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Some features of the solving of hydrodynamic equations for solitary waves in the open water channel

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The opportunity of use of an impulse equation special form for the solving of a problem of solitary waves (solitons) occurrence in the open water channel is considered. It is shown that the used of an impulse equation allows take into account a role of surface tension and gravitational forces in formation of waves. Using of the continuity equation expansion into series on Rayleigh's method the system of the differential equations is received, one of which is nonlinear. Application of Dalembert's method for running waves for the solving of the nonlinear differential equation in a hydrodynamic problem of solitary waves spreading in the open water channel is considered. It is shown that as against Dalembert's theory for the linear hyperbolic equations where initial conditions completely determine the form of arising waves, for the nonlinear equations the form of waves is determined by character of the equation nonlinearity. Thus during the solution of equations the sum of the functions describing linear waves extending in opposite directions, in the Dalembert's method for nonlinear waves is replaced with the sum of the nonlinear differential equations.

Keywords: soliton, open water channel, surface tension, gravitational forces, nonlinear differential equation, Dalembert's method.

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Introduction

Since the first observation of solitary waves in a water channel by D.S. Russell in 1834 who accompanied a solitary wave on a horse, many different approaches arose in the theory of describing such waves [1–3], including for various external conditions [4–6]. For example, simulation of solitary waves or solitons in [1] made it possible to describe waves propagating both from left to right and from right to left. However, if the wave propagates from left to right, then it is absolutely true that its description follows from the dynamic momentum equation, but if the wave propagates from right to left, its description in [1] follows from the kinematic continuity equation, which raises questions. The point is that the rise of liquid in a solitary wave is determined by the hydrostatic pressure at the base of the solitary wave. There is no force parameter in the continuity equation, therefore this equation from the physical point of view cannot describe a solitary wave. In our opinion, both oppositely propagating waves should appear based on the momentum equation. This is possible if the original momentum equation is written more correctly.

The studied solitary waves are called solitons, since their interaction resembles the interaction of particles. They can bounce off a solid boundary like particles. When interacting with each other, solitons diverge, keeping their structure unchanged. Such preservation of the structure is determined by the balance of nonlinear effects and dispersion at the leading and trailing edges of the solitons.

The derivation of the equation for a soliton on the surface of a water channel of a constant depth, the so-called Korteweg–de Vries equation, usually begins with an analysis of linear waves on water [3]. The obtained transcendental dispersion relation is replaced by a polynomial of the third order, which, together with the equations of hydrodynamics, leads to the Korteweg–de Vries equation.

A somewhat different derivation of soliton equations based on an asymptotic analysis of the original system of hydrodynamic equations is used in this paper.

1. The role of surface tension forces and gravitational forces in the generation of waves on the water surface of the channel

When solving hydrodynamic problems for an ideal fluid with open or elastic boundaries, it is often more correct to use the momentum equation in the form [7,8]:

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial X} + W\frac{\partial W}{\partial X} = -\frac{\partial (PS)}{\rho S \partial X}, \tag{1}$$

where ρ — fluid density, V and W — longitudinal and transverse (vertical) components of fluid velocity, X — longitudinal coordinate, t — time, S — flow cross section. Pressure in liquid P.

In particular, we will consider the flow of fluid in an open water channel (Fig. 1).

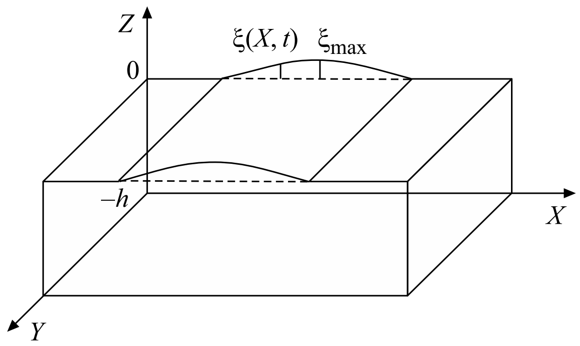

Figure 1. Formation of a solitary wave (soliton) in a water channel.

A change in the flow parameters is observed only along the X and Z axes. There is no fluid flow along the Y-axis. Let us assume that the width of the channel along the Y axis is equal to unity.

It is also assumed that no vortices are formed in the flow, the flow is potential [9], i.e. $\mathrm{rot}\,\mathbf{V} = 0$, where $\mathbf{V}$ — velocity vector, therefore,

$$\frac{\partial V}{\partial Z} = \frac{\partial W}{\partial X}.$$

1.1. Surface tension of a liquid in a solitary wave

Let us transform the equation (1) to the form

$$\rho\,\frac{\partial V}{\partial t} + \rho V\,\frac{\partial V}{\partial X} + \rho W\,\frac{\partial W}{\partial X} = -\frac{\partial(PS)}{S\partial X} = -\frac{\partial P}{\partial X} - P\,\frac{\partial X}{S\partial X}. \quad (2)$$

Let us assume that a solitary wave arises on the surface of a liquid, in which the main role is played by the forces of surface tension and gravitational forces.

Let us first consider the role of surface tension forces.

The balance between excess pressure in a liquid below the surface and surface tension forces obeys the law

$$(P - P_0)ld = 2(\sigma - \sigma_0)\delta l, \quad (3)$$

where l — length of the wave section along the Y axis, σ — mechanical stresses due to surface tension forces in the upper liquid layer with thickness δ, σ_0 — mechanical stresses in the upper liquid layer at atmospheric pressure P_0.

Therefore, the excess pressure under the surface of the liquid is equal to

$$P - P_0 = \frac{2(\sigma - \sigma_0)\delta}{d}.$$

The resulting ratio is called the Laplace formula. However, the Laplace formula is fundamentally inaccurate. Thermodynamic analysis shows that a more accurate formula is [10]:

$$\ln\frac{P}{P_0} = \frac{2(\sigma - \sigma_0)\delta}{Dd}, \quad (4)$$

where the quantity D characterizes not only the interaction of molecules leading to the effect of surface tension in the liquid, but also depends on the geometry of the wave and the thickness of the surface layer.

Expanding the exponent in (4) into a series, we obtain

$$P - P_0 = P_0\,\frac{2(\sigma - \sigma_0)\delta}{Dd} + \frac{1}{2}P_0\left(\frac{2(\sigma - \sigma_0)\delta}{Dd}\right)^2. \quad (5)$$

From the first term on the right side, as a first approximation (Laplace formula), it follows $P_0 = D$.

The slight compressibility of a liquid, like a solid body, obeys the Hooke law

$$P - P_0 = -E\,\frac{\Delta V}{V_0}, \quad (6)$$

where ΔV — fluid volume change, V_0 — initial volume, E — volumetric modulus of fluid elasticity (for water $E = 2\cdot 10^9$, $\frac{\mathrm{N}}{\mathrm{m}^2}$). The minus sign, since an increase in pressure $P - P_0 > 0$, leads to a decrease in the volume of the liquid, $\Delta V < 0$ [11].

Considering (6), we use the Hooke law to relate the change in pressure under the surface of the liquid in the wave and the increase in the flow area in the wave (Fig. 2) in the form similar to the equation for an elastic pipeline [12]

$$\partial P = -D\,\frac{\partial S}{S}, \quad (7)$$

where the value is $D = E\,\frac{\delta}{d}$.

In this case, equation (2) is transformed to the form

$$\rho\,\frac{\partial V}{\partial t} + \frac{\partial}{\partial X}\left(\frac{\rho V^2}{2} + \frac{\rho W^2}{2} + P - \frac{P^2}{2D}\right) = 0. \quad (8)$$

If the fluid flow is stationary, i.e. $\frac{\partial V}{\partial t} = 0$, then equation (8) can be integrated

$$\frac{\rho V^2}{2} + \frac{\rho W^2}{2} + P - \frac{P^2}{2D} = \mathrm{const}. \quad (9)$$

Let us analyze in more detail the reason for the appearance of the last term in (9). Let us show that it

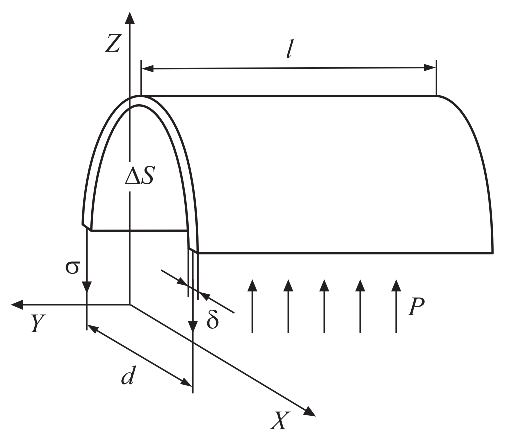

Figure 2. Area of the surface layer of a solitary wave in a channel.

is a consequence of the Hooke law in the form (7), which, for the convenience of transformations, we write in the form

$$P - P_0 = D\,\frac{\Delta S}{S},$$

where the plus sign is used, since with increasing pressure P the area ΔS also increases.

Taking into account the Hooke law in the form

$$\sigma - \sigma_0 = E\varepsilon = \frac{E\Delta d}{d},$$

where $\varepsilon = \frac{\Delta d}{d}$ — relative deformation of the wave base length d (Fig. 2), whose shape is approximately circular, we obtain equation (5) in the form

$$P - P_0 = \frac{2E\Delta d\delta}{d^2} + \frac{1}{2D}\left(\frac{2E\Delta d\delta}{d^2}\right)^2. \qquad (10)$$

Let us find the relationship between the relative change in the flow area in the wave $\frac{\Delta S}{S}$ and the relative change in the wave base length $\frac{\Delta d}{d}$. Taking into account the relationship between the cross-sectional area and the circle diameter $S = \frac{\pi d^2}{4}$, we find the derivative

$$\frac{dS}{d(d)} \approx \frac{\Delta S}{\Delta d} = \frac{\pi d}{2},$$

therefore,

$$\frac{\Delta S}{S} = 2\,\frac{\Delta d}{d}.$$

Therefore, for pressure in excess of P_0, we obtain

$$P - P_0 = \frac{E\delta}{d}\frac{\Delta S}{S} + \frac{1}{2D}\left(\frac{E\delta}{d}\frac{\Delta S}{S}\right)^2 = D\,\frac{\Delta S}{S} + \frac{1}{2D}\left(D\,\frac{\Delta S}{S}\right)^2 = D\,\frac{\Delta S}{S} + \frac{(P-P_0)^2}{2D} \qquad (11)$$

or

$$P - P_0 - \frac{(P-P_0)^2}{2D} = D\,\frac{\Delta S}{S}.$$

The result obtained (for $P_0 = 0$) shows that if we use the Hooke law in the form (7), then it is more correct to use the quantity $P - \frac{P^2}{2D}$ instead of pressure P, as it is accepted in the formula (9).

Therefore, the correctness of formula (1) for describing a wave in a channel that arises due to surface tension forces has been proven.

1.2. Gravity forces in a solitary wave

Consider the role of gravitational forces in wave formation.

Let us write the momentum equation in the traditional form [3] using separate writing of pressure forces and gravitational forces

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P - g\mathbf{j}, \qquad (12)$$

where g — gravitational acceleration, $\mathbf{j}$ — unit vector directed downwards along axis Z.

Let us investigate the relationship between equations (1) and (12). The last term on the left side of (1), in particular, at the bottom of the channel, can be transformed

$$\frac{1}{\rho S}\frac{\partial PS}{\partial X} = \frac{1}{\rho}\frac{\partial P}{\partial X} + \frac{P}{\rho S}\frac{\partial S}{\partial X} = \frac{1}{\rho}\frac{\partial P}{\partial X} + \frac{P}{\rho(h+\xi)}\frac{\partial(h+\xi)}{\partial X}$$
$$= \frac{1}{\rho}\frac{\partial P}{\partial X} + \frac{\rho g(h+\xi)}{\rho(h+\xi)}\frac{\partial(h+\xi)}{\partial X} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + g\,\frac{\partial(h+\xi)}{\partial X}. \qquad (13)$$

In (13) the formula for the cross section of the fluid flow in the channel $S = h + \xi$ (the channel width is taken equal to unity) is used, where h — channel depth with an undisturbed water surface in the channel, ξ — current height of a solitary wave above the undisturbed water surface (Fig. 1). It is assumed that the flow is inertial and the pressure at the bottom of the channel, taking into account the zero value on the surface $P = \rho g(h+\xi)$. The pressure in the problem under consideration has a gravitational character.

Replacing the real course of the curve $\xi(X)$ by two straight sections with the modulus of slope coefficients j by the formula $h + \xi \approx h \pm |j|X$, we arrive at the form of the right side of equation (12) in its algebraic form.

Thus, the momentum equation in the form (1) can describe the generation of waves on the channel surface both due to surface tension forces and due to gravitational forces.

2. Derivation of a system of differential equations for waves in an open channel

In further analysis, we will partially follow [1].

It can be assumed that when a perturbation occurs in any place of the channel, two nonlinear waves will appear, propagating in opposite directions from the place of the perturbation, the so-called Korteweg and de Vries solitons [1–3].

Let us find a differential equation, which is satisfied at once by both differently directed waves. The so far known Korteweg and de Vries equation (within the framework of a single-soliton solution) describes only a solitary wave propagating from left to right:

$$\frac{\partial u}{\partial t} + 6u\,\frac{\partial u}{\partial X} + \frac{\partial^3 u}{\partial X^3} = 0, \qquad (14)$$

where $u(X,t)$ — fluid velocity in the wave.

Let us write the primary system of equations of hydrodynamics in the form of the momentum equation and the continuity equation. From the physical point of view, for simplicity, we will consider the forces of gravitation to be the main effect in the appearance of waves.

We use the momentum equation in the form (1).

We use the continuity equation in the form

$$\nabla^2\varphi = 0, \qquad (15)$$

where φ — in this case, the velocity potential related to the velocity components by the formulas

$$V = \frac{\partial \varphi}{\partial X} \quad \text{and} \quad W = \frac{\partial \varphi}{\partial Z}.$$

2.1. Expansion of the velocity potential in a series

The method of expanding the potential into a series in terms of a small parameter was first proposed by Rayleigh [13].

For the convenience of further transformations, we use dimensionless variables. For this purpose, we scale the parameters of equations (1) and (15):

$$X = M_X X^*, \; Z = M_Z Z^*, \; \xi = M_\xi \xi^*, \; P = M_P P^*,$$

$$\varphi = M_\varphi \varphi^*, \; t = M_t t^*, \; V = M_V V^*, \; W = M_W W^*, \quad (16)$$

where M_i — scales of quantities, and the dimensionless quantities themselves are marked with asterisks.

To write specific values of the scales of quantities and, taking into account the relatively small value of the solitary wave amplitude $\xi_{\max}$ relative to the channel depth h, we introduce a small parameter $\varepsilon = \frac{\xi_{\max}}{h} \ll 1$.

Initially, we need only expressions for some scales. Accept

$$M_\xi = \varepsilon h = \xi_{\max}, \; M_X = \frac{h}{\sqrt{\varepsilon}}, \; M_Z = h, \; M_P = \varepsilon \rho g h. \quad (17)$$

Let us transform the continuity equation (4) into a dimensionless form.

Using

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial X^2} + \frac{\partial^2 \varphi}{\partial Z^2},$$

write down

$$\frac{\partial^2 \varphi^*}{\partial X^{*2}} \frac{M_\varphi}{(h/\sqrt{\varepsilon})^2} + \frac{\partial^2 \varphi^*}{\partial Z^{*2}} \frac{M_\varphi}{h^2} = 0,$$

therefore:

$$\varepsilon \frac{\partial^2 \varphi^*}{\partial X^{*2}} + \frac{\partial^2 \varphi^*}{\partial Z^{*2}} = 0. \quad (18)$$

Let us expand the velocity potential into a series

$$\varphi^* = \sum_{n=0}^{\infty} (Z^* + 1)^n \varphi_n^*(X^*, t^*), \quad (19)$$

where $\varphi_n^*(X^*, t^*)$ — dimensionless function depending only on dimensionless longitudinal coordinate X^* and dimensionless time t^*.

Let us substitute the potential expansion (19) into the dimensionless continuity equation in the form (18):

$$\sum_{n=0}^{\infty} \left(\varepsilon (Z^* + 1)^n \frac{\partial^2 \varphi_n^*}{\partial X^{*2}} + n(n-1)(Z^* + 1)^{n-2} \varphi_n^* \right) = 0. \quad (20)$$

We equate the coefficients at $(Z^* + 1)^n$ to zero, for which we replace n by $n + 2$ in the second term in brackets. As a result, we find the recurrent formula

$$\varphi_{n+2}^* = -\frac{\varepsilon}{(n+1)(n+2)} \frac{\partial^2 \varphi_n^*}{\partial X^{*2}}. \quad (21)$$

Thus, the expansion of the potential (19) taking into account (21) can be written in the form

$$\varphi^* = \varphi_0^* + (Z^* + 1)\varphi_1^* - \frac{\varepsilon}{2!}(Z^* + 1)^2 \frac{\partial^2 \varphi_0^*}{\partial X^{*2}} - \frac{\varepsilon}{3!}(Z^* + 1)^3 \frac{\partial^2 \varphi_1^*}{\partial X^{*2}} + \frac{\varepsilon^2}{4!}(Z^* + 1)^4 \frac{\partial^4 \varphi_0^*}{\partial X^{*4}} + \frac{\varepsilon^2}{5!}(Z^* + 1)^5 \frac{\partial^4 \varphi_1^*}{\partial X^{*4}} - \ldots, \quad (22)$$

where φ_0^* — dimensionless velocity potential at the channel bottom at $Z^* = -1$.

At the bottom of the channel at $Z^* = -1$ the vertical velocity of the liquid is $W = \frac{\partial \varphi}{\partial Z} = 0$. Therefore, starting from (22), $\left(\frac{\partial \varphi^*}{\partial Z^*}\right) = \varphi_1^* = 0$. Therefore, according to the recurrent formula (21), $\varphi_{2n+1}^* = 0$. But $\varphi_n^*(X^*, t^*) \neq f(Z^*)$, so $\varphi_{2n+1}^* = 0$ for any Z^*, and not only for $Z^* = -1$.

Consequently, expansion (22) can be rewritten in the form

$$\varphi^* = \varphi_0^* - \frac{\varepsilon}{2!}(Z^* + 1)^2 \frac{\partial^2 \varphi_0^*}{\partial X^{*2}} + \frac{\varepsilon^2}{4!}(Z^* + 1)^4 \frac{\partial^4 \varphi_0^*}{\partial X^{*4}} - \ldots. \quad (23)$$

On the water surface outside the solitary wave $Z^* = 0$, therefore:

$$\varphi^* = \varphi_0^* - \frac{\varepsilon}{2!} \frac{\partial^2 \varphi_0^*}{\partial X^{*2}} + \frac{\varepsilon^2}{4!} \frac{\partial^4 \varphi_0^*}{\partial X^{*4}} - \frac{\varepsilon^3}{6!} \frac{\partial^6 \varphi_0^*}{\partial X^{*6}} + \ldots. \quad (24)$$

Taking into account the small amplitude of the solitary wave, formula (24) can also be used on the surface of the solitary wave.

2.2. Obtaining a system of differential equations

The momentum equation (1) can be transformed using the formula for the cross section of the fluid flow in the channel $S = h + \xi = h(1 + \varepsilon \xi^*)$ (the channel width is assumed to be unity):

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial X} + W \frac{\partial W}{\partial X} + \frac{1}{\rho(1 + \varepsilon \xi^*)} \frac{\partial P(1 + \varepsilon \xi^*)}{\partial X} = 0. \quad (25)$$

Passing in (25) to dimensionless variables using scales (16), (17),

$$M_V = \frac{M_\varphi}{M_X}, \quad M_W = \frac{M_\varphi}{M_Z}$$

and composing the scales, we find

$$\frac{\partial V^*}{\partial t^*} + \varepsilon V^* \frac{M_\varphi M_t}{M_Z^2} \frac{\partial V^*}{\partial X^*} + W^* \frac{M_\varphi M_t}{M_Z^2} \frac{\partial W^*}{M_Z \partial X^*} = -\frac{1}{\rho(1 + \varepsilon \xi^*)} \frac{M_P M_t \partial P^*(1 + \varepsilon \xi^*)}{M_\varphi \partial X^*}. \quad (26)$$

Let us take the ratio of scales, which does not contradict (17), in the form

$$\frac{M_\varphi M_t}{M_Z^2}=\frac{M_P M_t}{\rho M_\varphi}=1. \quad (27)$$

As a result, we obtain the dimensionless momentum equation

$$\frac{\partial V^*}{\partial t^*}+\varepsilon V^*\frac{\partial V^*}{\partial X^*}+W^*\frac{\partial W^*}{\partial X^*}=-\frac{1}{(1+\varepsilon\xi^*)}\frac{\partial P^*(1+\varepsilon\xi^*)}{\partial X^*}. \quad (28)$$

Hereinafter, we will discard all small terms proportional to ε^2 and higher powers ε in equations.

Consider the sum

$$\varepsilon V^{*2}+W^{*2}=\varepsilon\left(\frac{\partial\varphi^*}{\partial X^*}\right)^2+\left(\frac{\partial\varphi^*}{\partial Z^*}\right)^2=\varepsilon\left(\frac{\partial\varphi^*}{\partial X^*}\right)^2,$$

since, according to (12):

$$\left(\frac{\partial\varphi^*}{\partial Z^*}\right)^2\sim\varepsilon^2.$$

Therefore, equation (28) can be rewritten in the form

$$(1+\varepsilon\xi^*)\left(\frac{\partial V^*}{\partial t^*}+\varepsilon V^*\frac{\partial V^*}{\partial X^*}\right)+\frac{\partial P^*}{\partial X^*}+\varepsilon\frac{\partial P^*\xi^*}{\partial X^*}=O(\varepsilon^2). \quad (29)$$

We assume that solitary waves on the water surface are determined only by gravitational forces. In this case, the pressure at the level of the unperturbed liquid at $Z=0$ is equal to $P=\rho g\xi$. In dimensionless form, this relation has the form $P^*=\xi^*$. Substituting it into (29), we find

$$\frac{\partial V^*}{\partial t^*}+\varepsilon V^*\frac{\partial V^*}{\partial X^*}+\varepsilon\xi^*\frac{\partial V^*}{\partial t^*}+\frac{\partial\xi^*}{\partial X^*}+\varepsilon\frac{\partial\xi^{*2}}{\partial X^*}=O(\varepsilon^2). \quad (30)$$

Next, we use the expansion (24) in the form

$$\varphi^*=\varphi_0^*-\frac{\varepsilon}{2}\frac{\partial^2\varphi_0^*}{\partial X^{*2}}+O(\varepsilon^2). \quad (31)$$

Let us pass in (31) to the dimensionless longitudinal velocity at the level of the channel bottom $V_0=\frac{\partial\varphi_0^*}{\partial X^*}$ (we do not use the asterisk in this case)

$$V^*=\frac{\partial\varphi^*}{\partial X^*}=V_0-\frac{\varepsilon}{2}\frac{\partial^2 V_0}{\partial X^{*2}}+O(\varepsilon^2). \quad (32)$$

The change in the longitudinal velocity along the height of the liquid in the channel can, according to (23), be found by the formula

$$V^*=\frac{\partial\varphi^*}{\partial X^*}=V_0-\frac{\varepsilon}{2}(Z^*+1)^2\frac{\partial^2 V_0}{\partial X^{*2}}+O(\varepsilon^2). \quad (33)$$

Hereinafter, we use the dimensionless fluid velocity at the level of the channel bottom V_0 in the equations.

Substituting (32) into (30) and rearranging the terms as the sum of the principal and first order with respect to the small parameter ε, we find

$$\left(\frac{\partial V_0}{\partial t^*}+\frac{\partial\xi^*}{\partial X^*}\right)+\varepsilon\left(\xi^*\frac{\partial V_0}{\partial t^*}+V_0\frac{\partial V_0}{\partial X^*}+\frac{\partial\xi^{*2}}{\partial X^*}-\frac{1}{2}\frac{\partial^3 V_0}{\partial t^*\partial X^{*2}}\right)=O(\varepsilon^2). \quad (34)$$

The first linear term in the leading order (34) is determined by the velocity of the fluid (its acceleration), the second term is determined by the rate of change of the solitary wave fronts. Due to the large difference between the velocities of fluid motion and propagation of a solitary wave, in the leading order for analysis we introduce the so-called „slow time" $\tau=\frac{\varepsilon}{2}t^*$. Making the substitution in the main order (34)

$$\frac{\partial}{\partial t^*}\to\frac{\partial}{\partial t^*}+\frac{\varepsilon}{2}\frac{\partial}{\partial\tau}\ [1],$$

we write (34) in the form

$$\left(\frac{\partial V_0}{\partial t^*}+\frac{\partial\xi^*}{\partial X^*}\right)+\varepsilon\left(\frac{1}{2}\frac{\partial V_0}{\partial\tau}+\xi^*\frac{\partial V_0}{\partial t^*}+V_0\frac{\partial V_0}{\partial X^*}+\frac{\partial\xi^{*2}}{\partial X^*}-\frac{1}{2}\frac{\partial^3 V_0}{\partial t^*\partial X^{*2}}\right)=O(\varepsilon^2). \quad (35)$$

Further, we show that the non-linear differential equation written up to a small parameter ε in the second bracket (35):

$$\frac{1}{2}\frac{\partial V_0}{\partial\tau}+\xi^*\frac{\partial V_0}{\partial t^*}+V_0\frac{\partial V_0}{\partial X^*}+\frac{\partial\xi^{*2}}{\partial X^*}-\frac{1}{2}\frac{\partial^3 V_0}{\partial t^*\partial X^{*2}}=0 \quad (36)$$

can be solved by the d'Alembert method. In this case, the solution will satisfy the linear equation obtained in the first bracket (35):

$$\frac{\partial V_0}{\partial t^*}+\frac{\partial\xi^*}{\partial X^*}=0. \quad (37)$$

3. Solution of a nonlinear differential equation by the d'Alembert method

We will seek the solution of the system of equations (36) and (37) by introducing new arguments:

$$r=X^*-t^*,\quad l=X^*+t^*. \quad (38)$$

With respect to these arguments, the functions in the system of equations (36) and (37) are sought by introducing auxiliary functions $f(r,\tau)$ and $g(l,\tau)$:

$$\xi^*=\beta\big[f(r,\tau)+g(l,\tau)\big],\quad V_0=\beta\big[f(r,\tau)-g(l,\tau)\big], \quad (39)$$

where the constant coefficient β will be determined later.

In the new variables, the linear equation (37) is identically satisfied

$$\frac{\partial V_0}{\partial t^*}+\frac{\partial \xi^*}{\partial X^*}=\beta\,\frac{\partial(f-g)}{\partial t^*}+\beta\,\frac{\partial(f+g)}{\partial X^*}$$
$$=\beta\left(\frac{\partial f}{\partial t^*}+\frac{\partial f}{\partial X^*}-\frac{\partial g}{\partial t^*}+\frac{\partial g}{\partial X^*}\right)$$
$$=\beta\left(\frac{\partial f}{\partial r}\,\frac{\partial r}{\partial t^*}+\frac{\partial f}{\partial r}\,\frac{\partial r}{\partial X^*}-\frac{\partial g}{\partial l}\,\frac{\partial l}{\partial t^*}+\frac{\partial g}{\partial l}\,\frac{\partial l}{\partial X^*}\right)$$
$$=\beta\left(-\frac{\partial f}{\partial r}+\frac{\partial f}{\partial r}-\frac{\partial g}{\partial l}+\frac{\partial g}{\partial l}\right)=0. \tag{40}$$

Substitute (39) into equation (36):

$$\frac{1}{2}\beta\,\frac{\partial f}{\partial \tau}-\frac{1}{2}\beta\,\frac{\partial g}{\partial \tau}-\beta^2(f+g)\,\frac{\partial f}{\partial r}-\beta^2(f+g)\,\frac{\partial g}{\partial l}$$
$$+\beta^2(f-g)\,\frac{\partial f}{\partial r}-\beta^2(f-g)\,\frac{\partial g}{\partial l}+2\beta^2(f+g)\,\frac{\partial f}{\partial r}$$
$$+2\beta^2(f+g)\,\frac{\partial g}{\partial l}+\frac{1}{2}\beta\,\frac{\partial^3 f}{\partial r^3}+\frac{1}{2}\beta\,\frac{\partial^3 g}{\partial l^3}=0. \tag{41}$$

Carrying out simple transformations, we find

$$\frac{1}{2}\beta\left(\frac{\partial f}{\partial \tau}-\frac{\partial g}{\partial \tau}+4\beta f\,\frac{\partial f}{\partial r}+4\beta g\,\frac{\partial g}{\partial l}+\frac{\partial^3 f}{\partial r^3}+\frac{\partial^3 g}{\partial l^3}\right)=0. \tag{42}$$

Relation (42) can be considered as an equation for the functions f and g. In order for this equation to be the sum of two standard Korteweg and de Vries equations[1], we choose $\beta=\frac{3}{2}$:

$$\left(\frac{\partial f}{\partial \tau}+6f\,\frac{\partial f}{\partial r}+\frac{\partial^3 f}{\partial r^3}\right)+\left(-\frac{\partial g}{\partial \tau}+6g\,\frac{\partial g}{\partial l}+\frac{\partial^3 g}{\partial l^3}\right)=0. \tag{43}$$

Equation (43) describes two solitary Korteweg and de Vries waves for auxiliary functions f and g. The first bracket describes a wave propagating from left to right, the second bracket describes a wave propagating from right to left. Thus, these waves propagate in opposite directions, gradually moving away from each other, which is typical for the d'Alembert formula [14].

The d'Alembert's theory refers to linear hyperbolic equations. One of the main conclusions of this theory is that the initial conditions completely determine the shape of the emerging waves [14]. For non-linear equations (43) this is not true, since the shape of the solitons is invariable and is determined by the nature of the nonlinearity of the equations.

Let us assume that the waves initially arise due to a disturbance linear along the Y axis (Fig. 2) on the water surface in the form of $\delta(X_0^*)$ — the Dirac function in the XZ plane.

The characteristics of wave f after the wave form is established are the straight lines $X^*-c^*t^*=\text{const}$, and the waves g — direct lines $X^*+c^*t^*=\text{const}$, where c^* — dimensionless wave velocity. However, both families of

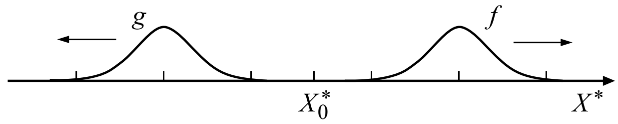

Figure 3. The emergence of two oppositely directed waves in a water channel.

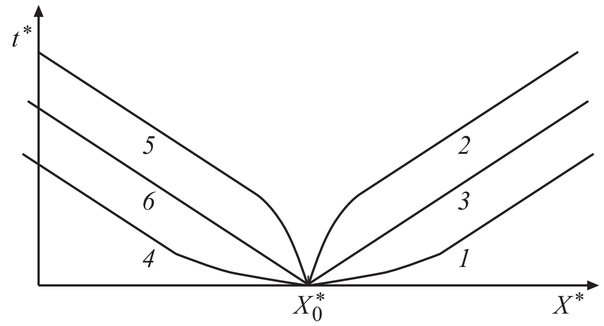

Figure 4. Characteristics of two oppositely directed waves in a water channel.

these characteristics must begin at the point X_0^* (Fig. 3). Therefore, all characteristics before the waveforms are established cannot be straight lines (Fig. 4).

Characteristics *1* and *2* reflect the leading and trailing edges of the wave f propagating from left to right, and characteristics *4* and *5*, respectively, of the wave g propagating from right to left. Rectilinear characteristics *3* and *6* reflect the propagation of wave maxima.

Fig. 3 shows the positions of the f and g waves after the waveforms have been established. The introduction of arguments (38) leads to equation (43) reflecting the formed waves f and g. On non-linear sections of the characteristics (Fig. 4), such arguments cannot be introduced. However, in these sections, in the transient process of establishing the waveform, the exact solution of the system of equations (35) and (37) is hardly possible.

The waves f and g are related to the original functions by the formulas following from (39):

$$f=\frac{\xi^*+V_0}{2\beta}\quad\text{and}\quad g=\frac{\xi^*-V_0}{2\beta}. \tag{44}$$

If a wave propagating from left to right — a direct wave f and a wave from right to left — a backward wave g are equal, then $f=g=\frac{\xi^*}{2\beta}$. If there is no reverse wave ($g=0$), then $\xi^*=V_0$, i.e., the waves of water rise in the channel and the water velocity are similar. In the general case $f\geq g$.

Some time after the appearance of the wave, f and g cease to influence each other, so they can be considered separately. Consider the wave moving from left to right, while the wave propagating from right to left is assumed to be absent, $g=0$. In this case, the solution of the system of equations (36) and (37) will have the form $\xi^*=V_0=\beta f(r,\tau)$, where the function $f(r,\tau)$ satisfies to the Korteweg and de Vries equation:

$$\frac{\partial f}{\partial \tau} + 6f\,\frac{\partial f}{\partial r} + \frac{\partial^3 f}{\partial r^3} = 0. \quad (45)$$

The one-wave (single-soliton) solution of equation (45) is well known [1]:

$$f = \frac{2k^{*2}}{\mathrm{ch}^2\big(k^*(r - 4k^{*2}\tau - r_0)\big)} = \frac{2k^{*2}}{\mathrm{ch}^2\big[k^*\big(X^* - (1 + 2\varepsilon k^{*2})t^* - r_0\big)\big]}, \quad (46)$$

where k^* — dimensionless wave number, $2k^{*2}$ — solitary wave amplitude, $c^* = (1 + 2\varepsilon k^{*2})$ — solitary wave velocity, r_0 — constant value. When writing (46), $r = X^* - t^*$ and $\tau = \frac{\xi}{2}t^*$ are taken into account.

Dimensionless solitary wave velocity

$$c^* = \frac{d\omega^*}{dk^*} = 1 + 2\varepsilon k^{*2}, \quad (47)$$

where ω^* — dimensionless cyclic frequency of the wave. Taking into account the condition that at $k^* = 0$ the value $\omega^* = 0$, integrating (47), we find the dispersion relation for the wave in the form

$$\omega^* = k^* + \frac{2}{3}\,\varepsilon k^{*3}. \quad (48)$$

To plot a solitary wave using formula (46), let's move on to dimensional variables. For the magnitude of the rise in the liquid level in a solitary wave above an undisturbed surface (Fig. 1), we write

$$\xi^* = \frac{\xi}{\xi_{\max}} = \beta f(r,\tau) = \frac{3k^{*2}}{\mathrm{ch}^2\big[k^*(X^* - c^*t^* - r_0)\big]}. \quad (49)$$

It follows from formula (49) that $3k^{*2} = 1$ and $k^* = \frac{1}{\sqrt{3}}$, which indicates the decisive influence of nonlinearity (and non initial conditions, as in linear waves) on waveform formation.

The dimensionless wave velocity can be found in the form

$$c^* = 1 + 2\varepsilon k^{*2} = 1 + \frac{2}{3}\,\frac{\xi_{\max}}{h}. \quad (50)$$

To find the time scale, we obtain from (27) the scale of the stream function

$$M_\varphi = \sqrt{\frac{M_Z^2 M_P}{\rho}} = \sqrt{\frac{h^2 \varepsilon g h \rho}{\rho}} = h\sqrt{\xi_{\max} g}. \quad (51)$$

Therefore, in accordance with (27), the time scale is equal to

$$M_t = \frac{M_Z^2}{M_\varphi} = \frac{h}{\sqrt{g\xi_{\max}}}. \quad (52)$$

Using (50), scales

$$M_X = \frac{h}{\sqrt{\xi_{\max}/h}}$$

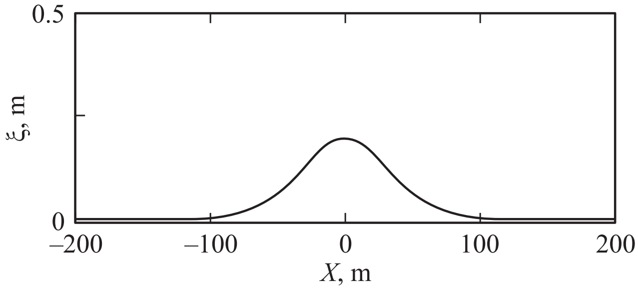

Figure 5. Graph of liquid rise in a solitary wave (soliton).

(17) and (52) for the formula (49) we find the expression

$$\xi = \frac{\xi_{\max}}{\mathrm{ch}^2\Big(\frac{1}{\sqrt{3}}\Big(\frac{X}{M_X} - \big(1 + \frac{2}{3}\frac{\xi_{\max}}{h}\big)\frac{t}{M_t}\Big) - \delta\Big)} = \frac{\xi_{\max}}{\mathrm{ch}^2\Big(\sqrt{\frac{\xi_{\max}}{3h^3}}\Big(X - t\big(1 + \frac{2}{3}\frac{\xi_{\max}}{h}\big)\sqrt{gh}\Big) - \delta\Big)}, \quad (53)$$

where the constant part of the phase is $\delta = k^* r_0$.

Similarly, we find the fluid velocity in a solitary wave. Using the longitudinal speed scale

$$M_V = \frac{M_\varphi}{M_X} = \frac{\xi_{\max}\sqrt{gh}}{h} = V_{0\,\max},$$

defining the relationship between the maximum liquid rise in the wave $\xi_{\max}$ and the maximum liquid velocity in it at the level of the channel bottom $V_{0\,\max}$, we find

$$V_0 = \frac{V_{0\,\max}}{\mathrm{ch}^2\Big(\sqrt{\frac{\xi_{\max}}{3h^3}}\Big(X - t\big(1 + \frac{2}{3}\frac{\xi_{\max}}{h}\big)\sqrt{gh}\Big) - \delta\Big)}. \quad (54)$$

The resulting formula for the fluid velocity at the level of the channel bottom is completely analogous to the formula (53) for the rise of the fluid in a solitary wave. In the formula (54), the dimensional velocities of the liquid are used. Thus, all liquid layers in the channel, and not just the surface layer, participate in the formation of a solitary wave.

Fig. 5 shows a graph of liquid rise in a solitary wave, constructed for a conditional time $t = 0$ and for the following model parameters: $h = 5\,\mathrm{m}$, $\xi_{\max} = 0.2\,\mathrm{m}$, $g = 9.8\,\mathrm{m/s^2}$.

In this case, the maximum fluid velocity in a solitary wave at the channel bottom

$$V_{0\,\max} = \frac{\xi_{\max}\sqrt{gh}}{h} = 0.28\,\mathrm{m/s},$$

and the speed of the solitary wave

$$c = \Big(1 + \frac{2}{3}\,\frac{\xi_{\max}}{h}\Big)\sqrt{gh} = 7.93\,\mathrm{m/s} = 28.5\,\mathrm{km/h}.$$

Therefore, D.S. Russell could accompany on horseback a solitary wave in a water channel.

The liquid velocity on the channel surface can be found by the formula (32), and the distribution of the liquid velocity along the channel height — by the formula (33), but this is not the purpose of this work. We only note that these velocities are less than the liquid velocity at the level of the channel bottom.

Conclusion

Finding the equation for solitary waves (Korteweg and de Vries solitary matter waves) in an open water channel is a well-known problem of hydrodynamics. So far, the derivation of an equation, for example, in [3], has been carried out according to the following scheme. Based on the standard momentum equation and linearized boundary conditions on the free surface of the liquid, a dispersion relation is sought, which has a transcendental (hyperbolic tangent) character. Then this dispersion relation is expanded into a polynomial series up to the third power of the wave number inclusive. The resulting dispersion relation leads to the Korteweg–de Vries equation. In our opinion, it seems more logical to derive the Korteweg–de Vries equation, regardless of the dispersion relation, which should be the result of solving this equation.

The paper considers the use of the momentum equation in the form (1) with a special form of writing the force term. It is shown that this form of the momentum equation makes it possible to take into account both the action of surface tension forces and the action of gravitational forces in the formation of waves on the surface of an open channel. In addition, when solving the momentum equation in the form (1), two solitary waves appear at once, propagating in opposite directions.

The proposed method for solving the equations of hydrodynamics is as follows. First, we find the series expansion in terms of the small parameter of the velocity potential in the continuity equation. The relative (relative to the channel depth) height of the solitary wave is used as a small parameter. Second, the resulting expansion is substituted into the momentum equation. Then, only terms that are linear with respect to a small parameter are retained in the momentum equation. As a result, a nonlinear equation arises for the fluid velocity, which is solved by a method similar to the d'Alembert method for linear hyperbolic waves.

The solution method used made it possible to obtain two solitary Korteweg–de Vries waves propagating in opposite directions: a wave propagating from left to right and a wave propagating from right to left. Moreover, in contrast to the linear waves obtained in the

d'Alembert method, where they immediately arise in an explicit form due to the initial conditions, in the case of the obtained nonlinear equation, the formation of two Korteweg–de Vries waves, which are not related to the initial conditions, is observed.

Further analysis of the two obtained nonlinear Korteweg–de Vries wave equations made it possible to find all the characteristics of these waves: their propagation velocity, wave shape, fluid velocity in the waves, dispersion relation etc.

Conflict of interest

The author declares that he has no conflict of interest.

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