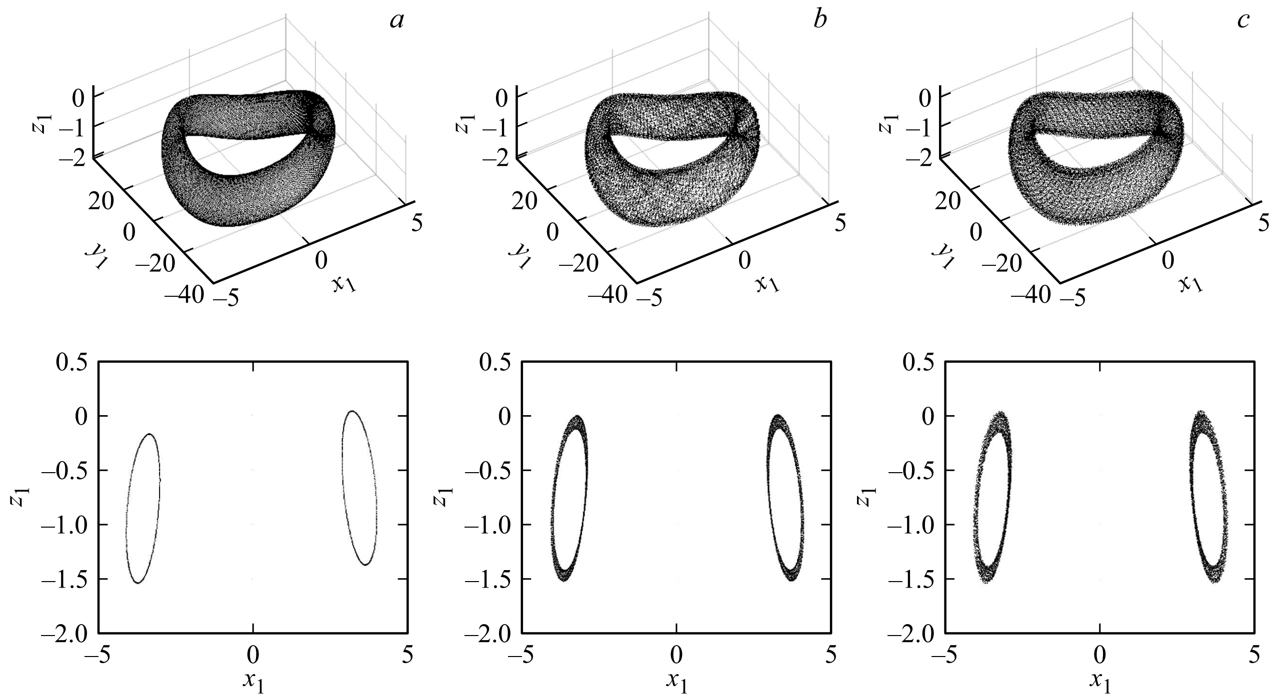

Figure 4. Projections of phase portraits (upper row) and double Poincaré sections (lower row) for three-, four- and five-frequency tori of the system (1) for selected points of the chart shown in Fig. 1, e. Parameter values: a — $\omega = 2.25\pi$, $a = 80$; b — $\omega = 4.15\pi$, $a = 110$; c — $\omega = 4.75\pi$, $a = 30$.

phase space of system (1) is presented in Fig. 4 on the bottom line. The double section of a three-frequency torus consists of two closed ovals, in contrast to four- and five-frequency tori (to distinguish the latter, double Poincaré sections are not informative and it is advisable to apply a triple section, for example, using an additional cutting plane $z_1 = -1$).

Finally, Fig. 1, f shows the Lyapunov chart for the case of the chaotic dynamics of the system without signal. In this case, for small amplitudes of signal, as expected, a chaotic regime is observed. However, with increase in the amplitude of the signal, similar [13] effect of chaos suppression by an external signal with the appearance of a periodic (synchronous) mode, is observed. Feature of the system under consideration is that the suppression of chaos leads not only to a periodic mode, but also to modes of two- and three-frequency tori. This fact is illustrated in Fig. 5, which shows the phase portrait and the double Poincaré section for the three-frequency mode (a). Also, with an increase in the signal amplitude, one can observe the evolution of chaotic dynamics, the formation of the so-called hyperchaotic mode, which is characterized by two positive Lyapunov exponents in the spectrum. Illustrations of the chaotic and hyperchaotic modes are shown in Fig. 5, b

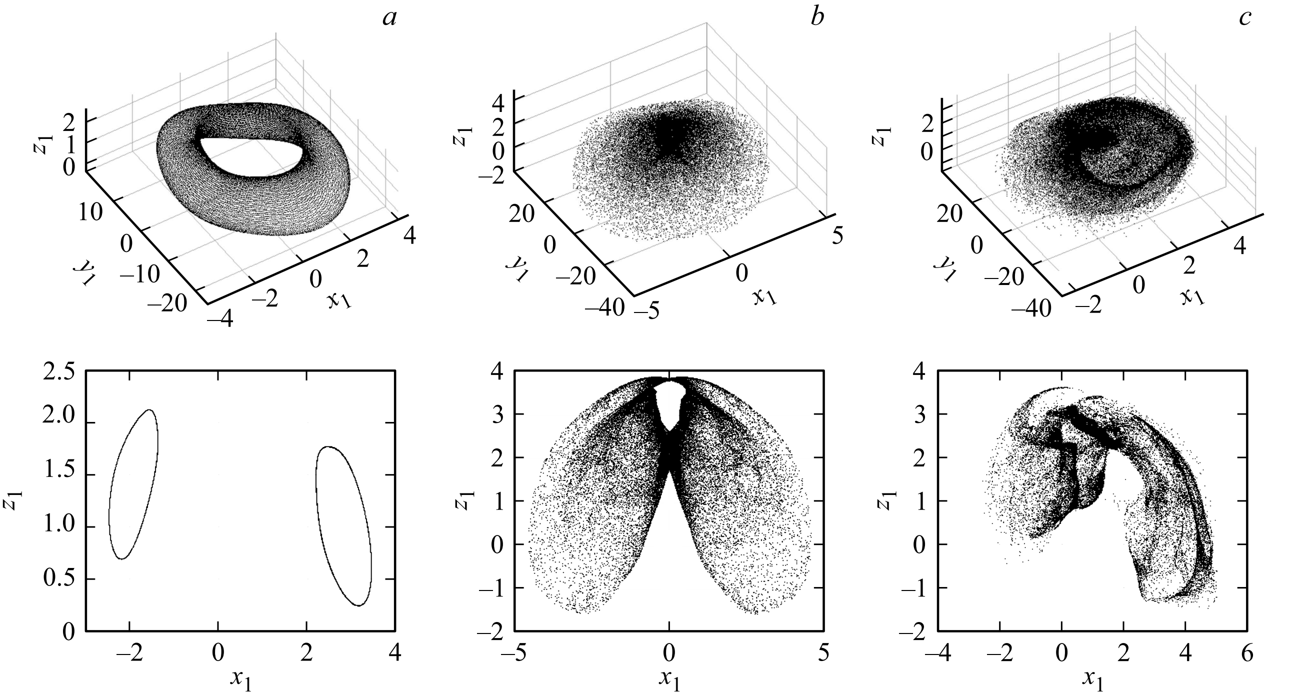

Figure 5. Projections of phase portraits at various points of the chart in Fig. 1,f (upper row) and the corresponding Poincar'e double sections (lower row). Parameter values: a — $\omega = 3.75\pi$, $a = 150$; b — $\omega = 4.5\pi$, $a = 20$; c — $\omega = 1.6\pi$, $a = 110$.

and c, respectively. Thus, with the help of an external signal, it is possible to control the chaotic behavior in a system of two coupled oscillators, and both to stabilize chaos and to complicate the evolution of hyperchaos.

In Fig. 1,f one can see that tongues of two-frequency tori are embedded in the region of three-frequency tori (upper right part of the chart, to the right of the main tongue of two-frequency tori). These tongues, however, „are inverted", they expand with a decrease in the amplitude of the signal and begin to overlap with the appearance of chaos. Note that in this case, chaos with one positive Lyapunov exponent, which is characteristic of the chaos that arose as a result of the loss of smoothness of the invariant curve, is mainly observed. Hyperchaos mainly develops to the left of the main tongue of two-frequency quasi-periodicity.

Conclusion

Thus, the system of excitable coupled quasi-periodic oscillators demonstrates a rich variety of modes based on invariant tori of different dimension. Mainly, the system of tongues of invariant tori immersed in the region of tori of higher dimension, is observed. The boundaries of tongues are saddle-node bifurcations of tori of the corresponding dimension. Bifurcations of tori doubling are observed inside tongues. External signal can initiate quasi-periodicity of different dimension in the case of chaotic dynamics of autonomous system. It is possible to control chaos: its stabilization or evolution into the hyperchaotic regime.

Funding

The study was supported by a grant from the Russian Science Foundation № 21-12-00121, https://rscf.ru/project/21-12-00121/.

Conflict of interest

The authors declare that they have no conflict of interest.

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