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Radiation from a Moving Bunch of Particles with a Variable Charge

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Received March 17, 2022
Revised May 26, 2022
Accepted May 28, 2022

We study the electromagnetic radiation of a charged particle bunch with small size moving at a constant velocity and having a variable charge. The environment medium is considered to be isotropic and homogeneous, and it may have frequency dispersion, but not spatial dispersion. The general solution of the problem is obtained. The main attention is paid to the case when the bunch charge, starting from a certain moment, decreases exponentially with time. The saddle point method is used to obtain the approximate expressions for the field components that are valid in the wave zone. The energy characteristics of the excited spherical wave are studied and compared with the case of a decelerating charge. In the case of excitation of Vavilov–Cherenkov radiation, we obtain the asymptotics which are valid in the entire wave zone, including the region in which the field cannot be divided into the spherical and cylindrical waves.

Keywords: Charged particle bunch, spherical wave, cylindrical wave, Vavilov–Cherenkov radiation.

DOI: 10.21883/TP.2022.09.54687.55-22

Introduction

A huge number of works are devoted to the problems of electromagnetic radiation of charged particles and their beams (bunches) in material media, among which are monographs [1–5], reviews [6,7] and textbooks [8]. Usually in such problems it is assumed that the number of particles in the beam does not change during its motion. An exception is the problem of radiation in dielectric waveguide structures, when the beam moves in a vacuum channel in a medium: in such a situation, the dynamics of the beam is often taken into account, associated with the effect of the radiation of the beam particles on them [9].

However, if the beam moves directly through the medium, then its particles interact with the particles of the medium, which leads to certain changes in the beam. In more or less dense media, this interaction is the main mechanism that determines the beam evolution. Various variants of this evolution have been described in many monographs and articles (see [10–14] and references therein). Depending on the mass of the particles, their energy, and the density of the medium, both a rapid deviation of particles from a rectilinear trajectory, leading to beam scattering, and an almost uniform motion of a relatively stable beam over most of the trajectory, followed by the deceleration of most of its particles over a relatively short segment of it, are possible. The last variant is typical for beams of protons and ions, decelerating mainly in the region of the known „Bragg peak“ [10–12]. Because of this feature, beams of heavy particles have found wide application in medicine (proton and ion therapy [10-13]).

As the beam moves through the medium, the number of particles in it changes. As a rule, it decreases: the particles of the beam are decelerated, actually turning from moving to stationary. It is of considerable interest to study the radiation of such a beam as a whole. It will be generated at wavelengths exceeding the size of the beam itself (for real beams, we are usually talking about wavelengths of about a millimeter or more). Of course, this radiation can be considered as bremsstrahlung of individual beam particles. However, from the point of view of macroscopic electrodynamics, it seems more natural from the very beginning to consider the beam as a whole, setting one or another law of its evolution.

The present work is devoted to studying the radiation of such a bunch of charged particles with a variable charge. The main attention will be paid to the case when the bunch charge was constant up to a certain moment of time, and then its exponential decrease occurs. In this case, to simplify the analysis, the environment will be considered homogeneous and unlimited. Such a model looks natural for a beam of heavy particles, since after entering the medium it moves almost without changes for quite a long time (the region of the Bragg peak is quite far from the boundary of the medium). For a beam of electrons that quickly lose energy after entering the medium, such a model can be implemented if the beam first moves in a vacuum channel in the medium, and at its end flies into the medium. If the channel diameter is small compared to the wavelengths under consideration, then the channel will not have a significant effect on the radiation [7] (with respect to such waves, the medium can be considered homogeneous everywhere).

The described model problem makes it possible to analyze the effects associated with a change in the beam charge, as well as the Vavilov–Cherenkov effect. In this case, we digress from the transition radiation that arises at the interface between media. Therefore, we can analyze the radiation associated with a change in the beam charge in its „pure" form: this radiation is unique if the beam velocity is less than the velocity of the waves in the medium (i.e., there is no Vavilov–Cherenkov radiation). However, initially we will not rule out the possibility of generating the Cherenkov radiation, and then we will consider two modes of charge motion — without this radiation and with it.

1. General solution of the problem

We will analyze radiation with wavelengths significantly exceeding the particle bunch size. In this case, it can be considered as a point charge, the value of which q depends on time. The charge velocity $\mathbf{v}$ will be considered constant. To satisfy the continuity equation $\operatorname{div}\mathbf{j} + \frac{\partial\rho}{\partial t} = 0$, it is necessary to introduce one more „additional" immobile source with a charge density ρ_1, which is a „trace" of the immobile charge. Combining the z axis with the bunch motion line, the total charge ρ_Σ and current $\mathbf{j}_\Sigma$ densities can be written as follows:

$$\begin{aligned} \rho_\Sigma &= \rho + \rho_1, \\ \rho &= q(t)\delta(x, y, z - vt), \\ \rho_1 &= -\frac{dq(t')}{v dt'}\Big|_{t'=z/v} \delta(x, y)\Theta(vt - z) \\ &= -\frac{dq(z/v)}{dz}\delta(x, y)\Theta(vt - z), \\ \mathbf{j}_\Sigma &= \mathbf{j} = v\rho\mathbf{e}_z. \end{aligned} \tag{1}$$

By substituting these expressions into the continuity equation, one can easily verify that it turns into an identity.

Physically, the formation of the „trace" means that the particles of the beam stop due to the interaction with the particles of the medium, i.e. from moving ones turn into stationary ones (as a result $\mathbf{j}_1 = 0$). In this case, from the point of view of macroscopic electrodynamics, a detailed description of this process is of no importance. For example, this can be the recombination of beam electrons with ions of the surrounding plasma, the stopping of particles due to collisions with neutral molecules, etc. What is important is the very fact of the formation of a filamentous charge in space, „additional" in relation to those charges that existed in the medium earlier (if any).

We consider the environment to be linear, homogeneous, stationary, isotropic and not having significant spatial dispersion (however, it may have frequency (temporal) dispersion). Let us recall some properties of such media. Such a medium is characterized by frequency-dependent ω dielectric (ε) and magnetic (μ) permeabilities, and its refractive index is equal to $n = \sqrt{\varepsilon\mu}$, and we will assume that $\operatorname{Im} n > 0$. Let us assume that in the range of propagating waves the real parts of both permeabilities are positive: $\varepsilon' > 0$, $\mu' > 0$ (thereby we exclude from consideration the so-called „left" media, for which these values are negative). We will also consider the environments to be "passive", i.e. incapable of generating electromagnetic energy. In such media, the signs of the imaginary parts of the permeabilities ε'', μ'' coincide with the sign of the frequency, and accordingly $\operatorname{sgn}(\operatorname{Im} n^2) = \operatorname{sgn}(\varepsilon'\mu'' + \varepsilon''\mu') = \operatorname{sgn}\omega$. In the case of relatively small absorption (and this is the situation we are interested in), we have $\operatorname{Re} n^2 = \varepsilon'\mu' - \varepsilon''\mu'' \approx \varepsilon'\mu' > 0$. Taking into account the imposed condition $\operatorname{Im} n > 0$, we see that the quantity n is either in the first or second quadrants of the complex plane, depending on the sign of the frequency, i.e. $\operatorname{sgn}(\operatorname{Re} n) = \operatorname{sgn}\omega$. Finally, we will be mainly interested in the case of negligible absorption when $\varepsilon'', \mu'' \to +0 \cdot \operatorname{sgn}\omega$, $n \to \operatorname{Re} n$, $\operatorname{sgn} n = \operatorname{sgn}\omega$.

When solving the problem, time Fourier transforms will be applied in the form

$$F_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{i\omega t}dt,\ F(t) = \int_{-\infty}^{\infty} F_\omega e^{i\omega t}dt. \tag{2}$$

Since for real functions on the real frequency axis the relation $F_{-\omega} = F_\omega^*$ is true (the asterisk means complex conjugation), we will further consider only positive frequencies $\omega > 0$. The corresponding spatial Fourier transforms have the form

$$\begin{aligned} F_{\omega,\mathbf{k}} &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} F_\omega(\mathbf{r})e^{-i\mathbf{kr}}d^3r, \\ F_\omega(\mathbf{r}) &= \int_{\mathbb{R}^3} F_{\omega,\mathbf{k}}e^{i\mathbf{kr}}d^3k. \end{aligned} \tag{3}$$

Let us use the vector $\mathbf{A}$ and scalar Φ potentials, in terms of which the field components are expressed by the formulas $\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi$, $\mathbf{B} = \operatorname{rot}\mathbf{A}$ (Gaussian units are used). When applying the Lorentz gauge, the time Fourier transforms of the potentials obey the Helmholtz equation:

$$(\Delta + k^2)\begin{Bmatrix}\mathbf{A}_\omega \\ \Phi_\omega\end{Bmatrix} = -4\pi\begin{Bmatrix}c^{-1}\ \mu\mathbf{j}_\omega \\ \varepsilon^{-1}\ \rho_{\Sigma\omega}\end{Bmatrix}, \tag{4}$$

where $k = \omega n/c$.

We will solve equation (4) by the Fourier method. For the space-time Fourier transforms of the charge and current densities, we have

$$\rho_{\Sigma\omega,\mathbf{k}} = \frac{1}{(2\pi)^3}\frac{vk_z}{\omega + i0}q_\Omega, \quad \mathbf{j}_{\omega,\mathbf{k}} = \frac{1}{(2\pi)^3}vq_\Omega\mathbf{e}_z, \tag{5}$$

where $q_\Omega = \frac{1}{2\pi}\int q(t)e^{i\Omega t}dt$, $\Omega = \omega - vk_z$. Note that the term „$+i0$" in the denominator provides the necessary detour around the pole, which gives expression (1) for ρ_Σ. Writing the potentials $\mathbf{A}_\omega$, Φ_ω as inverse Fourier integrals, substituting them into equations (4) and equating the integrands, we obtain fourfold Fourier transforms of the potentials. After passing to a cylindrical coordinate system both in the physical space (r, φ, z) and in the space of wave vectors (k_r, φ_k, k_z), for the time Fourier -images of potentials we get

$$\begin{Bmatrix}\mathbf{A}_\omega\\ \Phi_\omega\end{Bmatrix} = \frac{1}{2\pi^2}\int\limits_{-\infty}^{\infty} dk_z \int\limits_0^{\infty} dk_r \int\limits_0^{2\pi} d\varphi_k \begin{Bmatrix}\mu\boldsymbol{\beta}\\ \frac{k_z v}{(\omega+i0)\varepsilon}\end{Bmatrix} \times \frac{k_r q_\Omega \exp(ik_z z + ik_r r\cos(\varphi_k-\varphi))}{k_r^2+k_z^2-k^2}. \quad (6)$$

The integral over φ_k is [15]:

$$\int\limits_0^{2\pi} \exp(ik_r r\cos(\varphi_k-\varphi))d\varphi_k = 2\pi J_0(k_r r),$$

where $J_0(\xi)$ — is the Bessel function. To integrate over k_r, we first need to reduce the integral over a semi-infinite loop to an integral over an infinite loop:

$$\int\limits_0^{\infty} \frac{k_r J_0(k_r r)}{k_r^2+k_z^2-k^2}dk_r = \frac{1}{2}\int\limits_{e^{i\pi}\infty}^{\infty} \frac{k_r H_0^{(1)}(k_r r)}{k_r^2-\kappa^2}dk_r, \quad (7)$$

where $H_0^{(1)}(\xi)$ — is the Hankel function, and

$$\kappa = \sqrt{k^2-k_z^2} = \sqrt{\omega^2c^{-2}(\varepsilon'\mu'-\varepsilon''\mu'')-k_z^2+i\omega^2c^{-2}(\varepsilon'\mu''+\varepsilon''\mu')}. \quad (8)$$

For what follows, it is convenient to define the radical (8) by the rule $\mathrm{Im}\kappa > 0$. Taking into account that in the range of propagating waves $\varepsilon' > 0$, $\mu' > 0$, $\varepsilon'\mu' - \varepsilon''\mu'' > 0$, and $\mathrm{sgn}\varepsilon'' = \mathrm{sgn}\mu'' = \mathrm{sgn}\omega$, we see that the value under the radical in (8) is located in the first (for $\omega > 0$) or fourth (for $\omega < 0$) quadrants of the complex plane. So the requirement $\mathrm{Im}\kappa > 0$ results in $\mathrm{sgn}\,\mathrm{Re}\kappa = \mathrm{sgn}\omega$. This rule also determines the sign κ in the limiting case of a nonabsorbing medium.

Since $r > 0$, the integration contour in (7) can be completed to a closed semicircle of infinite radius located in the region $\mathrm{Im}\kappa > 0$. After that, the integral (7) is easily calculated by calculating the residue at the only pole $k_r = \kappa$. As a result, we obtain the following expressions for the Fourier transforms of the potentials:

$$\begin{Bmatrix}\mathbf{A}_\omega\\ \Phi_\omega\end{Bmatrix} = \frac{i}{2}\int\limits_{-\infty}^{\infty} \begin{Bmatrix}\mu\boldsymbol{\beta}\\ \frac{k_z v}{(\omega+i0)\varepsilon}\end{Bmatrix} q_\Omega H_0^{(1)}(\kappa r)e^{ik_z z}dk_z. \quad (9)$$

Calculating the Fourier transforms of the field components, we find

$$E_{r\omega} = \frac{iv}{2}\int\limits_{-\infty}^{+\infty} q_\Omega \frac{k_z\kappa}{(\omega+i0)\varepsilon} H_1^{(1)}(\kappa r)e^{ik_z z}dk_z,$$

$$E_{z\omega} = \frac{1}{2c}\int\limits_{-\infty}^{+\infty} q_\Omega \frac{vck_z^2-\omega^2n^2\beta}{(\omega+i0)\varepsilon} H_0^{(1)}(\kappa r)e^{ik_z z}dk_z,$$

$$H_{\varphi\omega} = \frac{i\beta}{2}\int\limits_{-\infty}^{+\infty} q_\Omega \kappa H_1^{(1)}(\kappa r)e^{ik_z z}dk_z. \quad (10)$$

Note that the branch point $k_z = k$ of the function $\kappa(k_z) = \sqrt{k^2-k_z^2}$ lies above the contour integration, and $k_z = -k$ — below. We emphasize that the obtained expressions are the components of the full field, i.e. the field of a source consisting of a moving variable charge and its filamentous „trace".

2. Asymptotic calculation of field components

We assume that at negative times the bunch charge was constant $(q = q_0)$, and starting from the moment $t = 0$ it decreases. The process of charge „melting" can occur according to different laws. We assume that for any small time interval the bunch loses the same fraction of charge, i.e. $dq/dt = q/\tau$, where $\tau = \text{const}$. Solving this equation gives

$$q(t) = \begin{cases} q_0, & t < 0\\ q_0 e^{-t/\tau}, & t \ge 0\end{cases} = q_0[\Theta(-t) + \Theta(t)e^{-t/\tau}], \quad (11)$$

where $\Theta(\xi)$ – is the Heaviside step function. Experimental data show that the exponential law of charge decrease is close to that which takes place, in particular, for proton and ion beams [10,11]. The Fourier transform of function (11) is equal to

$$q_\Omega = q_0\left(\delta(\Omega) - \frac{i}{2\pi(\Omega+i0)} + \frac{1}{2\pi}\frac{\tau}{1-i\Omega\tau}\right). \quad (12)$$

Let us calculate the integral using the example of the component E_r. Substituting (12) into (10), for the Fourier transform of $E_{r\omega}$ we obtain

$$E_{r\omega} = \frac{iq_0}{2v}\frac{s}{\varepsilon}H_1^{(1)}(sr)e^{i\omega z/v} + \frac{ivq_0}{2(\omega+i0)\varepsilon}I, \quad (13)$$

$$I = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} k_z\kappa\left(\frac{\tau}{1-i\Omega\tau} - \frac{i}{\Omega+i0}\right)H_1^{(1)}(\kappa r)e^{ik_z z}dk_z, \quad (14)$$

where $\Omega = \omega - vk_z$, $s = \sqrt{\omega^2v^{-2}(n^2\beta^2-1)}$ $(\mathrm{Im}s > 0)$. To analyze the integral (14), it is expedient to introduce a new integration variable χ: $k_z = k\cos\chi$, $\kappa = k\sin\chi$, as well as spherical coordinates R, θ $(r = R\sin\theta,\ z = R\cos\theta)$. In this case, the branch points in the integrand are eliminated, and the integral takes the form

$$I = \frac{k^3}{2\pi}\int\limits_C \left(\frac{i}{(\Omega+i0)} - \frac{\tau}{1-i\Omega\tau}\right)\cos\chi\sin^2\chi \times H_1^{(1)}(kR\sin\theta\sin\chi)\exp(ikR\cos\theta\cos\chi)d\chi, \quad (15)$$

where the contour C is shown in Fig. 1.

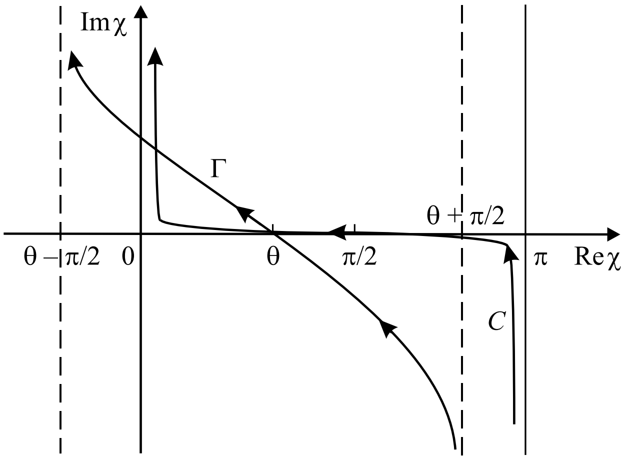

Figure 1. The original integration contour C and the fastest descent Γ contour on the plane χ for $\omega > 0$.

An approximate calculation of this integral in the wave zone ($|k|R \gg 1$) can be carried out using the saddle point method. The method of this calculation is standard [16], so we will focus only on the main points. First, it is advisable to transform the original integration contour to the fastest descent contour Γ (Fig. 1), after which, using the asymptotics of the Hankel function, we can write the integral in the form

$$I = \int_\Gamma f(\chi) e^{ikR\cos(\chi-\theta)} d\chi, \tag{16}$$

where $f(\chi) = \frac{k^3 \cos\chi \sin^2\chi}{\sqrt{2\pi^3 kR \sin\theta \sin\chi}} e^{-i3\pi/4} \left(\frac{i}{\Omega + i0} - \frac{\tau}{1 - i\Omega\tau}\right)$. The integrand has a saddle point $\chi = \theta$. When transforming the initial contour C to the contour Γ, the poles of the function $f(\chi)$ may intersect. The contribution of the pole, defined by the equation $i\Omega\tau = 1$, decreases exponentially with distance, i.e. is not a part of the radiation field (we do not take into account such contributions below). The pole defined by the equation $\Omega = 0$ is located at the point $\chi = \chi_0 = \arccos(1/(n\beta))$. For $n\beta < 1$ its contribution is exponentially small, but for $n\beta > 1$ it is significant. After an approximate calculation of the integral (16) by the saddle [16] method, taking into account the contribution of the pole χ_0, we obtain

$$I \approx \sqrt{\frac{2\pi}{kR\left|d^2\Phi(\chi)/d\chi^2|_{\chi=\theta}\right|}} f(\theta) e^{ikR + 3i\pi/4}$$

$$+ \frac{\omega s}{v^2}\sqrt{\frac{2}{\pi s r}}\Theta(\chi_0 - \theta)\Theta(n\beta - 1) e^{i\omega z/v + isr + i\pi/4}. \tag{17}$$

This result is true when the function $\exp(ikR\cos(\chi - \theta))$ on the steepest descent contour changes rapidly compared to the function $f(\chi)$. This condition is satisfied if the saddle point is far enough from the poles of the integrand, i.e. $|kR(\theta - \chi_0)| \gg 1$. By substituting (17) into (13) using the Hankel function asymptotics in (13), one can easily verify that the second term in (17) compensates the first term in (13) in the region $\theta < \chi_0$.

The Fourier transforms of the E_z and H_φ components are calculated similarly. As a result, we obtain a field in the form of a sum of spherical (I) and cylindrical (II) waves:

$$\mathbf{E} = \mathbf{E}^I + \mathbf{E}^{II}, \qquad \mathbf{H} = \mathbf{H}^I + \mathbf{H}^{II} \tag{18}$$

$$\begin{Bmatrix} E^I_{r\omega} \\ E^I_{z\omega} \\ H^I_{\varphi\omega} \end{Bmatrix} = \frac{q_0 k\beta \sin\theta}{2\pi} \begin{Bmatrix} -\sqrt{\mu/\varepsilon}\cos\theta \\ \sqrt{\mu/\varepsilon}\sin\theta \\ -1 \end{Bmatrix}$$

$$\times \left[\frac{1}{\omega(1 - n\beta\cos\theta)} + \frac{i\tau}{1 - i\omega(1 - n\beta\cos\theta)\tau}\right] \frac{e^{ikR}}{R}, \tag{19}$$

$$\begin{Bmatrix} E^{II}_{r\omega} \\ E^{II}_{z\omega} \\ H^{II}_{\varphi\omega} \end{Bmatrix} = \frac{q_0 s}{c} \begin{Bmatrix} (\beta\varepsilon)^{-1} \\ -cs(\omega\varepsilon)^{-1} \\ 1 \end{Bmatrix}$$

$$\times \frac{\exp(i\omega z/v + isr - i\pi/4)}{\sqrt{2\pi s r}} \Theta(\theta - \chi_0)\Theta(n\beta - 1). \tag{20}$$

The spherical wave (19) is due to the process of reducing the charge of the bunch and the simultaneous formation of a „trace“. The cylindrical wave — is the Vavilov−Cherenkov radiation. As expected, it exists only when the charge velocity exceeds the phase velocity of the waves in the medium, i.e. provided $v > c/n$, or $n\beta > 1$. According to „non-uniform“ asymptotics, this wave exists only in the region $\theta > \chi_0$ (Fig. 2). However, it should be kept in mind that for $|kR(\theta - \chi_0)| \sim 1$ the obtained asymptotics are inapplicable — in this region, the separation of the wave field into cylindrical and spherical waves is impossible. More precise („uniform“) asymptotics allow us to describe the behavior of the field in this transition region as well (see Sec. 4).

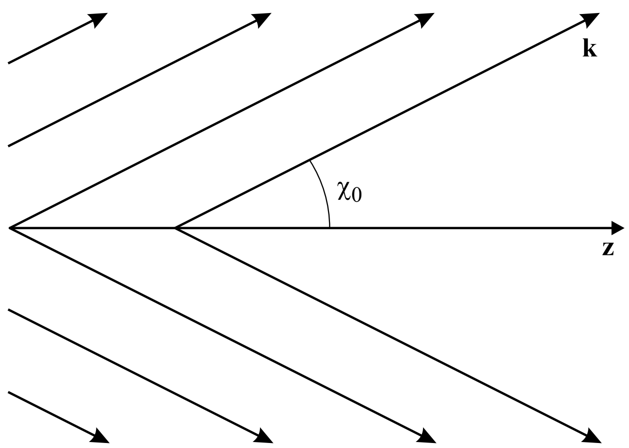

Figure 2. Region of existence of the Vavilov−Cherenkov radiation.

Let us briefly talk about the role of the two parts that make up the source: it is the point charge itself with densities ρ $\mathbf{j}$, and its „trace“ with densities ρ_1, $\mathbf{j}_1 = 0$. First of all, we note that the division of the field into two contributions from these parts does not make much physical sense. The reason for this is that the law of conservation of charge holds only for the total source, but not separately for the point charge and „trace“. Therefore, if we consider such sources separately, then the system of Maxwell's equations becomes unsolvable (the number of independent equations exceeds the number of unknowns).

Nevertheless, one can formally single out in strict expressions (10) and in asymptotics (18)$-$(20) the contributions of the charge itself and its „trace“. Analysis shows that in the electric field of a spherical wave (19) the charge and „trace“ make comparable contributions (despite the fact that „trace“ is not a point object). This is explained by the fact that the rate of change of the charge and its „trace“ is determined by the same parameter τ. In particular, at $\omega\tau \gg 1$ both components of the electric field are proportional to $(\omega\tau)^{-1}$ (like the entire field of the spherical wave (19)). At the same time, „trace“ does not give any addition to the magnetic field, which is natural due to the absence of charge movement in it. We emphasize that the „correct“ asymptotics (19) (transverse spherical wave) is obtained only when both parts of the source are taken into account. As for the cylindrical wave (20), it can be shown that „trace“ does not affect it (physically, this is also due to the absence of movement of charges of the „trace“).

3. Spherical wave

In spherical coordinates R, θ, φ the spherical wave (19) has only two nonzero components:

$$E^I_{\theta\omega} = \sqrt{\frac{\mu}{\varepsilon}} H^I_{\varphi\omega} = -\frac{q_0\mu}{2\pi c}\frac{\exp(ikR)}{R} F(\beta, \theta, \omega), \quad (21)$$

$$F(\beta, \theta, \omega) = \frac{\beta \sin\theta}{(1 - n\beta\cos\theta)[1 - i\omega\tau(1 - n\beta\cos\theta)]}. \quad (22)$$

As expected, we have obtained a transverse wave, which is „quasi-plane“, since the radius of curvature R of the constant phase surface is much greater than the wavelength. The factor $F(\beta, \theta, \omega)$ determines the dependence of the spherical wave amplitude on the bunch velocity, the observation point angle, and the considered frequency.

The angular distribution of the energy of a spherical wave can be written as

$$\frac{dW}{d\Omega} = R^2 \frac{c}{4\pi} \int_{-\infty}^{\infty} E^I_\theta H^I_\varphi dt$$

$$= R^2 \frac{c}{4\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} E^I_{\theta\omega} e^{-i\omega t} d\omega \int_{-\infty}^{\infty} H^I_{\varphi\omega'} e^{-i\omega' t} d\omega'$$

$$= cR^2 \int_0^{\infty} E^I_{\theta\omega} H^{I^*}_{\varphi\omega} d\omega. \quad (23)$$

As we can see, the spectral-angular density of radiation energy is determined by the expression

$$\frac{d^2W}{d\Omega d\omega} = cR^2 E^I_{\theta\omega} H^{I^*}_{\varphi\omega} = cR^2 \sqrt{\frac{\varepsilon}{\mu}} |E^I_{\theta\omega}|^2 = \frac{q_0^2}{4\pi^2 c}\sqrt{\varepsilon\mu^3}|F|^2, \quad (24)$$

$$|F|^2 = \frac{\beta^2 \sin^2\theta}{(1 - n\beta\cos\theta)^2[1 + \omega^2\tau^2(1 - n\beta\cos\theta)^2]}. \quad (25)$$

If $\varepsilon = \mu = n = 1$ and $\tau = 0$, i.e. particle moves in vacuum and instantly loses all charge, then the spectral-angular density of radiation energy is equal to

$$\frac{d^2W}{d\Omega d\omega} = \frac{q_0^2}{4\pi^2 c} \frac{\beta^2 \sin^2\theta}{(1 - \beta\cos\theta)^2}. \quad (26)$$

This expression coincides with the spectral-angular energy density of the radiation of an instantly decelerating charge in vacuum (it can be obtained from the general expression for an arbitrarily moving charge given, for example, in [17]). Thus, the case of an instantaneously vanishing charge is equivalent to the case of an instantaneously decelerating charge, which is quite natural.

The total angular density of radiation energy is obtained by integrating expression (24) over frequency. If the frequency dispersion of the medium is neglected (i.e., ε and μ are assumed to be frequency-independent), then it is easy to obtain the following result:

$$\frac{dW}{d\Omega} = \int_0^{\infty} \frac{d^2W}{d\Omega d\omega} d\omega = \frac{q_0^2}{8\pi c\tau}\sqrt{\varepsilon\mu^3} F_0,$$

$$F_0 = \frac{\beta^2 \sin^2\theta}{\left|1 - n\beta\cos\theta\right|^3}. \quad (27)$$

Examples of dependences of $|F|$ and F_0 on angles for different values n and β are shown in Fig. 3 (on a logarithmic scale). The left graphs correspond to the case of vacuum, and the right graphs correspond to the case of a medium for which $n = 2$. As the charge velocity increases from zero to c/n, the maxima of both quantities increase and shift towards smaller angles, tending to $\theta = 0$ at $v \to c/n$. In the case of $v > c/n$ for $\theta \to \chi_0$, the quantities $|F|$ and

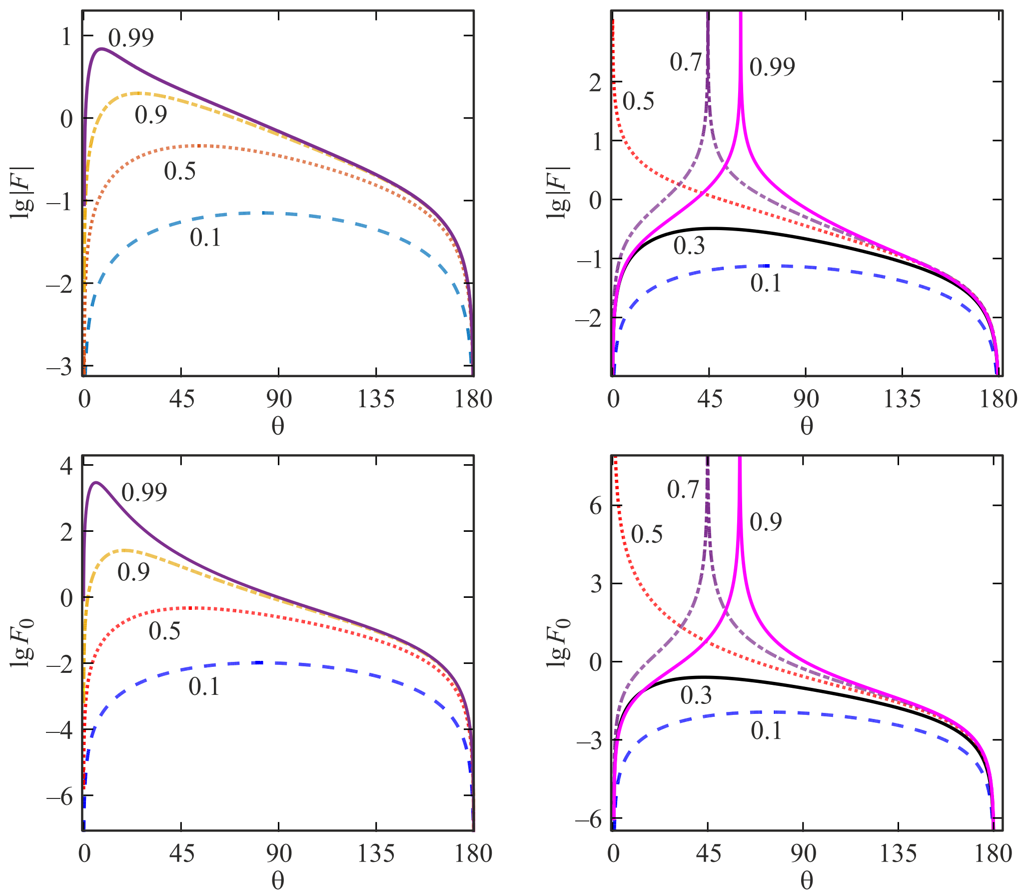

Figure 3. Dependence of $\lg|F|$ at $\omega\tau = 1$ (top) and $\lg F_0$ (bottom) on the angle θ (deg). On the left — the case of vacuum, on the right — the case of a medium with permeabilities $\varepsilon = 4$, $\mu = 1$. Values β are indicated near the curves.

F_0 increase with the angle θ until $\theta < \chi_0$, and decrease at $\theta > \chi_0$. For $\theta \to \chi_0$, both quantities formally tend to infinity, but this is due to the inapplicability of the asymptotics (18)–(20) in the vicinity of the angle $\theta = \chi_0$.

It is interesting to compare the radiation in the case under consideration with the radiation of a charge whose magnitude does not change, but it is decelerated according to an exponential law with the same characteristic time τ. The speed of this charge is

$$\beta_b(t) = \begin{cases} \beta, & t < 0, \\ \beta e^{-t/\tau}, & t \geq 0. \end{cases} \tag{28}$$

Let us make such a comparison for the case when the electrodynamic characteristics of the medium are practically indistinguishable from the vacuum ones ($\varepsilon = \mu = 1$). In the case of a charge decelerating in vacuum, the angular density of radiation power is determined by the formula [17]

$$\frac{dP_b}{d\Omega} = \frac{d^2W_b}{d\Omega dt} = \frac{q^2}{4\pi c}\left(\frac{d\beta_b}{dt}\right)^2 \frac{\sin^2\theta}{(1-\beta_b\cos\theta)^6}. \tag{29}$$

Integrating this expression over time, taking into account (28), we obtain the total angular density of radiation energy:

$$\frac{dW_b}{d\Omega} = \frac{q^2}{8\pi c\tau}\tilde{F}_0,$$

$$\tilde{F}_0 = \frac{\beta^2\sin^2\theta}{(1-\beta\cos\theta)^4}\left(1 + \frac{1}{10}\beta^2\cos^2\theta\frac{5-\beta\cos\theta}{1-\beta\cos\theta}\right). \tag{30}$$

Examples of the dependences of $\lg\tilde{F}_0$ and $\lg F_0$ on the angle θ at different speeds are shown in Fig. 4. At a nonrelativistic velocity ($\beta \ll 1$), the contribution to (30) practically makes only the first term, and in this case the exponentially decelerating and exponentially decreasing charges radiate approximately the same way. At $\beta \sim 1$, due to the second term in (30) and a different degree of brackets in the denominator, the radiation of the decelerating charge significantly exceeds the radiation of the decreasing charge at acute angles θ. At $\beta \approx 1$, the maximum for the decelerating charge is much larger than the maximum for the decreasing charge.

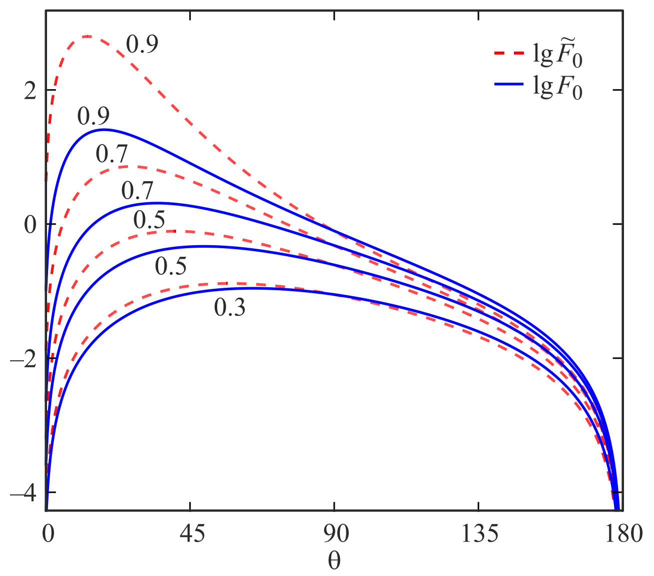

Figure 4. Dependence of $\lg F_0$ (solid curves) and $\lg \tilde{F}_0$ (dashed curves) on the angle θ at different velocities (shown on the graph) in vacuum.

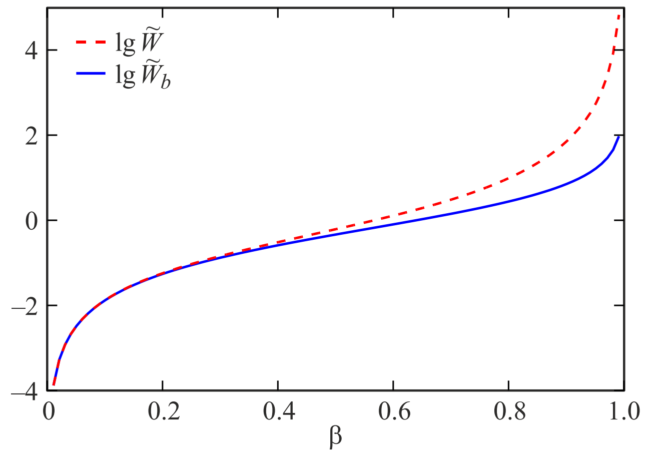

Figure 5. Dependence of $\lg \tilde{W}$ (solid curve) and $\lg \tilde{W}_b$ (dashed curve) on the velocity β in vacuum.

The total radiation energy is obtained by integrating the angular energy density:

$$W = \int_0^{2\pi} d\varphi \int_0^{\pi} \frac{dW}{d\Omega} \sin\theta d\theta.$$

For an exponentially decreasing charge in vacuum, it is equal to

$$W = \frac{q_0^2}{4c\tau}\tilde{W}, \quad \tilde{W} = \frac{2}{1-\beta^2} + \frac{1}{\beta}\ln\frac{1-\beta}{1+\beta}. \tag{31}$$

For a charge exponentially decelerating in vacuum, it is easy to obtain

$$W_b = \frac{q^2}{4c\tau}\tilde{W}_b, \quad \tilde{W}_b = \frac{4\beta^2}{3(1-\beta^2)}\left(1 + \frac{\beta^2}{10}\frac{1+3\beta^2}{1-\beta^2}\right). \tag{32}$$

The velocity dependences of $\lg \tilde{W}$ and $\lg \tilde{W}_b$ are shown in Fig. 5. As we can see, at a nonrelativistic velocity ($\beta \ll 1$), the decelerating and decreasing charges radiate approximately the same way: $W_b \approx W \approx \frac{q^2\beta^2}{3c\tau}$. In the ultrarelativistic case, when $\gamma \equiv (1-\beta^2)^{-1/2} \gg 1$, with an increase of γ the total radiation energy grows proportionally to γ^2 for a decreasing charge and γ^6 for a decelerating one. Thus, in the case of a large Lorentz factor, the decelerating charge radiates much more efficiently than the decreasing one.

Concluding this section, we note that in the case of a bunch of particles of size a, the radiation will be comparable to the radiation of a point charge if $ka \leq 1$, while for $ka \gg 1$ it will be much weaker. At the same time, according to (25), the range of radiation frequencies can be estimated by the inequality $\omega\tau \leq 1$. Thus, for real bunches, the radiation frequency range is bounded from above both due to the finiteness of the charge change time and due to the finiteness of its size: $\omega \leq \min(1/\tau, 1/\tau_a)$, where $\tau_a = na/c$ — the time during which the radiation travels the distance a. If $\tau > \tau_a$, then the charge can be approximately considered as a point charge in the entire significant range of radiation frequencies. Otherwise, the frequency range is limited due to the finite size of the bunch.

4. Field asymptotics valid for all viewing angles

To describe the behavior of the field in the entire wave zone $kR \gg 1$, including the range of angles $\theta \approx \chi_0$, one must use the uniform asymptotics of the integrals. This asymptotic behavior is valid for any location of the pole $\chi = \chi_0$ with respect to the saddle point $\chi = \theta$, including $\theta = \chi_0$. The coincidence of the pole with the saddle point corresponds to the conditional boundary of the region of the Cherenkov radiation, on which the non-uniform asymptotics (18)–(20) suffer a discontinuity. Uniform asymptotics correctly describe the field everywhere in the wave zone, including the given boundary itself and its neighborhood.

Without stopping on the transformations that are carried out according to the well-known procedure [16], we write down the final results valid for $kR \gg 1$:

$$E_{r\omega} = \frac{i}{2\varepsilon}I_\Gamma + \frac{q_0 s}{v\varepsilon}\frac{\exp(i\omega z/v + isr - i\pi/4)}{\sqrt{2\pi s r}}\Theta(\theta - \chi_0),$$

$$E_{z\omega} = \frac{I_\Gamma}{2\varepsilon} + \frac{q_0\omega(1-n^2\beta^2)}{c^2\varepsilon\beta^2}\frac{\exp(i\omega z/v + isr - i\pi/4)}{\sqrt{2\pi s r}}\Theta(\theta - \chi_0),$$

$$H_{\varphi\omega} = \frac{i}{2}I_\Gamma + \frac{q_0 s}{c}\frac{\exp(i\omega z/v + isr - i\pi/4)}{\sqrt{2\pi s r}}\Theta(\theta - \chi_0). \tag{33}$$

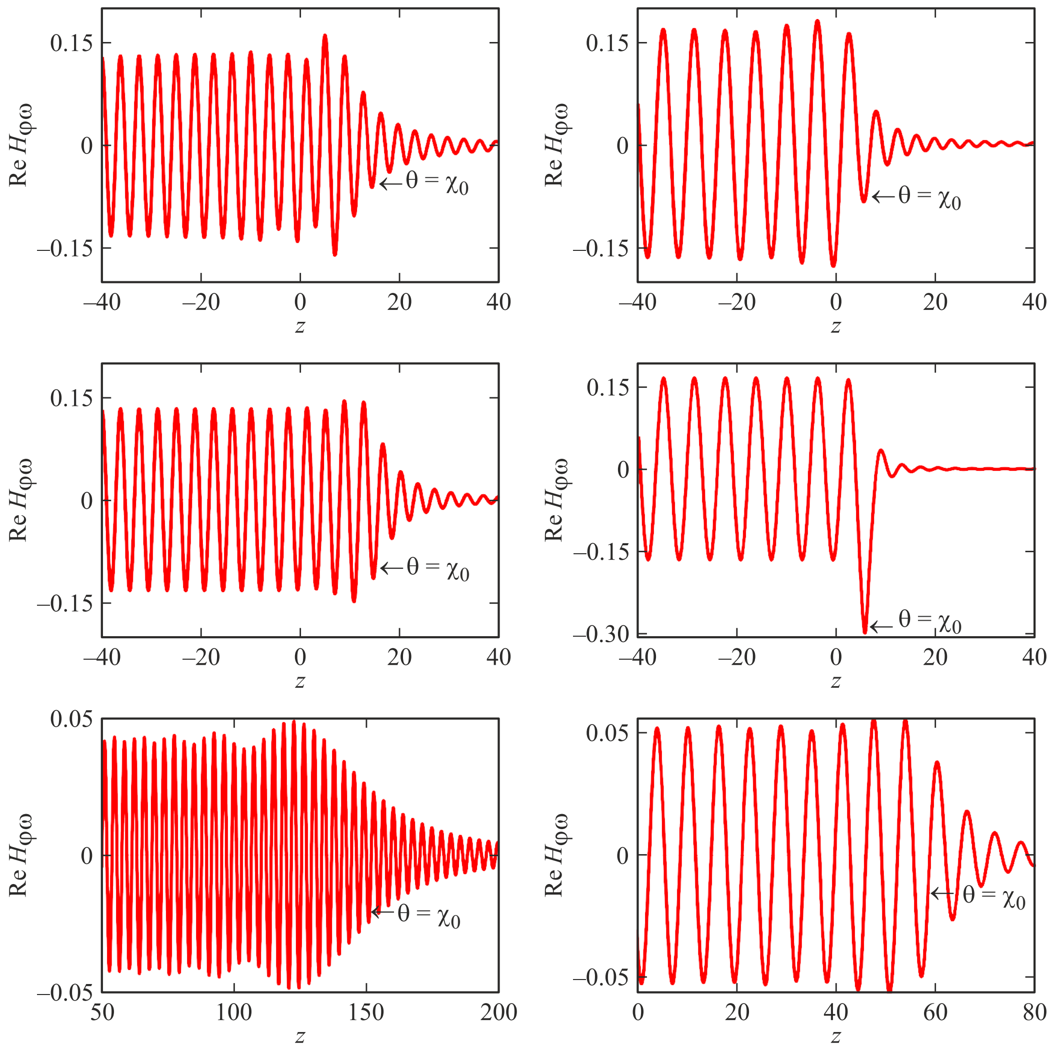

Figure 6. Dependence of the Fourier transform of the field $H_{\varphi\omega}$ (in units of $q_0\omega/c^2$) on z (in units of c/ω) for $\varepsilon = 4$, $\mu = 1$, $n = 2$. Left column: $\beta = 0.6$; right column: $\beta = 0.99$. Top row: $\tau = 0$, $r = 10c/\omega$; middle row: $\omega\tau = 10$, $r = 10c/\omega$; bottom row: $\omega\tau = 10$, $r = 100c/\omega$.

Here I_Γ is given by

$$I_\Gamma \approx 2ia\sqrt{\pi}\,\mathrm{sgn}(\mathrm{Im}b)Q\left(-\mathrm{sgn}(\mathrm{Im}b)ib\sqrt{kR}\right) \times \exp(ikR\cos(\chi_0-\theta)) + \sqrt{\frac{\pi}{kR}}\left(\sqrt{2}e^{3\pi i/4}f(\chi_s)+\frac{a}{b}\right)\exp(ikR), \tag{34}$$

where $b = e^{i\pi/4}\sqrt{2}\sin\left(\frac{\theta-\chi_0}{2}\right)$, $Q(y) = \int\limits_y^{+\infty} e^{-x^2}dx$, and the role of $f(\chi)$ is played by one of the following functions:

$$\begin{Bmatrix} f_r(\chi) \\ f_z(\chi) \\ f_h(\chi) \end{Bmatrix} = \frac{q_0\beta k^2}{\pi}\sqrt{\frac{\sin^3\chi}{2\pi kR\sin\theta}} \times \left[\frac{1}{i\omega(1-n\beta\cos\chi)} + \frac{\tau}{1-i\omega\tau(1-n\beta\cos\chi)}\right] \times \begin{Bmatrix} e^{i\pi/4}n\cos\chi \\ e^{-i\pi/4}n\sin\chi \\ e^{i\pi/4} \end{Bmatrix}. \tag{35}$$

Accordingly, as a it is necessary to substitute in (34) the amount of residue $a = a_{r,z,h} = \mathrm{Res}_{\chi=\chi_0} f_{r,z,h}(\chi)$:

$$\begin{Bmatrix} a_r \\ a_z \\ a_h \end{Bmatrix} = -\frac{iq_0\beta k\sqrt{\sin\chi_0}}{\pi v\sqrt{2\pi kR\sin\theta}} \begin{Bmatrix} e^{i\pi/4}n\cos\chi_0 \\ e^{-i\pi/4}n\sin\chi_0 \\ e^{i\pi/4} \end{Bmatrix}. \tag{36}$$

It is easy to show that under the condition $|b|\sqrt{kR} \gg 1$, i.e. $|\chi_0-\theta|\sqrt{kR} \gg 1$, the results give „non-uniform“ asymptotics (18)–(20).

In the case when the pole coincides with the saddle point ($\theta = \chi_0$), expressions (33) take the form

$$E_{r\omega} = -\frac{iv\tau q_0}{4\pi}\frac{k^2\sin(2\theta)}{\varepsilon\omega}\frac{e^{ikR}}{R} + \frac{q_0 s}{2\sqrt{2\pi}v\varepsilon}\frac{\exp(i\omega z/v + isr - i\pi/4)}{\sqrt{sr}},$$

$$E_{z\omega} = \frac{i\nu\tau q_0}{2\pi}\frac{k^2 \sin^2\theta}{\varepsilon\omega}\frac{e^{ikR}}{R} + \frac{q_0\omega(1-n^2\beta^2)}{2\sqrt{2\pi}c^2\varepsilon\beta^2}\frac{\exp(i\omega z/\nu + isr - i\pi/4)}{\sqrt{sr}},$$

$$H_{\varphi\omega} = -\frac{i\beta\tau q_0}{2\pi}k\sin\theta\frac{e^{ikR}}{R} + \frac{q_0 s}{2\sqrt{2\pi}c}\frac{\exp(i\omega z/\nu + isr - i\pi/4)}{\sqrt{sr}}. \quad (37)$$

As we see, for $\theta = \chi_0$ the amplitudes of the cylindrical wave components are equal to half the amplitudes of the Cherenkov radiation wave components (20). Thus, the use of uniform asymptotics provides a correct description of the transition zone between the region illuminated by the Vavilov−Cherenkov radiation and the region where it is absent. This transition zone has an angular width of the order of $\Delta\theta \sim 1/(kR)$.

Examples of the behavior of the Fourier transform of the full field H_φ are shown in Fig. 6. The left plot corresponds to the case $\beta = 0.6$, and the right one — to the case $\beta = 0.99$. The conditional boundary of the region of the Cherenkov radiation is shown at $\tau = 0$. For $\tau \neq 0$ this boundary shifts towards positive values of z. Recall that the origin of coordinates is located at the point where the bunch of particles begins to lose charge. Therefore, taking into account the decrease in the charge by e-times over a finite time τ, the boundary of the Cherenkov radiation region can be more accurately determined by the expression $\theta = \arctan\left(\frac{1}{1/\tan\chi_0 + \nu\tau/r}\right)$.

As expected, the field is a continuous function of the coordinates everywhere. Comparing the middle and lower figures (for $\beta = 0.6$ and $\beta = 0.99$), we see that if $z < 0$, then the field amplitude at $r = 100c/\omega$ is about 3 times less than at $r = 10c/\omega$ (in this region the main role is played by the Cherenkov radiation, whose amplitude decreases as $1/\sqrt{r}$). As z grows, starting from some value of z, the field decreases. In this case, the observation point falls into the transition zone between the region of the Cherenkov radiation and the region where it is insignificant. For sufficiently large values of z, practically only a spherical wave remains, decreasing as $R^{-1} = (r^2 + z^2)^{-1/2}$.

Conclusion

In this work, the electromagnetic field of a bunch of charged particles of small size moving at a constant speed and having a variable charge is studied. In doing so, it was taken into account that a filamentous „trail“, consisting of immobile charges, is formed behind the bunch. It was assumed that the environment is isotropic and homogeneous, and it may have frequency dispersion, but not spatial dispersion. The general solution of the problem is obtained. The main attention is paid to the case when the value of the charge, starting from a certain moment, decreases exponentially with time. The saddle point method is used to obtain asymptotic expressions for the field components that are valid in the wave zone.

For a spherical wave excited by a bunch, it is shown, in particular, that in vacuum the spectral-angular and angular radiation energy densities have a maximum at an acute angle, which decreases with increasing velocity. A comparison with the case of a decelerating charge is made. At a nonrelativistic velocity, the radiation energies of the decreasing and decelerating charges are approximately equal, and at an ultrarelativistic velocity, the radiation energy of the decelerating charge significantly exceeds the radiation energy of the decreasing charge.

In the case of charge motion in a medium, in addition to a spherical wave, a cylindrical wave (Cherenkov radiation) can be excited in a certain frequency range, which exists in a limited region of space. The division of the field into spherical and cylindrical waves is impossible in the vicinity of the conditional boundary of the Cherenkov radiation, where a correct description of the field is given by uniform asymptotics. Graphs of typical dependences of the Fourier transform of the field on the longitudinal coordinate are given for various parameters of the problem.

Funding

The study was supported by the Russian Science Foundation (project № 18-72-10137) and the Council of Scholarship Programs of China.

Conflict of interest

The authors declare that they have no conflict of interest.

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