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Focusing of the surface plasmon wave on the nanoapex of a scanning metal microtip near a plane-layered structure

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A generalized method of mirror reflections of electrostatics for a point charge located near a flat-layered structure is formulated and proved. The method is generalized to the case of an arbitrary system of charges. It is shown in detail how to apply the obtained method to finding the focal distribution of the electric field in the vicinity of the nanoapex of a metal micropoint located near the flat-layered structure, which is obtained when a surface plasmon TM wave converges to the nanoapex. The penetration of the field into the area of the surface layer of the flat-layered structure (photoresist) with a size of the order of the tip rounding radius is demonstrated.

Keywords: nanofocusing, surface plasmons, optical sensors.

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Introduction

Nanofocusing of light energy at the vertices of micropoints is the most important phenomenon underlying promising nanotechnological applications. Nanofocusing consists in an unusually sharp increase in the intensity of the surface plasmon-polariton wave, symmetrically excited at the base of a metal conical micropoint, as it converges to the nanovertex [1–3]. This phenomenon is explained by the fact that an axisymmetric electromagnetic standing wave with an electric field singularity at the vertex [4] can exist on a geometrically ideal metal tip. Experiments show [5,6] that this wave can be effectively excited by a surface plasmon-polariton TM wave converging to the vertex with the same axial field symmetry. The presence of a singularity of the electric field is well explained in the quasi-static approximation, which is valid in the vicinity of the vertex.

The real vertex of the micropoint is not perfect and has a rounded vertex. In the studies [7,8], to find the distribution of the electric field on the rounded vertex of a single micropoint, the surface of the vertex was approximated by a paraboloid of revolution. The problem was solved in the paraboloidal coordinate system. It was proved that the size of the focal distribution at the vertex decreases in proportion to the vertex radius, which fundamentally explains nanofocusing (when the tip radius decreases to nanometer sizes, the size of the focal region decreases in the same proportion).

If the nanovertex of the micropoint is located near the planar layered structure, then the need arises for a detailed description of the distribution of the field focused at the vertex both near the vertex tip and in the planar layered structure. In problems of radiation and propagation of electromagnetic fields in plane-layered media, matrix methods and Green's function methods for plane-layered media are widely used [9,10]. In the studies [11–13] an original version of the rigorous electromagnetic theory of radiation from an elementary dipole located at the boundary or inside a flat-layered structure was proposed, which is a development of the studies [14,15]. In particular, in [11–13] a method of analytical solution simplification was demonstrated, which has a potentially important general theoretical value. The generalization of this method for the case of an arbitrary number of films in a planar-layered structure [16] allowed to reduce the formulas for radiated fields to one-dimensional integrals, which significantly simplified the analysis of the problem and accelerated numerical calculations. In this study the methods of the studies [11–13,16] are first applied to finding a three-dimensional fundamental solution of electrostatics (quasi-electrostatics) in flat-layered media, i.e. to finding the field of a point charge in flat-layered media. On the basis of the results obtained, a generalization of the method of mirror reflections for a point charge located near a planar layered structure is given, and then this result is generalized to the case of an arbitrary distribution of charges near the planar layered structure. The application of the generalized method of reflections to the formulation of problems of nanofocusing of a surface plasmon wave at the vertex of a micropoint located near a photoresist nanolayer is discussed.

Problem formulation. Electric field of a point charge located inside a flat-layered structure

Let us consider the general problem of finding the electrostatic field from a point charge located inside a flat-

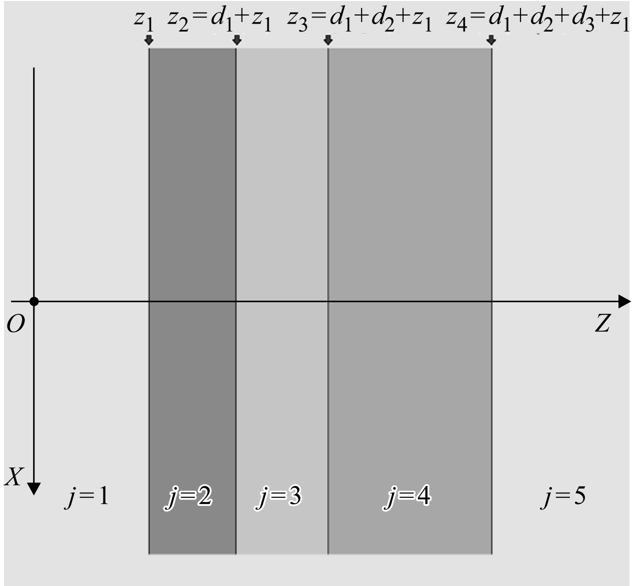

Figure 1. Geometry of a flat-layered structure consisting of three films.

layered structure. Let, for generality, this charge be located inside a flat layered structure consisting of several films and two half-spaces surrounding the layered structure. For definiteness we will first assume that the charge is located in one of the films, and then we generalize this problem to the case when the charge is located at their boundary or in one of the half-spaces.

Let the total number of films be N_f, the thickness of the mth film be d_m, and the total thickness of the layered structure be $d_{tot} = \sum_{m-1}^{N_f} d_m$. The total number of boundaries between films will be denoted as $N = N_f + 1$. Let us number the regions of the space $j = 1, \dots, (N + 1)$, (as an example, Fig. 1 shows the problem with $N = 4$ and $N_f = 3$). Let us assume that the films have absolute permittivities equal to ε_j, and in front of and behind the layered structure there are homogeneous half-spaces with permittivities ε_1 and ε_{N+1} . We also denote by z_j the coordinates N of the film boundaries along the axis Z as follows: $z_1 = z_1, z_j = z_1 + \sum_{m=1}^{j-1} d_m$ for $j = 2, \dots, N$.

The equations of electrostatics (or quasi-electrostatics) in the region with number j can be written in terms of the electric potential φ_j in the form: $\Delta\varphi_j = -q/\varepsilon_j$, where Δ – is the Laplace operator, ε_j – is the absolute permittivity of the j-th region. Solving the Laplace equations in each region, taking into account the boundary conditions, we find the electric field in all regions. Consider first the following auxiliary problem.

Electric field in a layer free of charges

Let there be no extraneous charges between the boundaries z_{j-1} and z_j in the region with the number j (Fig. 2). The permittivity of the medium in this film is ε_j.

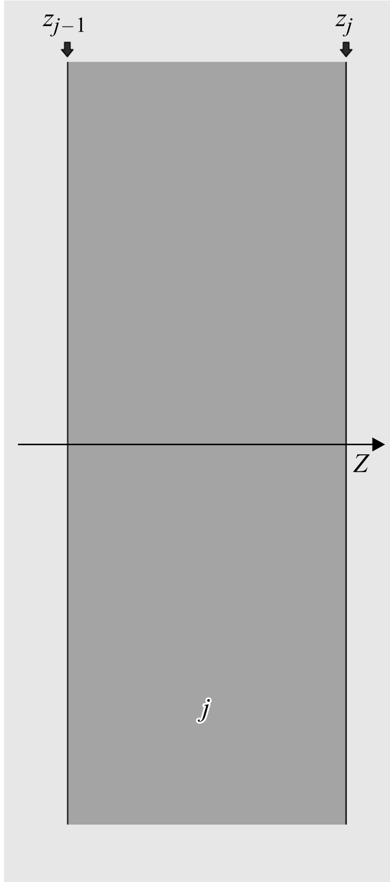

Figure 2. Film numbered j located between the boundaries z_{j-1} and z_j.

The electric potential can be represented as a Fourier expansion:

$$\varphi_j(x, y, z) = (2\pi)^{-2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\xi x + i\eta y} \hat{\varphi}_j(\xi, \eta, z) d\xi d\eta. \quad 1$$

Let us substitute the potential in the form of a Fourier expansion into the equation $\Delta\varphi_j = 0$, then in the region under consideration we can write

$$d^2\hat{\varphi}_j/dz^2 - \gamma^2\hat{\varphi}_j = 0, \quad (2)$$

where $\gamma = \sqrt{\xi^2 + \eta^2}$. The equations for fixed values of ξ and η are ordinary differential equations with respect to the variable z. The problem is to find the function $\hat{\varphi}_j$ from the equations in the region under consideration.

Linearly independent solutions of equations can be written as

$$\varphi_j^{\pm} = (2\pi)^{-2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{\varphi}_j^{\pm} e^{\mp\gamma z} e^{i(\xi x + \eta y)} d\xi d\eta. \quad (3)$$

We write the general solution of equations (2) in the region $[z_{j-1}, z_j]$ in the form

$$\varphi_j(z,y,z) = (2\pi)^{-2} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \hat{\varphi}_j^{+} e^{-\gamma(z-z_{j-1})} e^{i(\xi x+\eta y)} d\xi d\eta$$

$$+ (2\pi)^{-2} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \hat{\varphi}_j^{-} e^{\gamma(z-z_j)} e^{i(\xi x+\eta) y} d\xi d\eta, \tag{4}$$

where $\hat{\varphi}_j^{+}$ and $\hat{\varphi}_j^{-}$ are the functions of only ξ, η .

We note especially the following fundamentally important idea: the general solution of the Laplace equation, consisting of a linear combination of solutions, must be written in such a way that there are inverse Fourier transforms. Therefore, the form of writing the general solution for the field in the layer is not random, it highlights the physically correct solution. The first term on the right in the formula is the field from sources located to the left of the left boundary of the layer. In this case, the field will decrease when moving away to the right (when moving away from the sources to the left of the layer). The second term on the right in the formula is the field from sources located to the right of the right boundary of the layer (according to the condition of the problem, there are no sources inside the layer). This field will decrease as you move away to the left (when moving away from sources located to the right of the layer).

From we find the Fourier transform of the electric potential and the normal component of the electric field induction at the boundaries of the region j:

$$\begin{pmatrix} \hat{\varphi}_j \\ \hat{D}_{j,z} \end{pmatrix}\Bigg|_{z=z_{j-1}} = \begin{pmatrix} 1 & e^{-\gamma d_{j-1}} \\ \varepsilon_j \gamma & -\varepsilon_j \gamma e^{-\gamma d_{j-1}} \end{pmatrix} \begin{pmatrix} \hat{\varphi}_j^{+} \\ \hat{\varphi}_j^{-} \end{pmatrix},$$

$$\begin{pmatrix} \hat{\varphi}_j \\ \hat{D}_{j,z} \end{pmatrix}\Bigg|_{z=z_j} = \begin{pmatrix} e^{-\gamma d_{j-1}} & 1 \\ \varepsilon_j \gamma e^{-\gamma d_{j-1}} & -\varepsilon_j \gamma \end{pmatrix} \begin{pmatrix} \hat{\varphi}_j^{+} \\ \hat{\varphi}_j^{-} \end{pmatrix}, \tag{5}$$

where $d_{j-1} = z_j - z_{j-1}$. From we find the Fourier transform of the electric potential and the normal component of the electric field induction at the boundaries of the region j:

$$\begin{pmatrix} \hat{\varphi}_j \\ \hat{D}_{j,d} \end{pmatrix}\Bigg|_{z=z_{j-1}} = \mathbf{L}_j \times \hat{\mathscr{F}}_j, \quad \begin{pmatrix} \hat{\varphi}_j \\ \hat{D}_{j,d} \end{pmatrix}\Bigg|_{z=z_j} = \mathbf{R}_j \times \hat{\mathscr{F}}_j, \tag{6}$$

where the matrices $\mathbf{L}_j$ and $\mathbf{R}_j$ have the form

$$\mathbf{L}_j = \begin{pmatrix} 1 & e^{-\gamma d_{j-1}} \\ \varepsilon_j \gamma & -\varepsilon_j \gamma e^{-\gamma d_{j-1}} \end{pmatrix},$$

$$\mathbf{R}_j = \begin{pmatrix} e^{-\gamma d_{j-1}} & 1 \\ \varepsilon_j \gamma e^{-\gamma d_{j-1}} & -\varepsilon_j \gamma \end{pmatrix}, \tag{7}$$

Electric field in a multilayer structure free from external charges

Let us now consider a multilayer structure, inside which there are no extraneous charges. Let us consider the boundary $z = z_j$ between regions with numbers j and $j+1$. Continuity of the tangential components of the electric field strengths ($E_{j,x}$, E_{j+1}, x and $E_{j,y}$, $E_{j+1,y}$), and normal electric induction component ($D_{j,z} = -\varepsilon_j \partial\varphi_j/\partial z$ and $D_{j+1,z} = -\varepsilon_{j+1}\partial\varphi_{j+1}/\partial z$) on this boundary can be written in terms of the corresponding electric potentials φ_j and φ_{j+1} as follows:

$$\varphi_j\big|_{(x,y,z_j)} - \varphi_{j+1}\big|_{(x,y,z_j)} = 0$$

and

$$\varepsilon_j \partial\varphi_j/\partial z\big|_{(x,y,z_j)} - \varepsilon_{j+1}\partial\varphi_{j+1}/\partial z\big|_{(x,y,z_j)} = 0$$

where the electric potential φ_{j+1} in the region $j+1$ is expressed by the formula (4), in which the indices $j \to j+1$ are changed. Since the equations of electrostatics (quasi-electrostatics) are linear equations, the boundary conditions must be satisfied for each term of the Fourier expansion,i.e. boundary conditions must be satisfied for the Fourier transforms of the corresponding quantities:

$$\hat{\varphi}_j\big|_{(\xi,\eta,z_j)} - \hat{\varphi}_{j+1}\big|_{(\xi,\eta,z_j)} = 0,$$

$$\hat{D}_{j,z}\big|_{(\xi,\eta,z_j)} - \hat{D}_{j+1,z}\big|_{(\xi,\eta,z_j)} = 0. \tag{8}$$

Writing the boundary conditions (8) using expressions (6), we obtain a matrix equation on the boundary $z = z_j$:

$$\mathbf{R_j} \times \hat{\mathscr{F}}_j = \mathbf{L}_{j+1} \times \hat{\mathscr{F}}_{j+1}, \tag{9}$$

where $d_{j-1} = z_j - z_{j-1}$, $d_j = z_{j+1} - z_j$, and the matrices $\mathbf{R}_j$ and $\mathbf{L}_{j+1}$ are expressed by formulas (7), i.e.

$$\mathbf{L}_{j+1} = \begin{pmatrix} 1 & e^{-\gamma d_j} \\ \varepsilon_{j+1}\gamma & -\varepsilon_{j+1}\gamma e^{-\gamma d_j} \end{pmatrix},$$

$$\mathbf{R}_j = \begin{pmatrix} e^{-\gamma d_{j-1}} & 1 \\ \varepsilon_j \gamma e^{-\gamma d_{j-1}} & -\varepsilon_j \gamma \end{pmatrix},$$

Equation (9) can be written for $j = 2, \ldots, (N-1)$, where $(N+1)$ — the total number of regions, N – the number of boundaries, i.e. for all boundaries except the first $(j = 1)$ and last $(j = N)$ boundaries. That is, excluding the boundaries z_1 and $z_N = d_{tot} = \sum_{m-1}^{N-1} d_m$, where d_{tot} — total thickness of the layered structure (the sum of the thicknesses of the films that make up the structure under consideration).

The general solution for the electric potential in the region $j = 1$, i.e. in the interval $(-\infty, z_1]$, we write it as

$$\varphi_1(x, y, z) = (2\pi)^{-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\varphi}_1^{+} e^{-\gamma(z-z_1)} e^{i(\xi x+\eta y)} d\xi d\eta$$

$$+ (2\pi)^{-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\varphi}_1^{-} e^{\gamma(z-z_1)} e^{i(\xi x+\eta y)} d\xi d\eta. \quad (10)$$

In the problem under consideration, in the expression only the second term is nonzero, which implies the presence of sources only to the right of the boundary $z = z_1$.

Then, taking into account that it follows from expression (10) that

$$\begin{pmatrix} \hat{\varphi}_1 \\ \hat{D}_{1,x} \end{pmatrix}\Bigg|_{z=z_1} = \begin{pmatrix} 1 & 1 \\ \varepsilon_1\gamma & -\varepsilon_1\gamma \end{pmatrix} \begin{pmatrix} 0 \\ \hat{\varphi}_1^{-} \end{pmatrix},$$

write the boundary conditions at the boundary $z = z_1$, denoting $\hat{\mathscr{F}}_1 = (0; \hat{\varphi}_1^{-})^T$, in the form

$$\begin{pmatrix} 1 & 1 \\ \varepsilon_1\gamma & -\varepsilon_1\gamma \end{pmatrix} \times \hat{\mathscr{F}} = \mathbf{L}_2 \times \hat{\mathscr{F}}_2. \quad (11)$$

Similarly, the general solution for the electric potential in the region $j = N + 1$, i.e. in the interval $[z_N + \infty)$, we write it as

$$\varphi_{N+1}(x, y, z) = (2\pi)^{-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\varphi}_{N+1}^{+} e^{-\gamma(z-z_N)} e^{i(\xi x+\eta y)} d\xi d\eta$$

$$+ (2\pi)^{-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\varphi}_{N+1}^{-} e^{\gamma(z-z_N)} e^{i(\xi x+\eta y)} d\xi d\eta. \quad (12)$$

In the problem under consideration, in the expression (12), only one term is nonzero, which implies the presence of sources only to the left of the $z = z_N$ boundary.

Then, taking into account that it follows from

$$\begin{pmatrix} \hat{\varphi}_{N+1} \\ \hat{D}_{N+1,z} \end{pmatrix}\Bigg|_{z=z_N} = \begin{pmatrix} 1 & 1 \\ \varepsilon_{N+1}\gamma & -\varepsilon_{N+1}\gamma \end{pmatrix} \begin{pmatrix} \hat{\varphi}_{N+1}^{+} \\ 0 \end{pmatrix},$$

we write the boundary conditions on the boundary $z = z_N$ in the form

$$\mathbf{R}_N \times \hat{\mathscr{F}}_N = \begin{pmatrix} 1 & 1 \\ \varepsilon_{N+1}\gamma & -\varepsilon_{N+1}\gamma \end{pmatrix} \times \hat{\mathscr{F}}_{N+1}, \quad (13)$$

where $\hat{\mathscr{F}}_{N+1} = (\hat{\varphi}_{N+1}^{+}; 0)$. Equations (9), (11) and (13) allow to relate the column vectors of the electric potential in the first and last regions of the problem (i.e. in half-spaces, outside the flat-layered structure):

$$\hat{\mathscr{F}}_1 = \begin{pmatrix} 1 & 1 \\ \varepsilon_1\gamma & -\varepsilon_1\gamma \end{pmatrix}^{-1} \times \left[\mathbf{L}_2 \times (\mathbf{R}_2)^{-1}\right] \times \ldots$$

$$\times \left[\mathbf{L}_N \times (\mathbf{R}_N)^{-1}\right] \times \begin{pmatrix} 1 & 1 \\ \varepsilon_{N+1}\gamma & -\varepsilon_{N+1}\gamma \end{pmatrix} \times \hat{\mathscr{F}}_{N+1}, \quad (14)$$

or $\hat{\mathscr{F}}_1 = \mathbf{M} \times \hat{\mathscr{F}}_{N+1}$, and the matrix $\mathbf{M}$ is as follows $\mathbf{M} = \mathbf{T}_1 \times \left(\Pi_{m-2}^{N} \mathbf{T}_m\right) \times \mathbf{T}_{N+1}$, where

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 1 \\ \varepsilon_1\gamma & -\varepsilon_1\gamma \end{pmatrix}^{-1} \quad \mathbf{T}_m = \mathbf{L}_m \times (\mathbf{R}_m)^{-1},$$

$$T_{N+1} = \begin{pmatrix} 1 & 1 \\ \varepsilon_{N+1}\gamma & -\varepsilon_{N+1}\gamma \end{pmatrix}.$$

If we know, for example, the potential of some charge distribution in the structure of films, then from equation (14), as a system of two equations, we can find two unknowns — the function $\hat{\varphi}_1^{-}$ and, therefore, the potential in the half-space $j = 1$ of charges, in a plane-layered structure according to formula (10), and the function $\hat{\varphi}_{N+1}^{+}$ of the – vector column $\hat{\mathscr{F}}_{N+1}$ and the potential from charges in the half-space $j = N + 1$, in a flat-layered structure according to formula (12).

Electric field in a multilayer structure from a point charge

Let there be a point external charge q located at the point $(0, 0, z_q)$ in the region with number s (Fig. 3).

Let this charge be determined by the density distribution,

$$\rho(x, y, z) = q\delta(x)\delta(y)\delta(z - z_q),$$

where $\delta(x)$ — the Dirac delta function. The Fourier transform of this distribution is given by the following expression:

$$\hat{\rho}(\xi, \eta, z) = q \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x)\delta(y)\delta(z - z_q) e^{-i(\xi x+\eta y)} dx dy$$

$$= q\delta(z - z_q).$$

Let this point charge be in an infinitely thin layer $(z_q - \Delta z/2, z_q + \Delta z/2)$. Then from the equations of electrostatics $rot\mathbf{E} = 0$ and $div\mathbf{D} = \rho$ for the Fourier transforms of the fields one can write (at $\Delta z \to 0$) the equations:

$$i\eta\hat{E}_{s,z} - \frac{\Delta\hat{E}_{s,y}}{\Delta z} = 0, \quad \frac{\Delta\hat{E}_{s,x}}{\Delta z} - i\xi\hat{E}_{s,z} = 0$$

$$i\xi\hat{E}_{s,y} - i\eta\hat{E}_{s,x} = 0, \quad (15)$$

$$i\xi\hat{D}_{s,x} + i\eta\hat{D}_{s,y} + \frac{\Delta\hat{D}_{s,z}}{\Delta z} = q\delta(z - z_q). \quad (16)$$

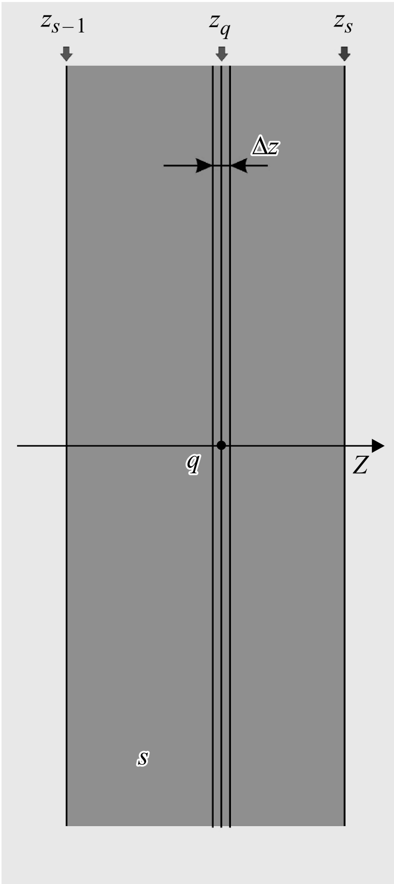

Figure 3. A point charge q located at the point $(0, 0z_d)$ in the region with the number s.

From which we get the following

$$\Delta\hat{E}_{s,y} = i\eta\hat{E}_{s,z}\Delta z, \quad \Delta\hat{E}_{s,x} = i\xi\hat{E}_{s,z}\Delta z,$$

$$\Delta\hat{D}_{s,x} = -(i\xi\hat{D}_{s,x} + i\eta\hat{D}_{s,y})\Delta z + q\delta(z - z_q)\Delta z.$$

Then, in the limit $\Delta z \to 0$, the jumps of the tangential components of the electric field strengths and the normal component of the electric field induction upon passing through an infinitely thin layer with a charge are equal to

$$\Delta\hat{E}_{s,x} \underset{\Delta z}{\longrightarrow} 0, \quad \Delta\hat{E}_{s,y} \underset{\Delta z}{\longrightarrow} 0, \quad \Delta\hat{D}_{s,z} \underset{\Delta z}{\longrightarrow} q.$$

In the matrix form these limit equations can be written in the following form:

$$\begin{pmatrix} \hat{E}_{s,x} \\ \hat{E}_{s,y} \\ \Delta D_{s,z} \end{pmatrix}\Bigg|_{z=z_q+0} - \begin{pmatrix} \hat{E}_{s,x} \\ \hat{E}_{s,y} \\ \Delta D_{s,z} \end{pmatrix}\Bigg|_{z=z_q-0} = \begin{pmatrix} 0 \\ 0 \\ q \end{pmatrix},$$

or, in an equivalent form, through the potential

$$\begin{pmatrix} \hat{\varphi}_s \\ \hat{D}_{s,z} \end{pmatrix}\Bigg|_{z=z_q+0} - \begin{pmatrix} \hat{\varphi}_s \\ \hat{D}_{s,z} \end{pmatrix}\Bigg|_{z=z_q-0} = \begin{pmatrix} 0 \\ q \end{pmatrix}. \quad (17)$$

Let us now express the left side of the boundary condition (17) in terms of the column vectors $\hat{\mathcal{F}}_1$ and $\hat{\mathcal{F}}_{N+1}$ of half-spaces outside the flat-layered structure. To do this, let's divide the region with number s into two regions and denote them by indices l and r (left and right, if you look at Fig. 3). Let us introduce the column vectors $\hat{\mathcal{F}}_l$ and $\hat{\mathcal{F}}_r$ in these regions. Then the terms to the left of the equal sign in (17) can be expressed as

$$\begin{pmatrix} \hat{\varphi}_s \\ \hat{D}_{s,z} \end{pmatrix}\Bigg|_{z=z_d-0} = \mathbf{R}_l \times \hat{\mathcal{F}}_l \text{ and } \begin{pmatrix} \hat{\varphi}_s \\ \hat{D}_{s,z} \end{pmatrix}\Bigg|_{z=z_d+0} = \mathbf{L}_r \times \hat{\mathcal{F}}_r. \quad (18)$$

In addition, from (9) it follows that

$$\hat{\mathcal{F}}_l = \mathbf{Q}_L \times \hat{\mathcal{F}}_l \quad \text{and} \quad \hat{\mathcal{F}}_r = \mathbf{Q}_R \times \hat{\mathcal{F}}_{N+1}, \quad (19)$$

where

$$\mathbf{Q}_L = \begin{pmatrix} 1 & 1 \\ \varepsilon_1\gamma & -\varepsilon_1\gamma \end{pmatrix}^{-1} \times \left[\mathbf{L}_2 \times (\mathbf{R}_2)^{-1}\right] \times \ldots$$

$$\times \times \left[\mathbf{L}_{s-1} \times (\mathbf{R}_{s-1})^{-1}\right] \times \mathbf{L}_l,$$

$$\mathbf{L}_l = \begin{pmatrix} 1 & e^{-\gamma(z_q - z_{s-1})} \\ \varepsilon_s\gamma & -\varepsilon_s\gamma e^{-\gamma(z_q - z_{s-1})} \end{pmatrix}$$

$$Q_R = (\mathbf{R}_r)^{-1} \times \left[\mathbf{L}_{s+1} \times (\mathbf{R}_{s+1})^{-1}\right] \times \ldots$$

$$\times \left[\mathbf{L}_N \times (\mathbf{R}_N)^{-1}\right] \times \begin{pmatrix} 1 & 1 \\ \varepsilon_{N+1}\gamma & -\varepsilon_{N+1}\gamma \end{pmatrix},$$

$$\mathbf{R}_r = \begin{pmatrix} e^{-\gamma(z_s - z_q)} & 1 \\ \varepsilon_s\gamma e^{-\gamma(z_s - z_q)} & -\varepsilon_s\gamma \end{pmatrix}$$

Substituting (19) into (18) and then the resulting expressions into (17), we obtain

$$\mathbf{H}_R \times \hat{\mathcal{F}}_{N+1} = \mathbf{H}_L \times \hat{\mathcal{F}}_1 + \mathbf{V}, \quad (20)$$

where $\mathbf{V} = (0,\ q)^T$ — is a column vector characterizing the exciting action of a point charge, and the matrices $\mathbf{H}_R$ and $\mathbf{H}_L$ — characterize the response to external excitation of the layered structure to the right and left of the charge and are expressed as follows:

$$\mathbf{H}_R = \mathbf{T}_R \times \left(\Pi_{m=s+1}^{N}\mathbf{T}_m\right) \times \mathbf{T}_{N+1},$$

$$\mathbf{H}_L = \left(\mathbf{T}_1 \times \left(\Pi_{m=2}^{s-1}\mathbf{T}_m\right) \times \mathbf{T}_L\right)^{-1},$$

where matrices $\mathbf{T}_m$ at $m \neq s$ are expressed by formula $\mathbf{T}_m = \mathbf{L}_m \times (\mathbf{R}_m)^{-1}$, and matrices $\mathbf{T}_L$ and $\mathbf{T}_R$ — by formulas $\mathbf{T}_L = \mathbf{L}_l \times (\mathbf{R}_l)^{-1}$, $\mathbf{T}_R = \mathbf{L}_r \times (\mathbf{R}_r)^{-1}$.

In the problem under consideration, the source of fields (a point charge) is located exclusively inside a flat-layered structure. Therefore, the $\hat{\mathcal{F}}_1$ and $\hat{\mathcal{F}}_{N+1}$ columns contain only the components that determine the waves coming from the flat-layered structure.

To obtain the remaining non-zero components $\hat{\mathscr{F}}_1$ and $\hat{\mathscr{F}}_{N+1}$, we split the matrices into $\mathbf{H}_R$ and $\mathbf{H}_L$ into H_{RA}, H_{RB}, H_{RC}, H_{RD} and H_{LA}, H_{LB}, H_{LC}, H_{LD}, then equation (20) takes the form

$$\begin{pmatrix} H_{RA} & H_{RB} \\ H_{RC} & H_{RD} \end{pmatrix} \times \begin{pmatrix} \hat{\varphi}^+_{N+1} \\ 0 \end{pmatrix} = \begin{pmatrix} H_{LA} & H_{LB} \\ H_{LC} & H_{LD} \end{pmatrix} \times \begin{pmatrix} 0 \\ \hat{\varphi}^-_1 \end{pmatrix} + \begin{pmatrix} 0 \\ q \end{pmatrix}. \quad (21)$$

Then it can be represented by the following equations:

$$H_{RA}\hat{\varphi}^+_{N+1} = H_{LB}\hat{\varphi}^-_1; \quad H_{RC}\hat{\varphi}^+_{N+1} = H_{LD}\hat{\varphi}^-_1 + q.$$

The resulting equations can be combined again into a single matrix 2×2 equation:

$$\begin{pmatrix} -H_{LB} & H_{RA} \\ -H_{LD} & H_{RC} \end{pmatrix} \hat{\mathscr{F}}_{out} = \begin{pmatrix} 0 \\ q \end{pmatrix}, \quad (22)$$

where the column vector $\hat{\mathscr{F}}_{out} = \left(\hat{\varphi}^-_1; \hat{\varphi}^+_{N+1}\right)^T$.

Solving this equation, we find $\hat{\varphi}^-_1$ and $\hat{\varphi}^+_{N+1}$, which means that the fields decrease with distance from the flat-layered structure:

$$\hat{\varphi}^-_1 = qH_{RA}/(H_{RC}H_{LB} - H_{RA}H_{LD}),$$

$$\hat{\varphi}^+_{N+1} = qH_{LB}/(H_{RC}H_{LB} - H_{RA}H_{LD}) \quad (23)$$

Then the field decreasing to the left in the half-space $J+1$ (in the direction $z \to -\infty$):

$$\varphi_1(x, y, z) = (2\pi)^{-2} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \hat{\varphi}^-_1 e^{\gamma(z-z_1)} e^{i(\xi x + \eta y)} d\xi d\eta, \quad (24)$$

and the field decreasing to the right in the half-space $j = N+1$ (in the direction $z \to +\infty$):

$$\varphi_{N+1}(x, y, z) = (2\pi)^{-2} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \hat{\varphi}^+_{N+1} e^{-\gamma(z-z_N)} e^{i(\xi x + \eta y)} d\xi d\eta. \quad (25)$$

Finally, knowing $\hat{\mathscr{F}}_1$ and $\hat{\mathscr{F}}_{N+1}$, we can, if necessary, find the column vectors of potentials in any interior region $\hat{\mathscr{F}}_j$, since they are uniquely determined by the boundary conditions. After that, the electric potential in any of these areas can be found using the formula (4). This way the margins will be defined throughout the space.

Electric field of a point charge located at some distance from the film boundary in a half-space

Let us now consider the problem of finding the electric potential from a point charge q located in a medium with permittivity ε_f, at some distance from a film with permittivity ε_p deposited on a half-space with permittivity ε_d (Fig. 4).

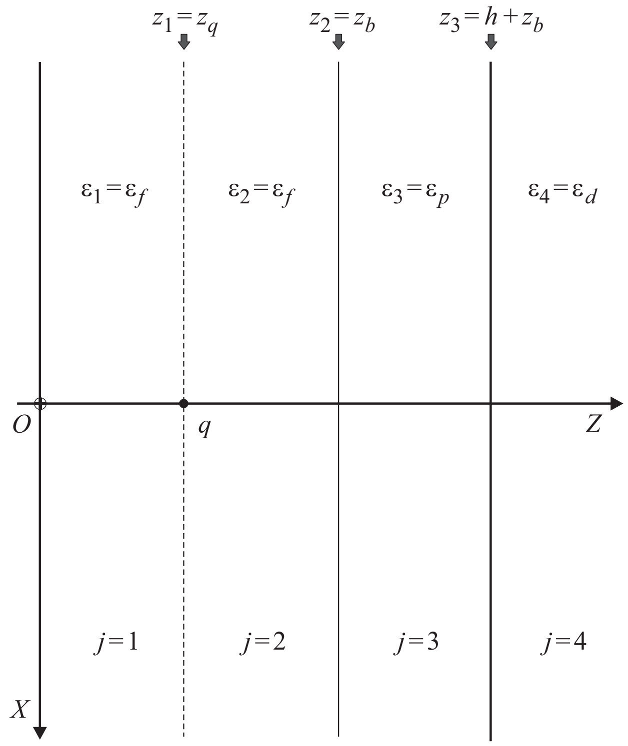

Figure 4. Point charge z_q located at a point with coordinate at a distance $()z_b - z_q$ from a film of thickness h and permittivity ε_p located on the boundary of the half space.

In the coordinate system of Fig. 4, the charge is located at a point with radius vector $\mathbf{r}_q = (0; 0; z_q)$ at a distance $z_{bq} = z_b - z_q$ along the Z axis from a h thick film.

This problem can be considered as a problem of finding the electric potential from a point charge located on the surface of an auxiliary film of thickness $z_{bq} = z_b - z_q$, and the dielectric constants of this auxiliary film and the half-space on the left are the same and equal ε_f.

Thus, there are four task areas. In the enumeration proposed above, the index $j = 1$ corresponds to the half-space with ε_f, $j = 2$ — to the auxiliary film with $\varepsilon_2 = \varepsilon_f$ and thickness equal to the distance from the charge to the film, i.e. $z_{bq} = z_b - z_q$, $j = 3$ — real film c $\varepsilon_3 = \varepsilon_p$ with thickness h, and $j = 4$ — to the half-space with $\varepsilon_4 = \varepsilon_d$ (Fig. 4).

Then $N = 3$, $z_1 = z_q$, $z_2 = z_b$, $z_3 = h + z_b$, $\mathbf{H}_R = \mathbf{T}_2 \times \mathbf{T}_3 \times \mathbf{T}_4$, $\mathbf{H}_L = (\mathbf{T}_1)^{-1}$ and equation (20) takes the form

$$(\mathbf{T}_2 \times \mathbf{T}_3 \times \mathbf{T}_4) \times \hat{\mathscr{F}}_4 = (\mathbf{T}_1)^{-1} \times \hat{\mathscr{F}}_1 + \mathbf{V}, \quad (26)$$

where the matrices are expressed by the following formulas:

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 1 \\ \varepsilon_f\gamma & -\varepsilon_f\gamma \end{pmatrix}^{-1},$$

$$\mathbf{T}_2 = \mathbf{L}_2 \times (\mathbf{R}_2)^{-1} = \begin{pmatrix} 1 & e^{-\gamma(z_b - z_q)} \\ \varepsilon_f \gamma & -\varepsilon_f \gamma e^{-\gamma(z_b - z_q)} \end{pmatrix} \times \begin{pmatrix} e^{-\gamma(z_b - z_q)} & 1 \\ \varepsilon_f \gamma e^{-\gamma(z_b - z_q)} & -\varepsilon_f \gamma \end{pmatrix}^{-1},$$

$$\mathbf{T}_3 = \mathbf{L}_3 \times (\mathbf{R}_3)^{-1} = \begin{pmatrix} 1 & e^{-\gamma h} \\ \varepsilon_p \gamma & -\varepsilon_p \gamma e^{-\gamma h} \end{pmatrix} \times \begin{pmatrix} e^{-\gamma h} & 1 \\ \varepsilon_p \gamma e^{-\gamma h} & -\varepsilon_p \gamma \end{pmatrix}^{-1},$$

$$\mathbf{T}_4 = \begin{pmatrix} 1 & 1 \\ \varepsilon_d \gamma & -\varepsilon_d \gamma \end{pmatrix},$$

and the column vector of a point charge is equal to

$$\mathbf{H}_L = \begin{pmatrix} H_{LA} & H_{LB} \\ H_{LC} & H_{LD} \end{pmatrix} = (\mathbf{T}_1)^{-1} = \begin{pmatrix} 1 & 1 \\ \varepsilon_f \gamma & -\varepsilon_f \gamma \end{pmatrix},$$

$$\mathbf{H}_R = \begin{pmatrix} 1 & e^{-\gamma(z_b - z_q)} \\ \varepsilon_f \gamma & -\varepsilon_f \gamma e^{-\gamma(z_b - z_q)} \end{pmatrix} \begin{pmatrix} e^{-\gamma(z_b - z_q)} & 1 \\ \varepsilon_f \gamma e^{-\gamma(z_b - z_q)} & -\varepsilon_f \gamma \end{pmatrix}^{-1} \times \begin{pmatrix} 1 & e^{-\gamma h} \\ \varepsilon_p \gamma & -\varepsilon_p \gamma e^{-\gamma h} \end{pmatrix} \begin{pmatrix} e^{-\gamma h} & 1 \\ \varepsilon_p \gamma e^{-\gamma h} & -\varepsilon_p \gamma \end{pmatrix}^{-1} \times \begin{pmatrix} 1 & 1 \\ \varepsilon_d \gamma & -\varepsilon_d \gamma \end{pmatrix}.$$

Introducing the column vector $\hat{\mathscr{F}}_{out} = (\hat{\varphi}_1^-; \hat{\varphi}_4^+)^T$, equation (22) for this problem takes the form

$$\begin{pmatrix} -1 & H_{RA} \\ \varepsilon_f \gamma & H_{RC} \end{pmatrix} \begin{pmatrix} \hat{\varphi}_1^- \\ \hat{\varphi}_4^+ \end{pmatrix} = \begin{pmatrix} 0 \\ q \end{pmatrix}. \quad (27)$$

From (27) we have

$$\hat{\varphi}_1^- = qH_{RA}/(H_{RC} + \varepsilon_f \gamma H_{RA}), \quad \hat{\varphi}_4^+ = q/(H_{RC} + \varepsilon_f \gamma H_{RA}). \quad (28)$$

Explicitly, expressions (28) can be obtained in terms of hyperbolic sines and cosines by substituting the expressions

$$H_{RA} = (\varepsilon_p \mathrm{ch}(\gamma h) + \varepsilon_d \mathrm{sh}(\gamma h))\mathrm{ch}(\gamma z_{bq})/\varepsilon_p + (\varepsilon_p \mathrm{sh}(\gamma h) + \varepsilon_d \mathrm{ch}(\gamma h))\mathrm{sh}(\gamma z_{bq})/\varepsilon_f, \quad (29)$$

$$H_{RC} = \gamma(\varepsilon_p \mathrm{ch}(\gamma h) + \varepsilon_d \mathrm{sh}(\gamma h))(\varepsilon_f \mathrm{sh}(\gamma z_{bq}) + \varepsilon_p \mathrm{ch}(\gamma z_{bq}))/\varepsilon_p. \quad (30)$$

Let us find the potential φ_1 in the half-space ($j = 1$) for $z \le z_q$ Substituting expressions (29) and (30) for H_{RA} and H_{RC} into expression (28) for $\hat{\varphi}_1^-$, we obtain

$$\hat{\varphi}_1^- = \frac{qH_{RA}}{H_{RC} + \varepsilon_f \gamma H_{RA}} = \frac{q}{2\gamma \varepsilon_f} + \frac{q}{2\gamma \varepsilon_f} e^{-2\gamma(z_b - z_q)} \Re(\gamma, h), \quad (31)$$

where

$$\Re(\gamma, h) = \frac{\varepsilon_f - \varepsilon_p}{\varepsilon_p + \varepsilon_p} + \frac{4\varepsilon_p \varepsilon_f}{(\varepsilon_p + \varepsilon_f)} \times \frac{(\varepsilon_p - \varepsilon_d)}{[(\varepsilon_p + \varepsilon_f)(\varepsilon_p + \varepsilon_d)e^{2\gamma h} + (\varepsilon_p - \varepsilon_d)(\varepsilon_f - \varepsilon_p)]}. \quad (32)$$

Then from the expression (24) using (31) we get

$$\varphi_1(x, y, z) = (2\pi)^{-2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{q}{2\gamma \varepsilon_f} e^{\gamma(z - z_q)} + \frac{q}{2\gamma \varepsilon_f} e^{\gamma(z - (2z_b - z_q))} \Re(\gamma, h) \right) \times e^{i(\xi x + \eta y)} d\xi d\eta.$$

Taking into account that $\gamma = \sqrt{\xi^2 + \eta^2}$, we use the identity

$$(2\pi)^{-2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(q e^{\gamma |z - a|} / 2\varepsilon_f \gamma \right) e^{i(\xi x + \eta y)} d\xi d\eta = q/4\pi \varepsilon_f \sqrt{x^2 + y^2 + (z - a)^2}. \quad (33)$$

Then

$$\varphi_1(x, y, z) = \frac{q}{4\pi \varepsilon_f \sqrt{x^2 + y^2 + (z - z_q)^2}} + \frac{\varepsilon_f - \varepsilon_p}{\varepsilon_p + \varepsilon_f} \frac{q}{4\pi \varepsilon_f \sqrt{x^2 + y^2 + (z - (2z_b - z_q))^2}} + \frac{q}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{2\varepsilon_p}{\gamma(\varepsilon_p + \varepsilon_f)} \times \frac{(\varepsilon_p - \varepsilon_d) e^{\gamma(z - (2z_b - z_q))}}{[(\varepsilon_p + \varepsilon_f)(\varepsilon_p + \varepsilon_d)e^{2\gamma h} + (\varepsilon_p - \varepsilon_d)(\varepsilon_f - \varepsilon_p)]} \right) \times e^{i(\xi x + \eta y)} d\xi d\eta. \quad (34)$$

Passing in the integral to polar coordinates in the planes (x, y) and (ξ, η) by the formulas

$$x = \rho \cos \psi, \quad y = \rho \sin \psi, \quad \text{and} \quad \xi = \lambda \cos \vartheta, \quad \eta = \lambda \sin \vartheta, \quad (35)$$

we will get

$$\varphi_1(\rho, \psi, z) \frac{q}{4\pi \varepsilon_f \sqrt{\rho^2 + (z - z_q)^2}} + \frac{\varepsilon_f - \varepsilon_p}{\varepsilon_p + \varepsilon_f} \frac{q}{4\pi \varepsilon_f \sqrt{\rho^2 + (z - (2z_b - z_q))^2}} + \frac{q}{(2\pi)^2} \frac{2\varepsilon_p(\varepsilon_p - \varepsilon_d)}{(\varepsilon_p + \varepsilon_f)} \int_0^{+\infty} \left(\int_0^{2\pi} e^{i\rho\lambda \cos(\psi - \vartheta)} d\vartheta \right) \times \left(\frac{e^{\lambda(z - (2z_b - z_q))}}{[(\varepsilon_p + \varepsilon_f)(\varepsilon_p + \varepsilon_d)e^{2\lambda h} + (\varepsilon_p - \varepsilon_d)(\varepsilon_f - \varepsilon_p)]} \right) d\lambda.$$

Using the identity $\int_0^{2\pi} e^{i\rho\lambda\cos(\psi-\vartheta)}d\vartheta = 2\pi J_0(\rho\lambda)$, we get

$$\varphi_1(\rho,\psi,z) = q/4\pi\varepsilon_f\sqrt{\rho^2+(z-z_q)^2}$$
$$+\frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p-\varepsilon_f}q/4\pi\varepsilon_f\sqrt{\rho^2+(z-(2z_b-z_q))^2}$$
$$+\frac{q\varepsilon_p(\varepsilon_p-\varepsilon_d)}{\pi(\varepsilon_p+\varepsilon_f)}$$
$$\times\int_0^{+\infty}\frac{J_0(\rho\lambda)e^{\lambda(z-(2z_b-z_q))}}{[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\lambda h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_v-\varepsilon_p)]}d\lambda. \quad (36)$$

Let us find the potential φ_4 in the half-space ($j=4$) for $z\geq h+z_b$. Substituting expressions (29) and (30) for H_{RA} and H_{RC} into expression (28) for $\hat{\varphi}_4^+$, we obtain

$$\hat{\varphi}_4^+ = \frac{q}{H_{RC}+\varepsilon_f\gamma H_{RA}}$$
$$=\frac{2q\varepsilon_p}{\gamma[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}$$
$$\times e^{-\gamma(z_b-z_q)}e^{\gamma h}. \quad (37)$$

From expression (25) , using (28) and substituting (37), we get

$$\varphi_4(x,y,z) = (2\pi)^{-2}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{2q\varepsilon_p e^{2\gamma h}}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}$$
$$\times\frac{e^{-\gamma(z-z_q)}}{\gamma}e^{i(\xi x+\eta y)}d\xi d\eta. \quad (38)$$

Passing to polar coordinates in the planes (x,y) and (ξ,η) by formulas (35) and again using the identity $\int_0^{2\pi} e^{i\rho\lambda\cos(\psi-\vartheta)} = 2\pi J_0(\rho\lambda)$, we get

$$\varphi_4(\rho,\psi,z)$$
$$=\frac{q}{\pi}\int_0^{+\infty}\frac{\varepsilon_p J_0(\rho\lambda)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\lambda h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}$$
$$\times e^{-\lambda(z-(z_q+2h))}d\lambda. \quad (39)$$

Let us now find the potential for $z_b\leq z\leq(h+z_b)$, i.e. in the layer $j=3$. Boundary conditions (9) on the plane can be written as

$$\begin{pmatrix} e^{-\gamma h} & 1\\ \varepsilon_p\gamma e^{-\gamma h} & -\varepsilon_p\gamma\end{pmatrix}\times\begin{pmatrix}\hat{\varphi}_3^+\\ \hat{\varphi}_3^-\end{pmatrix}=\begin{pmatrix}1 & 1\\ \varepsilon_d\gamma & -\varepsilon_d\gamma\end{pmatrix}\times\begin{pmatrix}\hat{\varphi}_4^+\\ 0\end{pmatrix}.$$

Solving this equation and using (37), we obtain the solution

$$\hat{\varphi}_3^+ = \frac{\varepsilon_p+\varepsilon_d}{2\varepsilon_p}e^{\gamma h}\hat{\varphi}_4^+ = \frac{\varepsilon_p+\varepsilon_d}{2\varepsilon_p}$$
$$\times\frac{2q\varepsilon_p}{\gamma[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}$$
$$\times e^{-\gamma(z_b-z_q-2h)},$$

$$\hat{\varphi}_3^- = \frac{\varepsilon_p-\varepsilon_d}{2\varepsilon_p}\hat{\varphi}_4^+ = \frac{\varepsilon_p-\varepsilon_d}{2\varepsilon_p}$$
$$\times\frac{2q\varepsilon_p}{\gamma[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}$$
$$\times e^{-\gamma(z_b-z_q-h)}.$$

The total potential of the field inside the layer $j=3$ is expressed by the formula (4)

$$\varphi_3(x,y,z) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\varphi}_3^+ e^{-\gamma(z-z_b)}e^{i(\xi x+\eta y)}d\xi d\eta$$
$$+(2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{\varphi}_3^- e^{\gamma(z-(z_b+h)}e^{i(\xi x+\eta y)}d\xi d\eta.$$

Substituting here the obtained expressions for $\hat{\varphi}_3^+$ and $\hat{\varphi}_3^-$, we obtain

$$\varphi_3(x,y,z) = (\varepsilon_p+\varepsilon_d)(2\pi)^{-2}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{qe^{-\gamma(z-(z_q+2h))}}{\gamma[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}$$
$$\times e^{i(\xi x+\eta y)}d\xi d\eta + (\varepsilon_p-\varepsilon_d)(2\pi)^{-2}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{qe^{\gamma(z-(2z_b-z_q))}}{\gamma[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}$$
$$\times e^{i(\xi x+\eta y)}d\xi d\eta. \quad (40)$$

Passing to polar coordinates in the planes (x,y) and (ξ,η) by formulas (35) and using the identity

$\int\limits_0^{2\pi} e^{i\rho\lambda\cos(\psi-\vartheta)}d\vartheta = 2\pi J_0(\rho\lambda)$, we get

$$\varphi_3(\rho,\psi,z) = \frac{q}{2\pi}\int\limits_0^{+\infty}\frac{J_0(\rho\lambda)(\varepsilon_p+\varepsilon_d)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\lambda h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}\times e^{-\lambda(z-(z_q+2h))}d\lambda + \frac{q}{2\pi}\times\int\limits_0^{+\infty}\frac{J_0(\rho\lambda)(\varepsilon_p-\varepsilon_d)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\lambda h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}\times e^{\lambda(z-(2z_b-z_q))}d\lambda. \quad (41)$$

Let us now find the potential for $z_q \le z \le z_b$, i.e. in the layer $j = 2$. Boundary conditions (9) on the plane $z = z_2 = z_b$ can be written as

$$\begin{pmatrix} e^{-\gamma(z_b-z_q)} & 1 \\ \varepsilon_f\gamma e^{-\gamma(z_b-z_q)} & -\varepsilon_f\gamma \end{pmatrix}\times\begin{pmatrix}\hat{\varphi}_2^+ \\ \hat{\varphi}_2^-\end{pmatrix} = \begin{pmatrix} 1 & e^{-\gamma h} \\ \varepsilon_p\gamma & -\varepsilon_p\gamma e^{-\gamma h}\end{pmatrix}\times\begin{pmatrix}\hat{\varphi}_3^+ \\ \hat{\varphi}_3^-\end{pmatrix}.$$

Then after simple calculations we get

$$\hat{\varphi}_2^+ = \frac{\varepsilon_f+\varepsilon_p}{2\varepsilon_f}e^{\gamma(z_b-z_q)}\hat{\varphi}_3^+ + \frac{\varepsilon_f-\varepsilon_p}{2\varepsilon_f}e^{\gamma(z_b-z_q-h)}\varphi_3^- = \frac{q}{2\gamma\varepsilon_f}, \quad (42)$$

$$\hat{\varphi}_2^- = \frac{\varepsilon_f-\varepsilon_p}{2\varepsilon_f}\hat{\varphi}_3^+ + \frac{\varepsilon_f+\varepsilon_p}{2\varepsilon_f}e^{-\gamma h}\hat{\varphi}_3^- = \Re(\gamma,h)\frac{q}{2\gamma\varepsilon_f}e^{-\gamma(z_b-z_q)}, \quad (43)$$

where $\Re(\gamma,h)$, as before, is expressed by formula (32).

The resulting expressions for $\hat{\varphi}_2^+$ and $\hat{\varphi}_2^-$ can also be obtained from $\hat{\varphi}_1^-$ and the boundary condition on the plane $z = z_1 = z_q$, taking into account the presence of a charge on the boundary [see (17)]

$$\begin{pmatrix} 1 & e^{-\gamma(z_b-z_q)} \\ \varepsilon_f\gamma & -\varepsilon_f\gamma e^{-\gamma(z_b-z_q)}\end{pmatrix}\times\begin{pmatrix}\hat{\varphi}_2^+ \\ \hat{\varphi}_2^-\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \varepsilon_f\gamma & -\varepsilon_f\gamma\end{pmatrix}\times\begin{pmatrix} 0 \\ \hat{\varphi}_1^-\end{pmatrix} + \begin{pmatrix} 0 \\ q\end{pmatrix}.$$

The total potential of the field inside the layer $j = 2$ is expressed by the formula (4)

$$\varphi_2(x,y,z) = (2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\hat{\varphi}_2^+ e^{-\gamma(z-z_q)}e^{i(\xi x+\eta y)}d\xi d\eta + (2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\hat{\varphi}_2^- e^{\gamma(z-z_q)}e^{i(\xi x+\eta y)}d\xi d\eta.$$

Inserting the expressions (42) and (43) we get

$$\varphi_2(x,y,z) = (2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{qe^{-\gamma(z-z_q)}}{2\gamma\varepsilon_f}e^{i(\xi x+\eta y)}d\xi d\eta + (2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\Re(\gamma,h)\frac{qe^{\gamma(z-(2z_b-z_q))}}{2\gamma\varepsilon_f}e^{i(\xi x-\eta y)}d\xi d\eta.$$

Substituting here the expression for $\Re(\gamma,h)$ and using identity (33), we obtain

$$\varphi_2(x,y,z) = q/4\pi\varepsilon_f\sqrt{x^2+y^2+(z-z_q)^2} + \frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}q/4\pi\varepsilon_f\sqrt{x^2+y^2+(z-(2z_b-z_q))^2} + \frac{q}{(2\pi)^2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\left(\frac{2\varepsilon_p}{\gamma(\varepsilon_p+\varepsilon_f)}\times\frac{\varepsilon_p-\varepsilon_d e^{\gamma(z-(2z_b-z_q))}}{[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}\right)\times e^{i(\xi x+\eta y)}d\xi d\eta. \quad (44)$$

Note that this is the same expression as (34) for $\varphi_1(x,y,z)$ in the half-space $j = 1$. The first term of this expression is the potential of a point charge q located at the point $\mathbf{r}_q = (0,0,z_q)$. The second term — the potential of a point charge of magnitude $q(\varepsilon_f-\varepsilon_p)(\varepsilon_p+\varepsilon_f)$ in a medium with permittivity ε_f, which is mirrored relative to the $z = z_b$ plane, at the point $\mathbf{r}_{ref} = (0;0;2z_b-z_q)$. The third term — the potential of the charge distributed over the plane $z = z_b$ with some surface density (below it will be calculated explicitly).

Passing to cylindrical coordinates (35) in the same way as when considering the solution in the half-space $j = 1$, we obtain from (44) a formula for φ_2, similar to (36) in cylindrical coordinates

$$\varphi_2(\rho,\psi,z) = q/4\pi\varepsilon_f\sqrt{\rho^2+(z-z_q)^2} + \frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}q/4\pi\varepsilon_f\sqrt{\rho^2+(z-(2z_b-z_q))^2} + \frac{q\varepsilon_p(\varepsilon_p-\varepsilon_d)}{\pi(\varepsilon_p+\varepsilon_f)} \times \int\limits_0^{+\infty}\frac{J_0(\rho\lambda)e^{\lambda(z-(2z_b-z_q))}}{[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\lambda h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}d\lambda. \quad (45)$$

To find φ_{ind} — the potential of the charges induced in the plane-layered structure under consideration, it is necessary to subtract the potential of the initial point charge from the total potential in this region in the region $z < z_b$. Then, denoting $z_{ref} = 2z_b - z_q$, we obtain from (44) and (45) expressions in Cartesian and cylindrical coordinate systems

$$\varphi_{ind}(x,y,z) = \frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}q/4\pi\varepsilon_f\sqrt{x^2+y^2+(z-z_{ref})^2} + \frac{q\varepsilon_p(\varepsilon_p-\varepsilon_d)}{2\pi^2(\varepsilon_p+\varepsilon_f)} \times \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{e^{\gamma(z-z_{ref})}e^{i(\xi x+\eta y)}}{[\gamma(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}d\xi d\eta, \quad (46)$$

$$\varphi_{ind}(\rho,\psi,z) = \frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}q/4\pi\varepsilon_f\sqrt{\rho^2+(z-z_{ref})^2} + \frac{q\varepsilon_p(\varepsilon_p-\varepsilon_d)}{\pi(\varepsilon_p+\varepsilon_f)} \times \int\limits_{-\infty}^{+\infty}\frac{J_0(\rho\lambda)e^{\lambda(z-z_{ref})}}{[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\lambda h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}d\lambda. \quad (47)$$

It can be seen from (45) that for $h \to +\infty$ the integral vanishes, and we obtain the well-known formula for the charge at the boundary of two half-spaces with the appropriate change of the permittivity symbols [17].

Generalization of the method of mirror reflections in electrostatics to the case of a point charge located near a flat-layered medium

Continuing the consideration of the problem with one film, we can find another form of writing for the potential $\varphi_{ind}(x,y,z)$ induced by a point charge q at the point $\mathbf{r}_q = (0,0,z_q)$. We use (33) and introduce the function $U(x,y,z) = q/4\pi\varepsilon_f\sqrt{x^2+y^2+z^2}$, which determines the potential of a point charge q located at the origin in a space with permittivity ε_f.

Let us write the expression (46) as

$$\varphi(x,y,z) = \frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}\frac{q}{4\pi\varepsilon_f\sqrt{x^2+y^2+(z-z_{ref})^2}} + \frac{q}{(2\pi)^2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\chi(\gamma,h)\frac{e^{\gamma(z-z_{ref})}e^{i(\xi x+\eta y)}}{2\varepsilon_f\gamma}d\xi d\eta, \quad (48)$$

where the function $\chi(\gamma,h) = \Re(\gamma,h) - (\varepsilon_f-\varepsilon_p)/(\varepsilon_p+\varepsilon_f)$ is defined by the formula

$$\chi(\gamma,h) = \frac{4\varepsilon_p\varepsilon_f}{(\varepsilon_p+\varepsilon_f)} \times \frac{(\varepsilon_p-\varepsilon_d)}{[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\gamma h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}. \quad (49)$$

Let us also introduce the function

$$r_{u,v} = (2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\chi\left(\sqrt{\xi^2+\eta^2},h\right)e^{i(\xi u+\eta v)}d\xi d\eta. \quad (50)$$

Applying the convolution theorem to the second term in (48), we can write $\varphi_{ind}(x,y,z)$ as

$$\varphi_{ind}(x,y,z) = \frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}\frac{q}{4\pi\varepsilon_f\sqrt{x^2+y^2+(z-z_{ref})^2}} + \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}r(u,v)U(x-u,y-v,z-z_{ref})du dv. \quad (51)$$

When φ_{ind} is found numerically using formula (51), the quadruple integration can be eliminated. This can be done by reducing the double integration in (50) to a single integration by passing to polar coordinates in the (u,v) and (ξ,η) planes using the formulas $u = \sigma\cos\psi$, $v = \sigma\sin\psi$ and $\xi = \lambda\cos\vartheta$, $\eta = \lambda\sin\vartheta$. Using the identity $\int\limits_0^{2\pi} e^{i\rho\lambda\cos(\psi-\vartheta)}d\vartheta = 2\pi J_0(\rho\lambda)$, we get

$$r(\sigma,\psi) = r(\sigma) = (2\pi)^{-1}\int\limits_0^{+\infty}J_0(\sigma\lambda)\chi(\lambda,h)\lambda d\lambda = \frac{2\varepsilon_p\varepsilon_f(\varepsilon_p-\varepsilon_d)}{\pi(\varepsilon_p+\varepsilon_f)} \times \int_0^{+\infty}\frac{\lambda J_0(\sigma\lambda)}{[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\lambda h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}d\lambda.$$

Then in the coordinates (u, ν) we get

$$r(u,\nu) = r\left(\sqrt{u^2+\nu^2}\right) = \frac{2\varepsilon_p\varepsilon_f(\varepsilon_p-\varepsilon_d)}{\pi(\varepsilon_p+\varepsilon_f)} \times \int_0^{+\infty} \frac{\lambda J_0\left(\lambda\sqrt{u^2+\nu^2}\right)}{[(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)e^{2\lambda h}+(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)]}d\lambda. \quad (52)$$

The function $r(\sigma)$ is a smooth function, it can be very easily approximated by splines, and in numerical integration in (51) this function can be used as a spline approximation.

Formula (51) gives the potential induced in a layered structure under the action of a point charge q located at the point $\mathbf{r}_q = (0, 0, q)$. Then for a point charge q located at an arbitrary point $\mathbf{r}_q = (x_q, y_q, z_q)$, the total potential $\varphi_1 = \varphi_2$ in the half-space $j = 1$ and $j = 2$ can be written as

$$\varphi_1(x,y,z) = \varphi_2(x,y,z) = = q/4\pi\varepsilon_f\sqrt{(x-x_q)^2+(y-y_q)^2+(z-z_q)^2} + \frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_f)}q/4\pi\varepsilon_f\sqrt{(x-x_q)^2+(y-y_q)^2+(z-z_{ref})^2} + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u,\nu)U(x-x_q-u, y-y_q-\nu, z-z_{ref})dud\nu, \quad (53)$$

where, as before, $z_{ref} = 2z_b - z_q$.

The resulting formula generalizes the method of specular reflections in electrostatics to the case when a point charge q is located in one half-space next to a film of thickness h located on the boundary of another half-space. The expression (53) has a simple physical meaning. The first term in (53) — the potential of the initial point charge q (field source), the second term — the potential of the virtual charge of the value $q(\varepsilon_f - \varepsilon_p)/(\varepsilon_p + \varepsilon_f)$, which is mirrored with respect to the nearest film boundary $z = z_b$, and the third term the potential of virtual charges distributed over the plane $z = z_{ref}$ with a density $q \cdot r(x, y)$ concentrated in a surface region with a size on the order of the film thickness h. It is easy to see from the expression (52) for $r(u, \nu)$ that the last term in (53) decreases quite rapidly with increasing film thickness (at $h \to \infty$).

It can be seen from the derivation of formula (53) that for the case of a plane-layered structure consisting of an arbitrary number of films, we can obtain a similar formula, but with a different distribution of the surface charge $q \cdot r(x, y)$ over the plane $z = z_{ref}$, this distribution will depend on the thicknesses of all films and their permittivities, as well as on the parameters of the half-space behind the planar-layered structure. Following the arguments proposed above, one can numerically or analytically find the function $r(x, y)$ for a multifilm structure and use this function in formula (53).

For further generalizations, it is easy to obtain formulas of the form (53) for the field potential of a point charge q located at an arbitrary point $\mathbf{r}_q = (x_q, y_q, z_q)$ in the film region ($j = 3$) and in the half-space behind the film ($j = 4$). Here is the calculation result:

$$\varphi_3(x,y,z) = \frac{2\varepsilon_f}{(\varepsilon_p+\varepsilon_f)}U(x-x_q, y-y_q, z-z_q) - \frac{(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u,\nu) \times U(x-x_q-u, y-y_q-\nu, z-z_q)dud\nu + \frac{2\varepsilon_f(\varepsilon_p-\varepsilon_d)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)} \times U(x-x_q, y-y_q, z-[2(z_f+h)-z_q]) - \frac{(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p(\varepsilon_p+\varepsilon_d)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u,\nu) \times U(x-x_q-u, y-y_q-\nu, z-[2(z_b+h)-z_q])dud\nu \quad (54)$$

$$\varphi_4(x,y,z) = \frac{4\varepsilon_f\varepsilon_p}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}U(x-x_q, y-y_q, z-z_q) - \frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_d)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u,\nu) \times U(x-x_q-u, y-y_q-\nu, z-z_q)dud\nu, \quad (55)$$

where the functions are defined (52).

Generalization of the mirror reflection method to the case of a system of source charges

Let us now generalize the obtained method of mirror reflections and find the potential of the total field $\Phi_{tot}(x, y, z)$ in the region $z < z_b$ in front of the film (Fig. 5) of an arbitrary compact system N_q of source charges q_k located at points with radius vectors $\mathbf{r}_{q,k} = (x_{q,k}, y_{q,k}, z_{q,k})$, where $k = 1, 2, \ldots, N_q$.

If there were no plane-layered structure, then the potential of this system of charges-sources would be represented by the formula

$$\Phi_s(x,y,z) = = \sum_{k=1}^{N_q} q_k/4\pi\varepsilon_f\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+(z-z_{q,k}^2)}. \quad (56)$$

Summing expressions (53) for each charge q_k over all N_q charges of the system and noticing that

$$\sum_{k=1}^{N_q} q_k/4\pi\varepsilon_f\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+(z-(z_b-z_{q,k}))^2}$$
$$=\sum_{k=1}^{N_q} q_k/4\pi\varepsilon_f\sqrt{(x-x_{q,k})^2+(y-y_{q,k})^2+((2z_b-z)-z_{q,k})^2}$$
$$=\Phi_s(x,y,2z_b-z),$$

we obtain a generalization of the method of mirror reflections in electrostatics for arbitrary charge distributions in the form

$$\Phi_{tot}(x,y,z)=\Phi_s(x,y,z)+\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_f)}\Phi_s(x,y,2z_b-z)$$
$$+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u,v)\Phi_s(x-u,y-v,2z_b-z)dudv. \quad (57)$$

A generalization of the specular reflection method was obtained earlier in studies [18,19], in a slightly different way, and was successfully applied to the problem of nanofocusing of a surface plasmon wave at the vertex of a metal tip. In this case, the field of the paraboloidal tip outside and inside was decomposed into complete systems of harmonic functions. The generalized method of mirror reflections allows to obtain an equation that determines these expansions. The solution of this nanofocusing problem allows to calculate the distribution of the focused field in a photoresist film, which is important for nanolithography problems.

It follows from the derivation of formula (57) that the potential of induced charges Φ_{ind}, which is generated by the potential Φ_s of distributions of charge-sources of the field, has the form

$$\Phi_{ind}(x,y,z)=\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_f)}\Phi_s(x,y,2z_b-z)$$
$$+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u,v)\Phi_s(x-u,y-v,2z_b-z)dudv. \quad (58)$$

Similarly, from (54) and (55) we find the potential of the total field Φ_3 in the film at $z_b \le z \le (z_b+h)$ $(j=3)$ and the potential Φ_4 of the full field in the half-space for $(z \ge z_b+h)$ $(j=4)$:

$$\Phi_3(x,y,z)=\frac{2\varepsilon_f}{(\varepsilon_p+\varepsilon_f)}\Phi_s(x,y,z)-\frac{(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u,v)\Phi_s(x-u,y-v,z)dudv$$
$$+\frac{2\varepsilon_f(\varepsilon_p-\varepsilon_d)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}\Phi_s(x,y,2(z_b+h)-z)$$
$$-\frac{(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p(\varepsilon_p+\varepsilon_d)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u,v)$$
$$\times\Phi_s(x-u,y-v,2(z_b+h)-z)dudv, \quad (59)$$

$$\Phi_4(x,y,z)=\frac{4\varepsilon_f\varepsilon_p}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}\Phi_s(x,y,z)-\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_d)}$$
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u,v)\Phi_s(x-u,y-v,z)dudv. \quad (60)$$

Distribution of the electric field at the nanovertex of a metal micropoint located near the flat boundary of a thin-film structure in the quasi-static approximation

Let us consider a metal tip with a nanosized vertex radius R. Let the surface of the tip near the vertex be represented (axisymmetric about the Z axis) paraboloid of revolution $z=R/2-(x^2+y^2)/2R$ (Fig. 6) . Let a film of thickness h with boundaries $z=z_b$ and $z=(z_b+h)$ be located near the tip. It is assumed that $z_b \ge R/2$. The complex permittivities of the metal of the tip, the external homogeneous medium, the film, and the semi-infinite medium behind the film are denoted by ε_m, ε_f, ε_p, and ε_d, respectively (Fig. 6).

Let us consider the distribution of a harmonic electric field, which will be established in the vicinity of the micropoint nanovertex upon convergence of a symmetric TM surface plasmon wave, which is used to focus the light field on the metal micropoint nanovertex in important applications [1].

When solving this problem, we will use the complex representation of fields with time dependence $e^{-i\omega t}$, where ω — cyclic frequency. We will assume that the characteristic geometric dimensions of the problem (on the order of ten nanometers) are much smaller than the length of an electromagnetic wave in vacuum corresponding to the frequency ω. It can be shown [7,20,21] that such a problem near the nanovertex can be solved in a quasi-electrostatic formulation, in which the complex potential of the electric field Φ satisfies the Laplace equation $\Delta\Phi=0$, and the

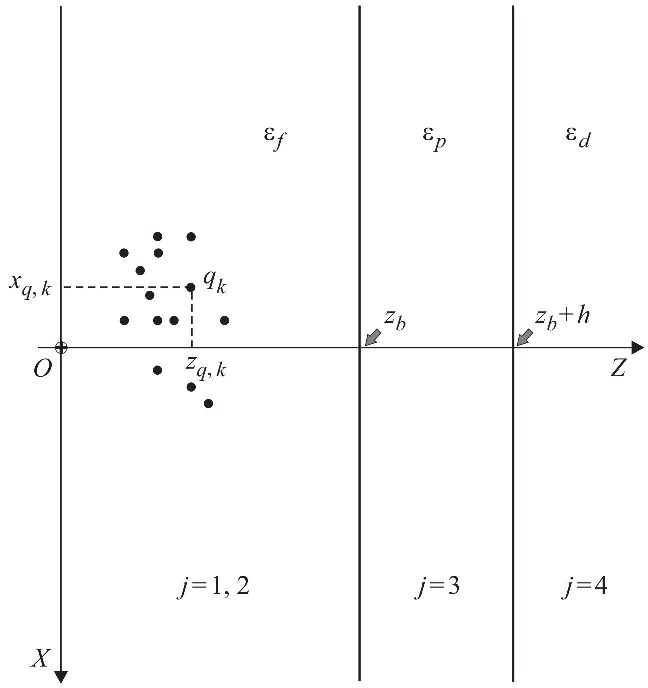

Figure 5. system of point charges q_k for a film located on the boundary of a half-space.

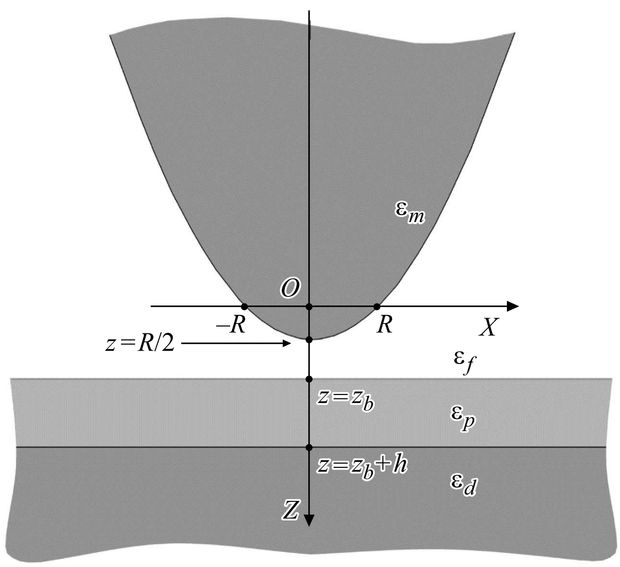

Figure 6. Vertex of the metal paraboloidal tip at the film located on the boundary of the half-space. Geometry of the problem.

normal and the tangential components of the electric field on the tip surface and on two flat film boundaries should correspond to the known boundary conditions on the tip surface:

$$\varepsilon_f E_{f,n} = \varepsilon_m E_{m,n} \text{ and } E_{f,\tau} = E_{m,\tau}, \tag{61}$$

on the film boundary $z = z_b$:

$$\varepsilon_p E_{3,n} = \varepsilon_f E_{f,n} \text{ and } E_{3,\tau} = E_{f,\tau}, \tag{62}$$

on the film boundary $z = z_b$:

$$\varepsilon_d E_{4,n} = \varepsilon_p E_{3,n} \text{ and } E_{4,\tau} = E_{3,\tau}. \tag{63}$$

We will look for an axisymmetric solution of the Laplace equation, which has a field maximum at the tip vertex, that corresponds to the focusing of a surface symmetric plasmon TM wave on it.

In addition, to automatically satisfy the boundary conditions on the flat film surfaces (62) and (63), we will use a special method, which will be described below. Let the potential of charges located on a paraboloidal metal tip in a space with a permeability ε_f be described by the function $\Phi_s(x, y, z)$. Let the potential of induced charges at the film boundaries (note that it follows from Maxwell's equations that there are no polarization charges inside a homogeneous dielectric, they can only be at the boundary) in a space with a permeability ε_f equal to $\Phi_{ind}(x, y, z)$. Then the total potential Φ_{tot} in a region filled with a dielectric with permittivity ε_f is equal to

$$\Phi_{tot}(x, y, z) = \Phi_s(x, y, z) + \Phi_{ind}(x, y, z). \tag{64}$$

Let the potential of the electric field excited by the charges of the tip in the regions with ε_p and ε_d, i.e. in the film and in the half-space behind the film are equal to $\Phi_3(x, y, z)$ and $\Phi_4(x, y, z)$, respectively.

Let us find the general form of the potential of a single tip without a layered structure nearby, which satisfies the Laplace equation in a homogeneous dielectric space outside $\Phi_s(x, y, z)$ and inside $\Phi_m(x, y, z)$ of the tip, and boundary conditions (61) should be satisfied at the tip boundary. Let us introduce paraboloidal coordinates [22] (a system of parabolic coordinates of rotation) (α, β, ψ), which are related to rectangular Cartesian coordinates (x, y, z) by the formulas

$$x = c\alpha\beta\cos\psi, \quad y = c\alpha\beta\sin\psi, \quad z = c(\beta^2 - \alpha^2)/2, \tag{65}$$

where c — scaling constant factor. In the coordinate system under consideration with origin at point 0 and axis Z (Fig. 6), the Laplace equation for the electric potential inside Φ_m or outside Φ_s of the tip at axial symmetries (Φ_m and Φ_s do not depend on ψ) can be written as follows [22]:

$$\Delta\Phi = c^{-2}(\alpha^2+\beta^2)^{-1}\left(\frac{\partial^2\Phi}{\partial\alpha^2}+\frac{\partial^2\Phi}{\partial\beta^2}+\frac{1}{\alpha}\frac{\partial\Phi}{\partial\alpha}+\frac{1}{\beta}\frac{\partial\Phi}{\partial\beta}\right) = 0. \tag{66}$$

The general axisymmetric solution (66) is known [22] and is determined by the expression

$$\Phi = \sum(B_1 J_0(p\alpha) + B_2 Y_0(p\alpha)) \times (C_1 I_0(p\beta) + C_2 K_0(p\beta)), \tag{67}$$

where p, B_1, B_2, C_1, C_2 are constants; J_0, Y_0 — Bessel functions of the first and second kind of zero order; I_0, K_0 — modified Bessel functions of the first and second kind of zero order. The summation is performed over the solutions with different constants values.

Let the boundary of the paraboloidal tip be determined by the equation $\beta = \beta$. It follows from (65) that the boundary of the tip $\beta = \beta_0$ in Cartesian coordinates (x, y, z) is determined by the equation $z = c\beta_0^2/2 - (x^2 + y^2)/2c\beta_0^2$. It is easy to show that the radius of curvature of the vertex of the tip is $R = c\beta_0^2$.

Bearing in mind the generality of the further presentation, we pass to dimensionless coordinates: $\tilde{x} = x/R$, $\tilde{y} = y/R$, $\tilde{z} = z/R$ and $\tilde{\alpha} = \alpha/\beta_0$, $\tilde{\beta} = \beta/\beta_0$. Dimensionless paraboloidal $(\tilde{\alpha}, \tilde{\beta})$ and Cartesian coordinates on the plane $(\tilde{x}, \tilde{z})$ are related by the formulas $\tilde{\alpha} = \sqrt{\sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} - \tilde{z}}$ and $\tilde{\beta} = \sqrt{\tilde{x}^2 + \tilde{y}^2}/\sqrt{\sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} - \tilde{z}}$. In these coordinates the tip boundary will be determined by the function $\tilde{z} = 1/2 - (\tilde{x}^2 + \tilde{y}^2)/2$. We normalize the potential to its value U at the field maximum at the tip vertex, then we can pass from the dimensional to the dimensionless potential $\tilde{\Phi} = \Phi/U$, for which equation (66) in dimensionless coordinates should be satisfied.

In the axisymmetric case, to fulfill the boundary conditions on the surface of the tip revolution, it is sufficient to satisfy them on the line of intersection of the tip surface with any plane of symmetry passing through the Z axis. As such a plane we choose the plane $(\tilde{x}, \tilde{z})$ for $\tilde{y} = 0$. More specifically, it suffices to satisfy the boundary conditions only on the boundary of the intersection of the half-plane $\tilde{y} = 0$ at $\tilde{x} \geq 0$ and the surface of the considered paraboloidal tip. In dimensionless coordinates, this will be the curve $\tilde{z} = 1/2 - \tilde{x}^2/2$, for $\tilde{y} = 0$ and $\tilde{x} \geq 0$. In normalized Cartesian coordinates in the plane $(\tilde{x}, \tilde{z})$ with $\tilde{y} = 0$, the components of the normalized electric field will have the form: $\tilde{E}_{\tilde{x}} = -\partial\tilde{\Phi}/\partial\tilde{x}$, $\tilde{E}_{\tilde{z}} = -\partial\tilde{\Phi}/\partial\tilde{z}$.

So, based on the general solution (67), we will look for a solution to the boundary value problem for the electric field in the vicinity of the tip, assuming that the potential outside is $\tilde{\Phi}_s$, at $\beta \geq \beta_0$, and the potential inside the metal tip $\tilde{\Phi}_m$, at $\beta \leq \beta_0$, have the form, respectively

$$\Phi_s = \sum_{j=1}^{N} A_j J_0(\lambda_j \tilde{\alpha}) K_0(\lambda_j \tilde{\beta}),$$

and

$$\Phi_m = \sum_{j=1}^{N} B_j J_0(\lambda_j \tilde{\alpha}) I_0(\lambda_j \tilde{\beta}), \tag{68}$$

where A_j, B_j and λ_j — constants. λ_i values can be chosen as $\lambda_j = \mu_j/L$, where μ_j, $j = 1, 2, \ldots, N$ — the first N roots of the Bessel equation $J_0(\mu_j) = 0$, and L — some dimensionless distance from the vertex, by which we will satisfy the boundary conditions on the surface of the tip. In the limit $N \to \infty$ of the function $J_0(\lambda_j \tilde{\alpha})$ with the above choice λ_j on the interval $0 \leq \tilde{\alpha} \leq L$ form a complete system of functions [23].

Note that the choice of functional dependencies (68) from the general solution (67) is due to natural requirements for the field concentrated at the vertex (which single out these dependencies uniquely):

a) Outside the tip, the field potential should decrease with distance from its surface, be finite and maximum at the tip vertex.

b) Inside the tip metal, the potential should be finite at the origin. In addition, the electrical potential should be continuous across the boundary.

Then the potential of the charges of the $\tilde{\Phi}$ tip in the coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ in the medium with ε_f, after substituting the expressions $\tilde{\alpha} = \sqrt{\sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} - \tilde{z}}$ and $\tilde{\beta} = \sqrt{\tilde{x}^2 + \tilde{y}^2}/\sqrt{\sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} - \tilde{z}}$ in (68) can be as follows

$$\tilde{\Phi}_s(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{j=1}^{N} A_j \mathfrak{F}_j(\tilde{x}, \tilde{y}, \tilde{z}), \tag{69}$$

where

$$\mathfrak{F}_j(\tilde{x}, \tilde{y}, \tilde{z}) = J_0\left(\lambda_j \sqrt{\sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} - \tilde{z}}\right)$$
$$\times K_0\left(\lambda_j \sqrt{\tilde{x}^2 + \tilde{y}^2}/\sqrt{\sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} - \tilde{z}}\right)$$

Similarly, the potential of induced charges in the flat-layered structure $\tilde{\Phi}_{ind}(\tilde{x}, \tilde{y}\tilde{z})$ can be represented, taking into account (58), as follows

$$\Phi_{ind}(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{j=1}^{N} A_j \mathfrak{P}_j(\tilde{x}, \tilde{y}, \tilde{z}). \tag{70}$$

where

$$\mathfrak{P}_j(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{(\varepsilon_f - \varepsilon_p)}{(\varepsilon_p + \varepsilon_f)} \mathfrak{F}_j(\tilde{x}, \tilde{y}, \tilde{z})$$
$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(u, v) \mathfrak{F}_j(\tilde{x} - u, \tilde{y} - v, 2\tilde{z}_b - \tilde{z}) du dv.$$

Substituting (69) and (70) into (64), we obtain the total potential in a medium with ε_f as follows

$$\Phi_{tot}(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{j=1}^{N} A_j \left(\mathfrak{F}_j(\tilde{x}, \tilde{y}, \tilde{z}) + \mathfrak{P}_j(\tilde{x}, \tilde{y}, \tilde{z})\right). \tag{71}$$

Similarly from (68) in the tip metal the potential can be represented as

$$\tilde{\Phi}_m(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{j=1}^{N} B_j \mathfrak{H}_j(\tilde{x}, \tilde{y}, \tilde{z}), \tag{72}$$

where

$$\mathfrak{H}_j(\tilde{x}, \tilde{y}, \tilde{z}) = J_0\left(\lambda_j \sqrt{\sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} - \tilde{z}}\right)$$
$$\times I_0\left(\lambda_j \sqrt{\tilde{x}^2 + \tilde{y}^2}/\sqrt{\sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} - \tilde{z}}\right)$$

On the tip boundary (for $\tilde{y}=0,\ \tilde{z}=1/2-\tilde{x}^2/2$), the unit vectors of the normal and tangent are represented by the formulas

$$\mathbf{n}=\mathbf{e}_{\tilde{x}}\left(\tilde{x}/\sqrt{1+\tilde{x}^2}\right)+\mathbf{e}_{\tilde{z}}\left(1/\sqrt{1+\tilde{x}^2}\right);$$

$$\tau=\mathbf{e}_{\tilde{x}}\left(1/\sqrt{1+\tilde{x}^2}\right)+\mathbf{e}_{\tilde{z}}\left(-\tilde{x}/\sqrt{1+\tilde{x}^2}\right).$$

Then the boundary conditions for normal and tangential fields on the tip surface can be written

$$-\varepsilon_f\partial\tilde{\Phi}_{tot}/\partial\mathbf{n}\Big|_{\substack{\tilde{y}=0\\ \tilde{z}=-1/2-\tilde{x}^2/2}}+\varepsilon_m\partial\tilde{\Phi}_m/\partial\mathbf{n}\Big|_{\substack{\tilde{y}=0\\ \tilde{z}=-1/2-\tilde{x}^2/2}}=0$$

and

$$-\partial\tilde{\Phi}_{tot}/\partial\tau\Big|_{\substack{\tilde{y}=0\\ \tilde{z}=1/2-\tilde{x}^2/2}}+\partial\tilde{\Phi}_m/\partial\tau\Big|_{\substack{\tilde{y}=0\\ \tilde{z}=1/2-\tilde{x}^2/2}}=0. \quad (73)$$

If we substitute expressions (71), (72) into (73), then we obtain linear equations with respect to A_j, B_j, which can be represented as

$$-\varepsilon_f\sum_{j=1}^{N}a_j(\tilde{x})A_j+\varepsilon_m\sum_{j=1}^{N}b_j(\tilde{x})B_j=0$$

and

$$\sum_{j=1}^{N}c_j(\tilde{x})A_j+\sum_{j=1}^{N}d_j(\tilde{x})B_j=0. \quad (74)$$

To save space, we will not write out the functions $a_j(\tilde{x})$, $b_j(\tilde{x})$, $c_j(\tilde{x})$*here* and $d_j(\tilde{x})$ in explicit form, which are obtained by the indicated trivial substitution.

In this study, equations (74) were solved by the collocation method [24]. These equations were written at $(N-1)$ points of the curve $\tilde{z}=1/2-\tilde{x}^2/2$ for $\tilde{y}=0$ and $\tilde{x}>0$ on the surface of the tip. As a result, $(2N-2)$ linear algebraic equations with $2N$ unknown coefficients A_j and B_j were obtained. To obtain a unique solution, two more equations were added: the potentials at the tip vertex were set equal to unity, outside $\tilde{\Phi}_{tot}(0,0,1/2)=1$ and inside $\tilde{\Phi}_m(0,0,1/2)=1$. As a result of solving the resulting system of $2N$ equations, find A_j and B_j, and the potential and electric field distributions over them in the entire space. This resulted in a normalized (to unity at the tip vertex) distribution $\tilde{\Phi}_{tot}(\tilde{x},\tilde{y},\tilde{z})$.

Knowing all A_j, one can find $\tilde{\Phi}_s(\tilde{x},\tilde{y},\tilde{z})$. Then, using formulas (59) and (60), one can find the potentials both in the film and in the free half-space as follows:

$$\tilde{\Phi}_3(\tilde{x},\tilde{y},\tilde{z})=\sum_{j=1}^{N}A_j\left(\frac{2\varepsilon_f}{(\varepsilon_p+\varepsilon_f)}\mathfrak{F}_j(\tilde{x},\tilde{y},\tilde{z})\right.$$
$$-\frac{(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p}\mathfrak{S}_j(\tilde{x},\tilde{y},\tilde{z})$$
$$+\frac{2\varepsilon_f(\varepsilon_p-\varepsilon_d)}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}\mathfrak{F}_j(\tilde{x},\tilde{y},2(\tilde{z}_b+\tilde{h})-\tilde{z})$$
$$\left.-\frac{(\varepsilon_p-\varepsilon_d)(\varepsilon_f-\varepsilon_p)}{2\varepsilon_p(\varepsilon_p+\varepsilon_d)}\mathfrak{S}_j(\tilde{x},\tilde{y},2(\tilde{z}_b+\tilde{h})-\tilde{z})\right), \quad (75)$$

$$\tilde{\Phi}_4(\tilde{x},\tilde{y},\tilde{z})=\sum_{j=1}^{N}A_j\left(\frac{4\varepsilon_f\varepsilon_p}{(\varepsilon_p+\varepsilon_f)(\varepsilon_p+\varepsilon_d)}\mathfrak{F}_j(\tilde{x},\tilde{y},\tilde{z})\right.$$
$$\left.-\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_d)}\mathfrak{S}_j(\tilde{x},\tilde{y},\tilde{z})\right), \quad (76)$$

where $\mathfrak{S}_j(\tilde{x},\tilde{y},\tilde{z})=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}r(u,v)\mathfrak{F}_j(\tilde{x}-u,\tilde{y}-v,\tilde{z})$ formulas (71), (75) and (76) solve the problem of finding the potential outside the tip and in the entire layered structure from known values A_j.

Generalization of the specular reflection method for the case of a plane-layered structure with an arbitrary number of films

If we analyze the obtained in the previous section generalization of the reflection method for a system of the charges located near the one film boundary, then an important point in the generalization was the representation of the solution of equation (27) for one charge in the form (31), i.e. as follows

$$\hat{\varphi}_1^-=\frac{q}{2\gamma\varepsilon_f}+\frac{q}{2\gamma\varepsilon_f}e^{2\gamma(z_b-z_q)}\mathfrak{R}(\gamma,h),$$

where $\mathfrak{R}(\gamma,h)=(\varepsilon_f-\varepsilon_p)/(\varepsilon_p+\varepsilon_f)+\chi(\gamma,h)$, while the function $\chi(\gamma,h)$ is determined for a plane-layered structure consisting of a single film by analytical formula (49) and depends on the film thickness and on all permittivities of the problem.

The function $\mathfrak{R}(\gamma,h)$ determines the potential $\varphi_{ind}(x,y,z)$ of induced charges, and the reflection from the film boundary $z=z_b$ is determined by the term $(\varepsilon_f-\varepsilon_p)/(\varepsilon_p+\varepsilon_f)$, and the influence of the parameters of the film and the half-space behind is determined by the function $\chi(\gamma,h)$ (see (48)).

Note that $\chi(\gamma,h)$ arises from the problem of finding the field from a point charge located near one film of a flat-layered structure. If there are several films in a plane layered structure, then it can be shown that the solution $\hat{\varphi}_1^-$ of an equation of type (27) for a point charge will have the same form

$$\hat{\varphi}_1^-=\frac{q}{2\gamma\varepsilon_f}+\frac{q}{2\gamma\varepsilon_f}e^{-2\gamma(z_b-z_q)}\left(\frac{(\varepsilon_f-\varepsilon_p)}{(\varepsilon_p+\varepsilon_f)}+\chi(\gamma)\right),$$

but the function $\chi(\gamma)$ will depend on γ and all parameters of the media and film thicknesses. For two films it is possible to obtain analytical formulas for $\chi(\gamma)$, but they turn out to be cumbersome to use. One can prove the above statement by induction, and also verify numerically that $\chi(\gamma)$ — is a rapidly decreasing function whose inverse Fourier transform exists and the integrals rapidly converge.

Thus, for fixed thicknesses and dielectric parameters of the problem, the function χ depends only on γ and can always be found numerically from the found $\hat{\varphi}_1^-$ (from (27)) in the following way:

$$\chi(\gamma)=\frac{2\gamma\varepsilon_f}{q}e^{2\gamma(z_b-z_q)}\left(\hat{\varphi}_1^-(\gamma)-\frac{q}{2\gamma\varepsilon_f}\right)-\left(\frac{\varepsilon_f-\varepsilon_p}{\varepsilon_p+\varepsilon_f}\right).$$

This function can be approximated by a one-dimensional spline in γ by calculating the values of $\chi(\gamma)$ at a finite number of points γ. Then the inverse Fourier transform of this function $r(u,\nu)) = (2\pi)^{-2}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\chi\left(\sqrt{\xi^2+\eta^2}), h\right)e^{i(\xi u+\eta\nu)}d\xi d\eta$ will have the simple form $r(u,\nu)=r\left(\sqrt{u^2+\nu^2}\right)$ and also be represented as a one-dimensional spline . Then for one point charge next to the multilayer planar structure $\varphi_1(x,y,z)$ and $\varphi_{ind}(x,y,z)$ will have the same form (53) and (51), but with another function $r(u,\nu)=r\left(\sqrt{u^2+\nu^2}\right)$ represented by a one-dimensional spline . We emphasize that the function $r(u,\nu)$ is determined from the problem for a point charge.

If the potential of a system of charges in free space is known without taking into account the flat-layered structure $\Phi_s(x,y,z)$, which is expressed by formula (56), then the representation (57) for the total potential $\Phi_{tot}(x,y,z)$ taking into account an arbitrary flat-layered structure in the free region of the half-space in front of it.

How can we now find the potentials inside a plane layered multifilm structure, since formulas (75) and (76) are suitable only for one film? The answer is as follows: knowing $\tilde{\Phi}_{tot}(\tilde{x},\tilde{y},\tilde{z})$, one can find $\tilde{\Phi}_{tot}(\tilde{x},\tilde{y},\tilde{z}_b)$and $\tilde{D}_{tot,x}(\tilde{x},\tilde{y},\tilde{z}_b)=-\varepsilon_f\partial\tilde{\Phi}_{tot}(\tilde{x},\tilde{y},\tilde{z})/\partial\tilde{z}|_{\tilde{z}=\tilde{z}_b}$, which means that the Fourier transforms of these functions can be found on this boundary:

$$\mathfrak{T}(\xi,\eta)=\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\tilde{\Phi}_{tot}(\tilde{x},\tilde{y},\tilde{z}_b)e^{-i\xi\tilde{x}-i\eta\tilde{y}}d\tilde{x}d\tilde{y},$$

$$\mathfrak{D}(\xi,\eta)=\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\tilde{D}_{tot,x}(\tilde{x},\tilde{y},\tilde{z}_b)e^{-i\xi\tilde{x}-i\eta\tilde{y}}d\tilde{x}d\tilde{y},$$

Due to the symmetry of the problem, $\tilde{\Phi}_{tot}(\tilde{x},\tilde{y},\tilde{z}_b)=\tilde{\Phi}_{tot}(\tilde{\rho},\tilde{z}_b)$, $\tilde{D}_{tot,z}(\tilde{x},\tilde{y},\tilde{z}_b)=\tilde{D}_{tot,z}(\tilde{\rho},\tilde{z}_b)$, $\mathfrak{T}(\xi,\eta)=\mathfrak{T}(\gamma)$, $\mathfrak{D}(\xi,\eta)=\mathfrak{D}(\gamma)$, where $\tilde{\rho}=\sqrt{\tilde{x}^2+\tilde{y}^2}$, $\gamma=\sqrt{\xi^2+\eta^2}$. Then from (6) and boundary conditions (8), we obtain

$$\begin{pmatrix}\mathfrak{T}(\gamma)\\ \mathfrak{D}(\gamma)\end{pmatrix}=\begin{pmatrix}1 & e^{-\gamma\tilde{h}_1}\\ \varepsilon_3\gamma & -\varepsilon_3\gamma e^{-\gamma\tilde{h}_1}\end{pmatrix}\times\hat{\mathscr{F}}_3$$

or

$$\hat{\mathscr{F}}_3=\begin{pmatrix}\hat{\varphi}_3^+\\ \hat{\varphi}_3^-\end{pmatrix}=\begin{pmatrix}1 & e^{-\gamma\tilde{h}_1}\\ \varepsilon_3\gamma & -\varepsilon_3\gamma e^{-\gamma\tilde{h}_1}\end{pmatrix}^{-1}\times\begin{pmatrix}\mathfrak{T}(\gamma)\\ \mathfrak{D}(\gamma)\end{pmatrix}, \quad (77)$$

where $\tilde{h}_1$ — the first film thickness, ε_3 — its permittivity.

From the functions $\hat{\varphi}_3^+(\gamma)$ and $\hat{\varphi}_3^-(\gamma)$ obtained from (77) and from the expression (4) find the potential in the first film (thickness $\tilde{h}_1$):

$$\tilde{\Phi}_3(\tilde{x},\tilde{y},\tilde{z})=(2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\hat{\varphi}_3^+(\gamma)e^{-\gamma(\tilde{z}-\tilde{z}_b)}e^{i(\xi\tilde{x}+\eta\tilde{y})}d\xi d\eta$$

$$+(2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\hat{\varphi}_3^+(\gamma)e^{\gamma(\tilde{z}-(\tilde{z}+\tilde{h}_1))}e^{i(\xi\tilde{x}+\eta\tilde{y})}d\xi d\eta.$$

In the next film with thickness $\tilde{h}_2$ and permittivity ε_4, we calculate the field from equation (9) relating $\hat{\mathscr{F}}_3$ and $\hat{\mathscr{F}}_4$, i.e. $\hat{\mathscr{F}}_4=\mathbf{L}_4^{-1}\times\mathbf{R}_3\times\hat{\mathscr{F}}_3$ or

$$\hat{\mathscr{F}}_4=\begin{pmatrix}\hat{\varphi}_4^+\\ \hat{\varphi}_4^-\end{pmatrix}=\begin{pmatrix}1 & e^{-\gamma\tilde{h}_2}\\ \varepsilon_4\gamma & -\varepsilon_4\gamma e^{-\gamma\tilde{h}_2}\end{pmatrix}\times\begin{pmatrix}e^{-\gamma\tilde{h}_1} & 1\\ \varepsilon_3\gamma e^{-\gamma\tilde{h}_1} & -\varepsilon_3\gamma\end{pmatrix}\times\hat{\mathscr{F}}_3, \quad (78)$$

here $\tilde{h}_1$ and $\tilde{h}_2$ — the thicknesses of the first and second films in normalized coordinates. Then, having calculated from (78) the functions $\hat{\varphi}_4^+$ and $\hat{\varphi}_4^-$, we obtain the potential in the second film:

$$\tilde{\Phi}_4(\tilde{x},\tilde{y},\tilde{z})=(2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\hat{\varphi}_4^+(\gamma)e^{-\gamma(\tilde{z}-\tilde{z}_3)}e^{i(\xi\tilde{x}+\eta\tilde{y})}d\xi d\eta$$

$$+(2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\hat{\varphi}_4^-(\gamma)e^{\gamma(\tilde{z}-\tilde{z}_4)}e^{i(\xi\tilde{x}+\eta\tilde{y})}d\xi d\eta.$$

Continuing in the same way, one can find the total potentials from the charged tip in all the films of the problem and in the half-space behind them. For definiteness, consider the problem with two films. Let us write the boundary condition on the last boundary of the second film and half-space: $\mathbf{L}_5\times\hat{\mathscr{F}}_5=\mathbf{R}_4\times\hat{\mathscr{F}}_4$, where

$$\mathbf{L}_5=\mathbf{T}_5=\begin{pmatrix}1 & 1\\ \varepsilon_5\gamma & -\varepsilon_5\gamma\end{pmatrix},$$

whence $\hat{\mathscr{F}}_5=\mathbf{L}_5^{-1}\times\mathbf{R}_4\times\hat{\mathscr{F}}_4$ or

$$\hat{\mathscr{F}}_5=\begin{pmatrix}\hat{\varphi}_5^+\\ 0\end{pmatrix}=\mathbf{T}_5^{-1}\times\mathbf{T}_4^{-1}\times\mathbf{T}_3^{-1}\times\begin{pmatrix}\mathfrak{T}(\gamma)\\ \mathfrak{D}(\gamma)\end{pmatrix},$$

where $\mathbf{T}_3^{-1}=\mathbf{R}_3\times(\mathbf{L}_3)^{-1}$, $\mathbf{T}_4^{-1}=\mathbf{R}_4\times(\mathbf{L}_4)^{-1}$, and $\mathbf{R}_3$, $\mathbf{L}_3$, $\mathbf{R}_4$, $\mathbf{L}_4$ are expressed by formulas (7). Having calculated $\hat{\varphi}_5^+$, we obtain the potential in the free half-space

$$\tilde{\Phi}_5(\tilde{x},\tilde{y},\tilde{z})=(2\pi)^{-2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\hat{\varphi}_5^+(\gamma)e^{-\gamma(\tilde{z}-\tilde{z}_4)}e^{i(\xi\tilde{x}+\eta\tilde{y})}d\xi d\eta.$$

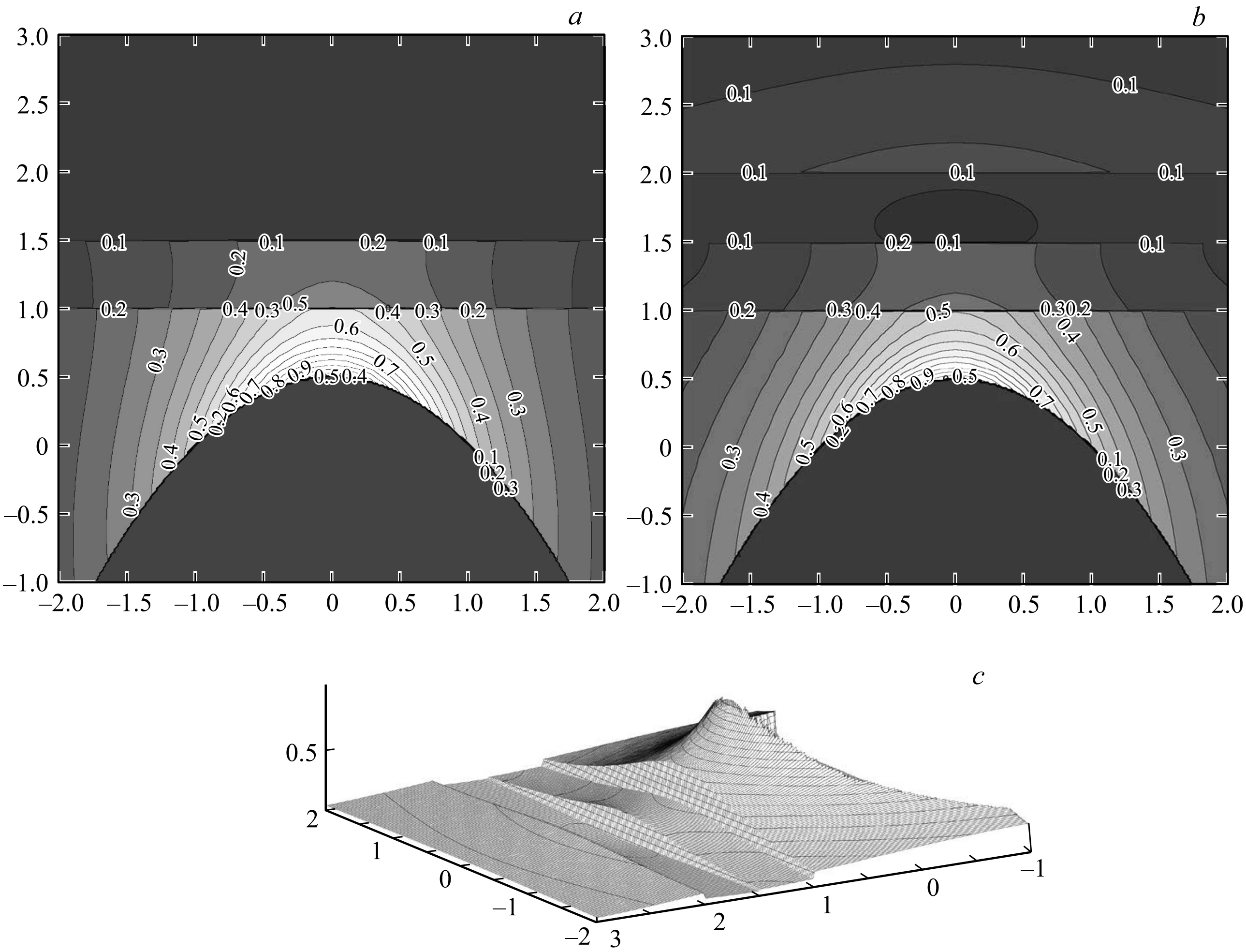

Figure 7. Distribution of the amplitude of the electric field normalized to the unity at the maximum (at the tip vertex): (a) for the film thickness $\tilde{h} = 0.5$, the permittivity of the tip, the film, and the half-space behind the film, respectively, $-11.6 + i1.2$, 2 and $-11.6 + i1.2$; (b) and (c) the same for two films of thickness $\tilde{h}_1\tilde{h}_2 = 0.5$ and dielectric constants of the tip , the first film, the second film and the half-space behind the film $-11.6 + i1.2$, 2, $011.6 + i1.2$ and 4.

Study of the focal distribution of the EM field near the nanovertex tip located next to the flat-layered structure

Numerical calculations were made of the amplitude distribution of the focused field for a gold tip located near a plane-layered structure, which occurs when a plasmon-polariton TM wave converges (along the surface of the tip to the vertex). The frequency of the focused wave corresponds to the wavelength in vacuum $\lambda_0 = 633$ nm. The permittivity of gold at this frequency was assumed to be $\varepsilon_m = -11.6 + i1.2$ [25]. The paraboloidal tip was located in a medium with $\varepsilon_f = 1$ next to a flat-layered structure consisting of a single film with a thickness equal to $\tilde{h} = 0.5$ (in units normalized to the curvature radius of the tip). The dielectric constant of the film is $\varepsilon_p = 2$, and the half-space behind the film is filled with gold $\varepsilon_d = \varepsilon_m$. The film boundaries were determined by the equations $\tilde{z} = \tilde{z}_2 = \tilde{z}_b = 1$ and $\tilde{z} = \tilde{z}_3 = \tilde{z}_b + \tilde{h} = 1.5$ (Fig. 4, 6). The vertex of the metal tip was located at a distance $\Delta\tilde{z} = 0.5$ from the film surface. The system of equations (74) was solved by the collocation method at individual points of the boundary near the vertex of the tip, as described above.

Figure 7, a shows the amplitude distribution of the total electric field in the $(\tilde{x}, \tilde{z})$ plane normalized to unity at the maximum for the above parameters. It can be seen that the field has the greatest value in free space (in the region c ε_f, Fig. 6) near the vertex. The field in the film with $\varepsilon_p = 2$ is less, and in the metal of the tip and half-space of the substrate it is much less than the maximum. This is due to the natural screening of the electric field in the dielectric and metal, which occurs due to the occurrence of induced charges. The greater the dielectric constant, the stronger the shielding. An important property of the resulting field distribution is that its characteristic size in normalized coordinates is the same and is equal in order of magnitude to the tip vertex radius. That is, at nanosized vertices and the focal field distribution in the photoresist dielectric film will be nanosized. The obtained solution theoretically rigorously proves the fundamental possibility of nanolithography based on the polymerization of a photoresist at the focus of the described excited metal tip, and with nanoscale accuracy.

The problem with two films was considered. The paraboloidal tip, as in previous calculations, was in a medium with $\varepsilon_f = 1$ next to a flat-layered structure consisting of two films. The first film, as in the previous example, had a thickness $\tilde{h}_1 = 0.5$ and a permittivity $\varepsilon_3 = \varepsilon_p = 2$. The second film had thickness $\tilde{h}_2 = 0.5$ and permittivity $\varepsilon_4 = \varepsilon_m$ (gold film). The half-space behind the second film had a permittivity $\varepsilon_5 = \varepsilon_d = 4$. The film boundaries were determined by the equations $\tilde{z} = \tilde{z}_2 = \tilde{z}_b = 1$, $\tilde{z} = \tilde{z}_3 = \tilde{z}_b + \tilde{h}_1 = 1.5$, and $\tilde{z} = \tilde{z}_4 = \tilde{z}_b + \tilde{h}_1 + \tilde{h}_2 = 2$. The vertex of the metal tip was located, as in the previous problem, at a distance of $\Delta\tilde{z} = 0.5$ from the film surface. The system of equations was solved by the method described in the previous section for two films. The Figure 7, b, c shows the results of calculations of the normalized (by one at the maximum at the tip vertex) distribution of the field amplitude in the $(\tilde{x}, \tilde{z})$ plane. It can be seen from the figures that qualitatively the conclusions made for the previous problem are preserved. The field does not penetrate well into a metal with a large modulus of permittivity. It is important that the field has a significant level in the dielectric film with the permittivity $\varepsilon_p = 2$, which models the photoresist layer. A thin metal film helps to localize the field in the dielectric film and can be used for auxiliary, technological purposes, for sharper focusing of the field in the photoresist.

Conclusion

The study proposes an original matrix technique for finding the fundamental solution of the Laplace operator for plane-layered media using the example of an electrostatic problem. A generalization of the specular reflection method for plane-layered media is proposed in a formulation that allows one to use the advantage of the proposed matrix technique. The main features of the developed method are logical simplicity and the possibility of generalizing the solution to multifilm structures and, in the limit, to gradient films.

The application of the method for finding the focal distribution of the electric field in the vicinity of the nanovertex of a metal micropoint located next to one photoresist film allows to describe the field penetration into the film and, thus, to solve practical problems of nanolithography. For the first time, the problem of finding the field distribution at the vertex of a metal tip located near two films (dielectric and metal) demonstrated the application possibilities of the proposed approaches.

References

[1] F. De Angelis, G. Das, P. Candeloro et al. Nature Nanotech., **5**, 67 (2010). DOI: 10.1038/nnano.2009.348
[2] H.G. Frey, F. Keilmann, A. Kriele, R. Guckenberger. Appl. Phys. Lett., **81**, 5030 (2002). DOI: 10.1063/1.1530736
[3] I. Stockman. Phys. Rev. Lett., **93**, 137404 (2004). DOI: 10.1103/PhysRevLett.93.137404
[4] A.B. Petrin. High Temp., **50**, 15 (2012). DOI: 10.1134/S0018151X12010129.
[5] A. Giugni, M. Allione, B. Torre et al. J. Opt., **16** (11), 114003 (2014). DOI: 10.1088/2040-8978/16/11/114003
[6] A. Giugni, B. Torre, A. Toma et al. Nature Nanotech., **8** (11), 845 (2013). DOI: 10.1038/nnano.2013.207
[7] A.B. Petrin. Uspekhi prikladnoi fiziki,**3** (3), 236 (2015). (in Russian),
[8] A.B. Petrin. Quantum Electronics, **45** (7), 658 (2015). DOI: 10.1070/QE2015v045n07ABEH015713.
[9] W.C. Chew. Waves and Fields in Inhomogeneous Media (IEEE Press, New York, 1995).
[10] A.G. Kyurkchan, S.A. Manenkov. Radiotekhnika i elektronika. (in Russian), **65** (7), 644 (2020).
[11] A.B. Petrin. Opt. Spectrosc., **128**, 1809 (2020). DOI: 10.1134/S0030400X20110193.
[12] A.B. Petrin. Opt. i spektr., **128** (12), 1874 (2020). (in Russian). DOI: 10.21883/EOS.2022.09.54838.3636-22
[13] A.B. Petrin. ZhETF, **159** (1), 35 (2021) (in Russian). DOI: 10.31857/S004445102101003X [A. B. Petrin. J. Exp. Theor. Phys., **132**, 27 (2021). DOI: 10.1134/S1063776120120055].
[14] R.W.P. King, G.S. Smith. Antennas in Matter (M.I.T. Press, Cambridge MA, 1981).
[15] R.W.P. King. IEEE Trans. Microwave Theory Tech., **36** (6), 1080 (1988). DOI: 10.1109/22.3635
[16] A.B. Petrin. Opt. Spectrosc., **129**, 72 (2021). DOI: 10.1134/S0030400X21010161.
[17] D.V. Sivukhin. *Obshchiy kurs fiziki, III „Elektrichestvo“* (Nauka, Moskva, 1977). (in Russian).
[18] A.B. Petrin. Uspekhi prikladnoi fiziki, **4** (4), 326 (2016). (in Russian).
[19] A.B. Petrin. High Temperature, **57** (1), 17 (2019). DOI: 10.1134%2FS0018151X1901019X.
[20] A.B. Petrin. Quantum Electronics, **45** (7), 658 (2015). DOI: 10.1070/QE2015v045n07ABEH015713.
[21] A.B. Petrin. Prikladnaya fizika, № 1, 11 (2016). (in Russian).
[22] A. Ango. Matematika dlya elektro- i radioinzhenerov (Nauka, Moskva, 1967).(in Russian).
[23] A.F. Nikiforov, V.B. Uvarov. *Spetsialnyye funktsii matematicheskoy fiziki* (Nauka. Glavnaya redakciya fiziko-matematicheskoy literatury, Moskva, 1984). (in Russian).
[24] *Chislennye metody teorii difrakcii (Matematika. Novoe v zarubezhnoy nauke. Vyp. 29): Sbornik statei.* Per. s angl., (Mir, Moskva, 1982).(in Russian).
[25] L. Novotny, B. Kheht.*Osnovy nanooptiki, pod red. Samarceva V.V.* (Fizmatlit, Moskva, 2009).(in Russian).