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Polarization singularities in the interference of three plane waves

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Polarization singularities arising from the interference of three plane monochromatic waves with close propagation directions in a vacuum or a linear isotropic medium are studied within the framework of the quasi-optical approximation. With a special choice of linear polarization of waves, the conditions for the formation of L-lines in the cross section, where the polarization is linear, and C-points, where the polarization is circular, are determined, and analytical expressions are obtained for the location of such polarization singularities.

Keywords: polarization singularities of radiation, L-lines, C-points.

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Introduction

Although the topological features of the structure of optical radiation were studied as early as the 1830-s [1], these features have been intensively developed at the present time, accompanied by the penetration of topological methods into physics and other sciences. With regard to coherent optical radiation, the singularities of its wave front (phase) and polarization [2–8] are of significant scientific and applied interest, which is caused by preservation of topological characteristics even with significant perturbations of the system. In this case, the isolated singularity of the phase (dislocation) corresponds to its uncertainty; it arises at the points of the beam cross-section, at which the radiation intensity turns to 0, and when passing along a closed contour around such a point, a phase incursion occurs that is a multiple of 2π. Similarly, polarization singularities correspond to degenerate cases of a general elliptical polarization. Here there are C-points with circular polarization, for which the concept of the main axis of the polarization ellipse is not defined, and L-lines with linear polarization (it is not defined whether the polarization is right or left) [9]; we exclude from consideration the so-called V-points at which the total radiation intensity turns to 0.

Topological radiation singularities arise both in linear and nonlinear optical media. As is known, isolated phase singularities can be obtained by interference of three plane monochromatic waves, if their real amplitudes satisfy „the triangle rule“ (each of them is greater than the difference of the other two amplitudes, but less than their sum [10]). The question naturally arises whether the formation of polarization singularities is possible with such interference. This article is devoted to the answer to this question.

Basic relations

Let us consider the interference of three monochromatic plane waves with the same frequency ω and close wave vectors, which allows us to use the quasi-optical (paraxial) approximation. Intensity of the total electric field is written as

$$\tilde{\mathbf{E}} = \mathrm{Re}[\mathbf{E}\exp(ik_0z - i\omega t)], \tag{1}$$

where z — longitudinal coordinate along the direction of predominant wave propagation, t — time, wavenumber in vacuum $k_0 = \omega/c$.

The field envelope $\mathbf{E}$ in the case under consideration is written in the form

$$\mathbf{E} = \sum_{m=1}^{3} \mathbf{E}_m \exp[i(\mathbf{k}_{\perp m}\mathbf{r}_\perp + \delta k_{zm}z)]. \tag{2}$$

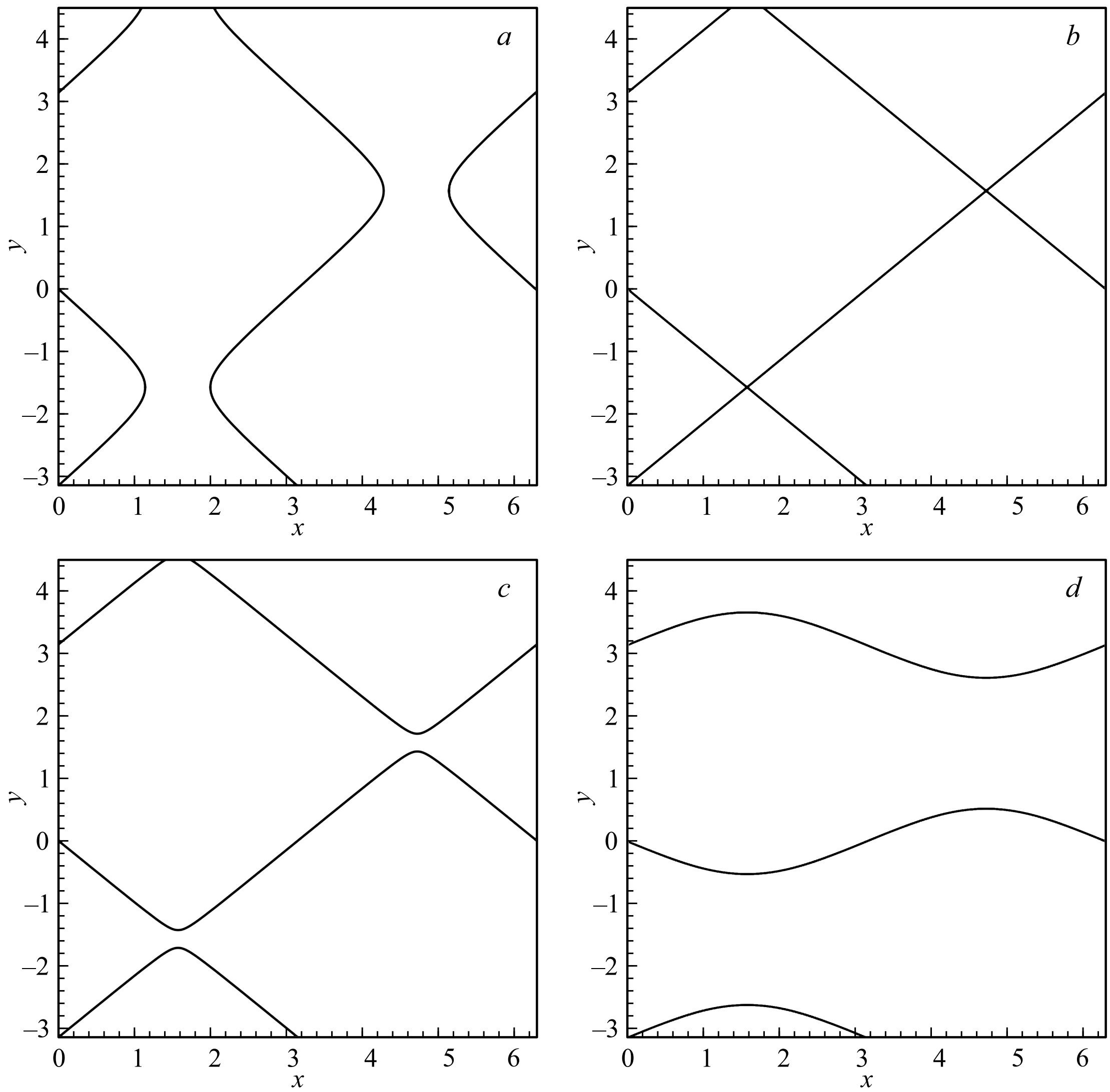
Here $\mathbf{E}_m$ — complex wave amplitudes, $\mathbf{r}_\perp$ — transverse coordinates, $\mathbf{k}_{\perp m}$ — transverse wave vectors of waves and $\delta k_{zm} = -k_{\perp m}^2/(2k_0)$.

By rotating the coordinate system, we can set $\mathbf{k}_{\perp 3} = 0$ and align the $\mathbf{k}_{\perp 1}$ direction with the x axis. To simplify calculations, let us consider the case when the vectors $\mathbf{k}_{\perp 1}$ and $\mathbf{k}_{\perp 2}$ are orthogonal, so that $\mathbf{k}_{\perp 2} = \ k_{\perp 2}\mathbf{e}_y$, where $\mathbf{e}_y$ — the unitary vector in the direction of the y axis. In addition, we introduce dimensionless coordinates $x = \mathbf{k}_{\perp 1}\mathbf{r}_\perp$, $y = \mathbf{k}_{\perp 2}\mathbf{r}_\perp$. Then the field envelope

$$\mathbf{E} = \mathbf{E}_1\exp(ix + i\delta k_{z1}z) + \mathbf{E}_2\exp(iy + i\delta k_{z2}z) + \mathbf{E}_3. \tag{3}$$

Obviously, the transverse structure of the field is periodic in x and y with period 2π. After the introduction of real wave amplitudes $\mathbf{E}_m = \mathbf{A}_m\exp(i\alpha_m)$, $\mathbf{A}_m = |\mathbf{E}_m|$, and new transverse coordinates $x + \delta k_{z1}z + \alpha_1 - \alpha_3 \to x$, $y + \delta k_{z2}z + \alpha_2 - \alpha_3 \to y$ we find

$$\mathbf{E} = [\mathbf{A}_1\exp(ix) + \mathbf{A}_2\exp(iy) + A_3]\exp(i\alpha_3). \tag{4}$$



L-lines for $A_1/A_2 = 1.1$ (a), 1 (b), 0.99 (c) and 0.5 (d). The lines are periodic in x and y with period 2π.

With a change in the longitudinal coordinate, only a linear shift in z of the transverse structure occurs, so it is sufficient to confine ourselves to considering only one of the transverse sections, for example, $z = 0$. As can be seen in (4), the field does not change with the simultaneous replacement

$$\mathbf{A}_1 \leftrightarrow \mathbf{A}_2, \quad x \leftrightarrow y. \tag{5}$$

It is convenient to determine the polarization structure using the Stokes parameters [9,11], which in the paraxial approximation have the form

$$s_0 = |E_x|^2 + |E_y|^2, \quad s_1 = |E_x|^2 - |E_y|^2,$$

$$s_2 = 2\,\mathrm{Re}(E_x^* E_y), \quad s_3 = 2\,\mathrm{Im}(E_x^* E_y). \tag{6}$$

For circular polarization (C-points) $s_1 = s_2 = 0$, $s_3 = \pm s_0$, and for linear polarization (L-lines) $s_3 = 0$.

Singularities of the transverse structure of the field

If each of the three waves has the same polarization, then their total field has the same structure. Therefore, a change in the polarization structure is possible only if at least one of the waves has polarization that differs from that of the other two waves. Let us consider a relatively simple case in which there are transverse changes in the polarization structure. Namely, let us assume

$$\mathbf{A}_1 = A_1 \mathbf{e}_x, \quad \mathbf{A}_2 = A_2 \mathbf{e}_x, \quad \mathbf{A}_3 = A_3 \mathbf{e}_y. \tag{7}$$

Here $\mathbf{e}_{x,y}$ — unitary vectors in the direction of the corresponding axes. Consequently

$$E_x = A_1 \exp(ix) + A_2 \exp(iy), \quad E_y = A_3. \tag{8}$$

Obviously, the Cartesian y-component of the intensity has no phase singularities. And for the x-component, they appear in the form of a straight line only if the amplitudes of the first and second waves are equal, $A_1 = A_2$ (edge dislocation). We emphasize that for $A_1 \neq A_2$ the Cartesian components of the field do not have phase singularities.

L-lines, on which the polarization is linear, are determined by the condition $s_3 = 0$, which turns out to be independent of the amplitude A_3:

$$y = -\mathrm{Arcsin}(a \sin x), \quad a = A_1/A_2. \tag{9}$$

Let us recall that the function Arcsin is multi-valued. L-lines look like deformed sinusoids (figure). In view of symmetry (5), the change $a \to 1/a$ leads to the same lines, but with the change $x \leftrightarrow y$.

If the wave amplitudes $A_1 = A_2$ are equal, they degenerate into two sets of straight lines

$$x + y = 2\pi n, \quad x - y = (2n - 1)\pi, \tag{10}$$

where n — integers. The lines of each set are parallel to each other, being orthogonal to the lines of the other set (Fig. *b*).

When crossing each L-line, the right/left elliptical polarization changes to left/right.

C-points are found based on the conditions $s_1 = s_2 = 0$, which in the case under consideration take the form

$$A_1^2 + A_2^2 - A_3^2 + 2A_1A_2 \cos(y - x) = 0,$$
$$A_1 \cos x + A_2 \cos y = 0. \tag{11}$$

We write the first of these conditions as follows:

$$\cos(y - x) = \beta. \tag{12}$$

Here

$$\beta = -\frac{A_1^2 + A_2^2 - A_3^2}{2A_1A_2}. \tag{13}$$

Solution (12) is possible only for $|\beta| < 1$, which corresponds to the „triangle rule“mentioned above. This solution has the form

$$y - x = \pm\Delta + 2\pi n, \tag{14}$$

where $\cos\Delta = \beta$, $\sin\Delta = \sqrt{1 - \beta^2}$.

Substituting this relation into the second condition (11), we find in the version of the upper sign in (14)

$$x = \mathrm{arctg}\,\frac{\beta + (A_1/A_2)}{\sqrt{1 - \beta^2}} + n\pi,$$
$$y = -\,\mathrm{arctg}\,\frac{\beta + (A_2/A_1)}{\sqrt{1 - \beta^2}} + n\pi. \tag{15}$$

In the second version (lower sign)

$$x = -\,\mathrm{arctg}\,\frac{(A_1/A_2) + \beta}{\sqrt{1 - \beta^2}} + n\pi,$$
$$y = \mathrm{arctg}\,\frac{\beta + (A_2/A_1)}{\sqrt{1 - \beta^2}} + n\pi. \tag{16}$$

On one period in x and y (period 2π) there are four C-points, two of which correspond to the right polarization and two — to the left. When the amplitude A_3 changes, the position of these points changes, but under no circumstances are they superimposed on the L-lines.

Discussion

Thus, the analysis shows that polarization singularities (L-lines and C-points) can already arise during the interference in vacuum of three plane monochromatic waves with linear polarization. L-lines can be formed at any ratio of wave amplitudes, whereas for the possibility of availability of C-points in the situation considered, the wave amplitudes must satisfy the „triangle rule“, known as the existence condition for phase singularities in the interference of scalar waves [10]. The singularities arising from the interference of plane waves are periodically repeated on the beam cross section unlimited number of times. In the transition from plane waves to beams, the number of significant ones (near which the intensity is above the noise level) is limited. The most stable phase and polarization singularities can be observed in dissipative (laser) solitons [12,13].

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Conflict of interest

The author declares that he has no conflict of interest.

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