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Exact two-dimensional solution for magnetic compression of a thin axisymmetric shell and neck formation in an X-pinch

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Received August 26, 2022
Revised September 13, 2022
Accepted September 13, 2022.

Exact two-dimensional solutions are constructed that describe the dynamics of magnetic compression of a hollow shell, which is a one-sheeted hyperboloid of revolution. The solutions are applicable for interpreting the results of experiments on the formation of necks in X-pinches. In particular, they make it possible to relate the main parameters of the problem: the axial scale of the neck, the time of its formation, and the geometric characteristics of the system.

Keywords: X-pinch, neck formation, thin shell model.

DOI: 10.21883/TPL.2022.11.54880.19349

A „hot spot" of the X-pinch is a powerful X-ray source [1]. In the initial state the X-pinch consists of two or more crossed thin wires (Fig. 1). When a current pulse is applied with an amplitude from tens of kiloamperes to megaamperes, a dense high-temperature plasma is formed in the crossing region, which is a source of X-ray radiation. The small size of the „hot spot" (units of micrometers) and the short duration of the radiation pulse (units of nanoseconds) are attractive for the implementation of pulsed probing in the soft X-ray spectral range [2]. Compared to other methods of creating soft X-ray microsources (for example, using femtosecond lasers [3]), the X-pinch has the advantages of relatively simple driver and efficiency of radiation generation. Four stages can be distinguished in the X-pinch dynamics: electrical explosion of wires, plasma expansion and neck formation, neck implosion, and „hot spot" formation. The first and fourth stages take up to a few nanoseconds, while the total duration of the process is hundreds of nanoseconds. In paper [4] a model of neck formation and implosion is developed, based on the assumption that the neck length is independent of other parameters, which agrees satisfactorily with experimental data. At the same time, the reasons for the constant length of the neck remain not exactly clear.

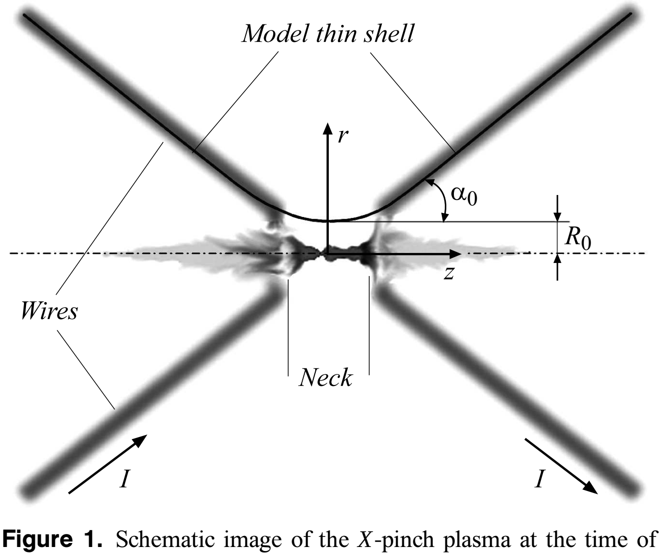

Figure 1. Schematic image of the X-pinch plasma at the time of generation of the radiation pulse and the model shape of the thin shell. The designations are explained in the text.

A specific feature of the X-pinch dynamics is its fundamental non-one-dimensionality: the plasma can flow from the neck region in the axial direction. Since the neck size significantly (by more than an order of magnitude) exceeds the diameter of the wires, its formation can be considered within the framework of a two-dimensional model of a thin hollow shell of the Ott model type [5,6] (see also the related model [7]). The purpose of the present paper is to construct exact two-dimensional solutions describing the magnetic compression of a thin axisymmetric shell and use them to interpret the results of experiments with X-pinches.

Let us define the shell geometry (it is shown schematically in Fig. 1) by a pair of functions that determine its radius and longitudinal coordinate: $r = R(\xi, t)$ and $z = Z(\xi, t)$. Here ξ is the Lagrangian coordinate, which is conveniently chosen so that it determines the mass distribution over the shell: $\xi = \int_0^z \rho_l(z, t)dz$ (ρ_l is linear density, or linear mass). The motion of ring shell element of mass $\Delta\xi$ with radial and axial dimensions ΔR and ΔZ under the action of external pressure P is described by Newtonian equations

$$\Delta\xi \frac{d^2R}{dt^2} = -\Delta s P \cos\alpha, \quad \Delta\xi \frac{d^2Z}{dt^2} = \Delta s P \sin\alpha,$$

where $\alpha = \arctan(\Delta R/\Delta Z)$ is angle of inclination of the surface element to the axis z, $\Delta s = 2\pi R \Delta Z \sqrt{1 + \tan^2\alpha}$ is its surface area. The magnetic pressure when electric current I flows through the shell is $P = \mu_0 I^2/(8\pi^2 R^2)$, where μ_0 is the vacuum magnetic permeability. Under X-pinch conditions [4], the neck is formed, as a rule, at the front of the current pulse, when we can take $I \propto t$. After passing to partial derivatives, we obtain

$$\frac{\partial^2 R}{\partial t^2} = -\frac{Ct^2}{R}\frac{\partial Z}{\partial \xi}, \quad \frac{\partial^2 Z}{\partial t^2} = \frac{Ct^2}{R}\frac{\partial R}{\partial \xi}, \tag{1}$$

where $C = \mu_0 (dI/dt)^2/4\pi$ is a constant. Model (1) is a generalization of the Ott model [5,6].

A non-trivial family of particular solutions of model (1) can be found by separation of variables. Let us perform the substitute $R = f(t)F(\xi)$ and $Z = g(t)G(\xi)$, where f, g, F, G are unknown functions. We obtain four equations connected through auxiliary constants s and q. The first pair — with respect to the variable t, and the second pair — with respect to ξ:

$$\frac{d^2 f}{dt^2} = -Cst^2\frac{g}{f}, \quad \frac{d^2 g}{dt^2} = Cqt^2, \tag{2}$$

$$\frac{dF}{d\xi} = qFG, \quad \frac{dG}{d\xi} = sF^2. \tag{3}$$

The solution of differential equations (3) gives in the interval $|\xi\sqrt{hq}| < \pi/2$

$$F = \sqrt{h/s}\cos^{-1}\big(\xi\sqrt{hq}\big), \quad G = \sqrt{h/q}\tan\big(\xi\sqrt{hq}\big), \tag{4}$$

where h is the integration constant (we assume that $s > 0$, $q > 0$, $h > 0$). Eliminating the Lagrangian coordinate ξ from (4), we find that the shell is a one-sheeted hyperboloid of revolution (Fig. 1) with parameters varying with time:

$$\left(\frac{r}{R_{\min}(t)}\right)^2 - \left(\frac{z\tan\alpha_0\rho_{\max}(t)}{R_0\rho_0}\right)^2 = 1. \tag{5}$$

The closeness of the hyperboloidal shell to the geometry of the X-pinch will allow us in the future to use the resulting solutions to describe the formation of necks. The solution corresponds to the following linear density distribution along z:

$$\rho_l \equiv \left(\frac{\partial Z}{\partial \xi}\right)^{-1} = \frac{R_0^2\rho_0^2\rho_{\max}(t)}{R_0^2\rho_0^2 + \tan^2\alpha_0\rho_{\max}^2(t)z^2}. \tag{6}$$

In expressions (5), (6) the following notations are introduced: $R_{\min}(t) = f(t)\sqrt{h/s}$ is shell radius in the section plane $z = 0$, where it is minimal, and $R_0 = R_{\min}(0)$ is its initial value; $\rho_{\max}(t) = 1/\big(hg(t)\big)$ is linear density in the section plane, where it is maximum, and $\rho_0 = \rho_{\max}(0)$; $\alpha_0 = \arctan\big(\sqrt{qh}R_0\rho_0\big)$ is initial angle of inclination of the shell to the axis z at $|z| \to \infty$ (the geometric parameters R_0 and α_0 are illustrated in Fig. 1). Linear density distribution (6) is bell-shaped. The width of the distribution (the axial size of the region in which $\rho_l > k\rho_{\max}$; k is the coefficient, which we take equal to 0.8) is characterized by the combination

$$L(t) = 2\sqrt{k^{-1}-1}R_0\rho_0/\big(\tan\alpha_0\rho_{\max}(t)\big). \tag{7}$$

Let us now consider equations (2). It is convenient to rewrite them using $R_{\min}$ and $\rho_{\max}$ functions:

$$\frac{d^2 R_{\min}}{dt^2} = -\frac{Ct^2}{R_{\min}\rho_{\max}},$$
$$\frac{d^2\rho_{\max}^{-1}}{dt^2} = \frac{Ct^2\tan^2\alpha_0}{R_0^2\rho_0^2}. \tag{8}$$

Initial conditions: $R_{\min}|_{t=0} = R_0$, $\rho_{\max}^{-1}|_{t=0} = \rho_0^{-1}$, $(dR_{\min}/dt)|_{t=0} = (d\rho_{\max}^{-1}/dt)|_{t=0} = 0$. For $\alpha_0 = 0$, the problem reduces to the trivial case of one-dimensional compression of a cylindrical shell (Z-pinch geometry). The second equation of system (8) is easily integrated. We find

$$\rho_{\max}^{-1}(t) = \rho_0^{-1} + Ct^4\tan^2\alpha_0/(12R_0^2\rho_0^2).$$

This shows that the linear density in the section $z = 0$ decreases with time due to the mass displacement from the region of the formed neck. As a result, system (8) reduces to a single equation with initial conditions $x|_{\tau=0} = 1$ and $(dx/d\tau)|_{\tau=0} = 0$:

$$\frac{d^2x}{d\tau^2} = -\frac{\tau^2}{x}\left(1 + \frac{\tan^2\alpha_0}{12}\tau^4\right), \tag{9}$$

where we introduced dimensionless variables $x = R_{\min}/R_0$ and $\tau = t/(R_0^2\rho_0 C^{-1})^{1/4}$. According to (9), the neck radius decreases to zero in a finite time τ_c (see insert in Fig. 2): the shell collapses. The calculated dependence τ_c on the angle α_0 is shown in Fig. 2. In the one-dimensional case ($\alpha_0 = 0$), the time is maximum and amounts to $\tau_0 \approx 1.728$. As α_0 increases, it decreases monotonically, which is explained by

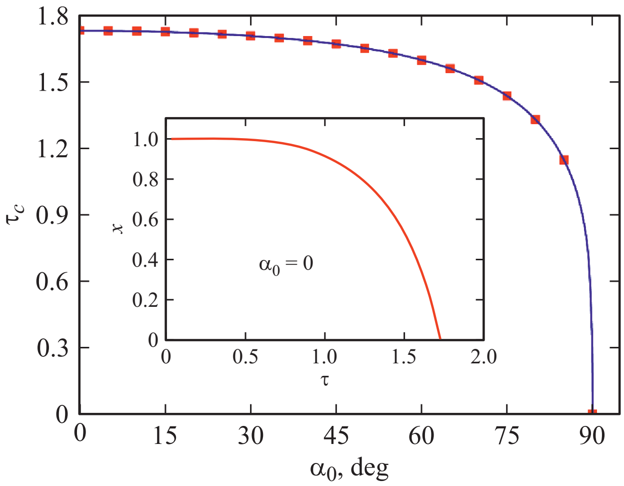

Figure 2. Dimensionless collapse time τ_c vs. angle α_0. Points — calculation, line — approximation (10). In the insert — dynamics of the neck radius at $\alpha_0 = 0$.

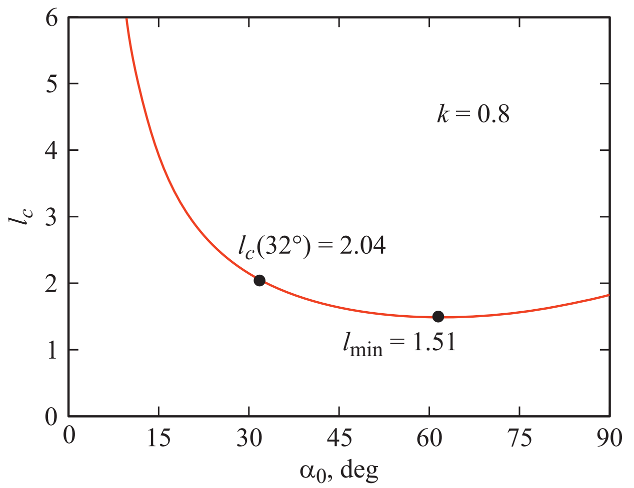

Figure 3. Dimensionless neck length l_c vs. angle α_0 for $k = 0.8$.

the more efficient mass displacement from the neck region. The following approximation is valid:

$$\tau_c(\alpha_0) \approx \tau_0(1 + 0.08\alpha_0^2 + 0.009\alpha_0^4)\cos^{1/4}\alpha_0,$$
$$0 \leqslant \alpha_0 < \pi/2. \qquad (10)$$

According to the obtained solution, by the time $T_c = (R_0^2\rho_0/C)^{1/4}\tau_c(\alpha_0)$, a radial collapse of the shell occurs along its entire length. We have $R|_{t=T_c} = 0$ for any ξ, despite the fact that that initially (for $t = 0$) the shell radius is the larger the greater the distance $|z|$. This is due to the heterogeneity of the linear density distribution (6). It decreases at the periphery as $1/z^2$, which allows the light „wings" of the shell to collapse in the same time as the heavy shell region close to the axis z near the section $z = 0$.

Let us consider how the obtained results can be used in relation to the problem of the neck formation in the X-pinch. It is clear that the pinch is characterized by an initially uniform distribution of the linear density: at the time $t = 0$, we can take $\rho_l = N\rho_w/\cos\alpha_0 = \text{const}$, where ρ_w is linear mass of one wire, N is their number, α_0 is angle of inclination (we identify it with the angle of the hyperboloidal shell). In such a situation, the collapse will not occur simultaneously along the entire axis of the system, but only in a „narrow" place where the wires cross. Let us ask ourselves what will happen to the solutions obtained above if, other things being equal, we take the initially homogeneous distribution of the linear density: $\rho_l(z, 0) = \rho_0$. The answer is obvious: weighted „wings" cannot collapse in T_c time. The shell collapses only in the vicinity of the section $z = 0$, in which the linear density (6) was close to ρ_0. The corresponding scale is determined by the width (7) of the linear density distribution, i.e. the experimental value of the neck length L_{exp} can be identified with the value $L_c = L(T_c)$:

$$L_c = l_c R_0, \quad l_c = \frac{2\sqrt{k^{-1} - 1}}{\tan\alpha_0}\left(1 + \frac{\tau_c^4(\alpha_0)\tan^2\alpha_0}{12}\right). \qquad (11)$$

Fig. 3 shows the dimensionless neck length $l_c \equiv L_c/R_0$ dependence on the angle α_0. It is nonmonotonic. The limit $\alpha_0 \to 0$ corresponds to the one-dimensional case — to the collapse of the cylindrical shell; as a consequence, we have $L_c \to \infty$. At $\alpha_0 \approx 62°$ the minimum is reached: $L_{\min} \approx 1.51R_0$. Further, as α_0 increases, the neck length begins to increase, reaching the value of $1.83R_0$ at $\alpha_0 = 90°$. The expansion of the neck region at large α_0 can be related to the axial displacement of the mass. Note that for the angle $\alpha_0 = 32°$ characteristic for the experiments [4], $L_c \approx 2.04R_0$.

The neck formation time as applied to the X-pinch is found by substituting $\rho_0 = N\rho_w/\cos\alpha_0$. We get $T_c = (R_0^2 N\rho_w/C\cos\alpha_0)^{1/4}\tau_c(\alpha_0)$. Eliminating the value R_0 using (11), we find the relationship between the main parameters of the problem:

$$\frac{\mu_0 T_c^4}{4\pi N\rho_w L_c^2}\left(\frac{dI}{dt}\right)^2 \approx \frac{\mu_0 T_{exp}^2 I_{exp}^2}{64\pi N\rho_w L_{exp}^2} \approx \frac{\tau_c^4(\alpha_0)}{l_c^2(\alpha_0)\cos\alpha_0}. \qquad (12)$$

Here we took into account that the length L_{exp}, the time of generation of the X-ray pulse T_{exp}, and the corresponding current $I_{exp} \approx (dI/dt)T_{exp}$ are the parameters recorded in the experiments. According to estimates [4], we can assume $T_{exp} \approx 2T_c$ (the durations of the neck formation and its subsequent compression with the formation of „hot spot" are comparable). Note that for $\alpha_0 = 32°$ the value of the right-hand side (12) is ~ 2.38.

The analysis of the experimental data [4] shows that the relation (12) is met with acceptable accuracy for a wide range of parameters ($I_{exp} \approx 70{-}200$ kA, $T_{exp} \approx 60{-}220$ ns, $\rho_0 \approx 30{-}550\,\mu$g/cm). As in [4], the dependence of the neck length on the linear density is rather weak: with mass increasing by 20 times, the neck length increases by 2 times only. Unlike the model [4], where the neck length was external parameter, in our model it was calculated and, as it was found, is related to the radius R_0 as $L_c = (1.5{-}2.1)R_0$ at experimentally realized angles α_0. In experiments, the radius R_0 is the radius at which the radial expansion of the plasma stops, and its compression begins [8]. The weakness of the R_0 dependence on the pinch parameters is not obvious and will be the subject of further studies, including those using magnetohydrodynamic simulation.

Conflict of interest

The authors declare that they have no conflict of interest.

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