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Operator form of the generalized optical theorem for wave problems

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The operator form of the generalized optical theorem (OT) obtained earlier for the scalar wave equation is extended to the case of electromagnetic radiation, as well as other wave equations. It is shown that in the transition from the operator to the usual OT, instead of the „forward"scattering amplitude, the diagonal matrix element of the T-scattering operator corresponding to the incident wave appears. The transitions to known forms of OT following from operator OT as special cases are also illustrated.

Keywords: energy conservation, generalized optical theorem, observations in the near and far zones, radiation losses.

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Introduction

The optical theorem (OT), one of the basic ones in scattering theory, relates extinction cross-section of a plane wave incident on a scatterer σ_{ext} with the scattering amplitude in the „forward" direction $A(0)$ by the known relation $\sigma_{\text{ext}} = (4\pi/k_0)\operatorname{Im} A(0)$, where k_0 is the wavenumber [1]. Also the generalized optical theorem (GOT), is known, which is an integral relation expressing the imaginary part of the scattering amplitude at different angles [2]. When the directions of incidence and scattering coincide, the GOT passes into the usual OT. Both OT and GOT follow directly from the law of conservation of energy. They are used in a wide range of wave scattering problems regardless of the nature of wave processes (the history of the problem is given in paper [3]). A general description of GOT for the case of acoustic, electromagnetic, quantum-mechanical and elastic waves was given in paper [4].

Both forms of the optical theorem refer to the description of the field in the far wave zone, where the field is approximately represented as a superposition of plane traveling waves. In the author's papers [5,6], using the scalar wave equation as an example, an operator form of GOT was proposed, suitable for arbitrary radiation sources. This form makes it possible, in particular, to consider the case of point sources, as well as the field in the near zone, including excitation of the scatterer by radiation beams and evanescent waves. In this case, in [5] the case of a lossless scatterer is considered, and in [6] of the scatterer with absorption. In this paper the results [5,6] are extended to the case of scattering of electromagnetic radiation, as well as other wave equations that satisfy certain conditions formulated below. The main one is the possibility of separating the dissipative part describing absorption in the scatterer in the original differential (or in other words, local) formulation of the problem. It turns out that such separation can also be preserved when passing from the original differential equation to an integral one, in which additional conditions are taken into account that ensure the uniqueness of the problem solution. In the most general formulation the generalized optical theorem makes it possible to estimate the changes introduced by the used scatterer into the results of the original problem which does not contain this scatterer.

The traditional approach to obtaining the optical theorem is based on the expression of the balance of energy fluxes flowing through selected closed surfaces. In the case of plane waves a sphere of infinitely large radius is usually taken as such surface. This approach is described in most details in the papers [7,8], where the general case of radiation sources localized at an arbitrary distance from the scatterer is considered for the electrodynamic problem. This allowed the authors [7,8] to obtain some new results by describing in details the energy balance conditions for flows through surfaces enclosing the scatterer, sources, and the scatterer together with sources. By contrast with such „energy" approach, the operator method used below is simple and allows one to obtain not only the usual OT associated with energy flows and corresponding to the diagonal elements of the corresponding operators, but also the GOT that connects their off-diagonal elements.

In this case, the approach considered below is not uniquely tied to the electrodynamic case. The specific statement of the problem is chosen only as an illustration of the general approach, so that no detailed description of the electrodynamic consequences is included in the purpose of this paper. Due to the rejection of rigorous mathematical definitions and the use of operators at the „physical" level of rigor, it is possible to obtain a generalized OT in the form (11) (see below), suitable for a wide class of wave problems. Previously, the universality of GOT was illustrated in the cited paper [4], but using a more complex approach for a specific (albeit quite general) model of the original differential equation, and only for the case of plane waves (far field).

The following Sections present the main results for the forms of the operator optical theorem. Simple, but somewhat cumbersome calculations are given in the Appendix.

1. Maxwell's equations and the integral formulation of scattering problem

Let us consider a three-dimensional stationary problem of the scattering of monochromatic electromagnetic radiation by a scatterer located in a lossless medium. The electric field vector **E** is determined by the equation following from the system of Maxwell's equations (we omit the multiplier $e^{-i\omega t}$ everywhere) [9]:

$$[\nabla \times \nabla \times -k_0^2 \varepsilon(\mathbf{r})]\mathbf{E}(\mathbf{r}) = i\omega\mu_0 \mathbf{j}(\mathbf{r}). \qquad (1)$$

Here, $k_0 = 2\pi/\lambda$ is wavenumber, λ is wavelength, $\varepsilon(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ describe, respectively, the distribution of the permittivity of the medium and currents, and μ_0 is the magnetic permeability of the vacuum. The permittivity $\varepsilon(\mathbf{r})$ in (1) is represented as the sum $\varepsilon(\mathbf{r}) = \varepsilon_0(\mathbf{r}) + \chi_e(\mathbf{r})$, function $\chi_e = \chi_e(\mathbf{r})$ is non-zero only inside the scatterer, and in the general case can be a tensor value that takes into account the anisotropic characteristics of the scatterer. In this case, $\varepsilon(\mathbf{r})$ will also be tensor.

It is assumed that $\varepsilon_0(\mathbf{r})$ has a form that admits a complete solution, i.e. finding the Green's operator in the absence of the scatterer. This condition significantly limits the class of admissible problems. Two examples of this kind will be considered below. This is the case of a homogeneous lossless medium, $\varepsilon_0(\mathbf{r}) = \varepsilon_0 = \text{const}$, and also the case of scatterer near a homogeneous lossless half-space.

The homogeneous equation (1) corresponding to the absence of sources $\mathbf{j} = 0$ has non-zero solutions. Therefore, to make the solutions unique (1) is supplemented with known radiation conditions, and in the presence of discontinuous changes in $\varepsilon(\mathbf{r})$ also with continuity conditions for the tangential components of the electric **E** and magnetic **H** fields. From a mathematical point of view, these conditions limit the class of functions admissible for consideration, thereby highlighting the domain of definition of the operators considered below. A detailed description of these conditions can be found in the textbooks of electrodynamics. Bearing in mind that these conditions are satisfied, we can pass from the differential equation (1) written symbolically in the form (see, for example, [10])

$$(L_0 - V)u = q, \qquad (2)$$

to the integral form of the equation for the field u:

$$u = G^0 q + G^0 V u \equiv Gq. \qquad (3)$$

Here we use abbreviated notation for the field $u = E(\mathbf{r})$ and the sources $q = i\omega\mu_0 \mathbf{j}(\mathbf{r})$, $L_0 = \nabla \times \nabla \times -k_0^2 \varepsilon_0(\mathbf{r})$. In this case, u and q are considered as functions of the argument $x = (i, \mathbf{r})$, which contains a discrete tensor index i and a spatial argument $\mathbf{r}$, and integration over x is always understood as integration over $\mathbf{r}$, supplemented by summation over the corresponding tensor index i, which is not written explicitly.

The Green's operator with tensor kernel $G(\mathbf{r}, \mathbf{r}_0)$ included in (3) acts according to the rule

$$Gq = \int G(\mathbf{r}, \mathbf{r}_0) q(\mathbf{r}_0) d\mathbf{r}_0,$$

and $G^0 = \left(\nabla \times \nabla \times -k_0^2 \varepsilon_0(\mathbf{r})\right)^{-1}$ is matrix operator of „free propagation", describes the problem in the absence of scatterer and is considered to be set. The perturbation operator V in the case under consideration is reduced to multiplication by $\nu(\mathbf{r}) = k_0^2 \chi_e(\mathbf{r})$ and has kernel $V(\mathbf{r}, \mathbf{r}_0) = \nu(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_0)$ (symbols of unit operators and matrices are omitted everywhere).

In the general case, G^0 does not necessarily refer to the case of free propagation. It is assumed that G^0 describes some original „unperturbed" problem whose solution is actually completely known. As such problem in addition to the simplest case of free space, for example, the case of waves near a half-space (see below) can be used, or more complex problems associated with resonators and waveguides, which are not considered in this paper. In this case, the use of the optical theorem makes it possible, in particular, to simplify some estimates of the changes introduced by the scatterer into the propagation of energy fluxes.

2. Operator form of the scattering problem

The transition from the Helmholtz vector equation (1) with additional conditions to the integral formulation of the problem (2) is sophisticated in the general case, since the Green operator kernel contains a strong singularity requiring the use of the concept of excluded volume [11,12]. It turns out that the very form of the integral equation (3) implicitly contains a lot of physics, making it possible to distinguish the dissipative terms in (3), to which inevitable radiation losses are also added.

Substituting $u = Gq$ into (3) gives the relation

$$Gq = G^0 q + G^0 V G q,$$

wherefrom, after reducing by q the operator equation for G follows

$$G = G^0 + G^0 V G. \qquad (4)$$

We define the operator T by the usual relation [10]:

$$VG = TG^0, \qquad (5)$$

so

$$G = G^0 + G^0 T G^0. \qquad (6)$$

Thus, the complete solution of the scattering problem, i.e. the calculation of G is reduced to finding the T-operator, since the free propagation operator G^0 is assumed to be known.

Multiplying both parts (6) in the left side by V and taking into account (5), after simple reductions we obtain the equation for T- operator

$$T = V + VG^0 T. \quad (7)$$

According to (3):

$$u = u^0 + u^{\mathrm{sc}},$$

where u^0 and u^{sc} represent the incident and scattered waves, respectively:

$$u^0 = G^0 q, \quad u^{\mathrm{sc}} = G^{\mathrm{sc}} q, \quad (8)$$

and the scattering operator G^{sc} is expressed as

$$G^{\mathrm{sc}} = G^0 V G = G^0 T G^0.$$

We will consider the introduced linear operators as infinite-dimensional matrices acting in the unitary space of functions $u = u(x)$ with scalar product of the form

$$u^\dagger u \equiv \int u^*(x) u(x) dx,$$

where the symbol † means the Hermitian conjugation, and the integral over $x = (i, \mathbf{r})$ is understood as the integral over $\mathbf{r}$ supplemented by summation over the tensor index i (in used in quantum physics „bra-ket" notations $u = |u>$, $u^\dagger =< u|$, so that $u^\dagger u =< u|u>$ is a number, and $u\,u^\dagger = |u>< u|$ is a matrix operator). Such an approach allows us to use the general properties of operators without being bound in advance to any or fixed basis (see, for example, [13]). At that the kernel of the adjoint operator $V^\dagger$ is expressed as $V^\dagger(x, x_0) = V^*(x_0, x)$.

3. Operator form of the generalized optical theorem

The key point for what follows is to write the perturbation operator V as the sum of the dissipative and conservative parts. In the general case, the perturbation operator V, like any linear operator, can be decomposed into Hermitian (V^h) and anti-Hermitian (iV^a) components[1]

$$V \equiv V^h + iV^a, \quad V^h \equiv (V + V^\dagger)/2, \quad V^a \equiv (V - V^\dagger)/2i. \quad (9)$$

[1] Instead of the Hermitian operator V^a in (9) it would be possible to use the anti-emitter operator $V^{a'} \equiv iV^a$, thereby excluding the "excessive„ imaginary unit from consideration, but the record (9), similar to the decomposition of a complex number into real and imaginary parts, seems to be more illustrative. For the electromagnetic problem the algebraic properties of linear operators are described in more details in Appendices E, F of paper [8].

It is known that in the electromagnetic case considered here, the time-averaged power dissipated in the scatterer can be written as [14]:

$$P_{abs} = \frac{\omega}{2} \operatorname{Im} \int E^\dagger \varepsilon E d\mathbf{r}_0 = \frac{\omega}{2} \int E^\dagger \varepsilon^a E \, d\mathbf{r}_0,$$

where $\varepsilon^a = (\varepsilon - \varepsilon^\dagger)/2i$, $a^\dagger$ is the usual Hermitian conjugation of vectors and matrices. In accordance with the above said, this relation can be abbreviated as

$$P_{abs} = b \operatorname{Im} u^\dagger V u = b u^\dagger V^a u, \quad (10)$$

where the coefficient $b = \frac{c^2}{2\omega}$ depends on the choice of the system of units and, for simplicity, is assumed to be equal to one.

Thus, according to (10), the time-averaged absorbed power up to a multiplier is expressed as the matrix element $u^\dagger V^a u$ of the anti-Hermitian, or otherwise, dissipative part of the operator V^a corresponding to the field u. In this case, the field $u = u(x)$ is considered as one of the basis vectors in the space of functions depending on x, which can be completed to a basis covering the entire space of functions under consideration.

Using the scalar wave equation as an example, it was shown in [6] that the following operator relation follows from the operator equations (6) or (7) in the case of a scatterer with absorption:

$$G^{0\dagger} T^a G^0 = G^{0\dagger} T^\dagger G^{0a} T G^0 + G V^a G \quad (11)$$

(for the particular case $V^a = 0$, the equivalent (11) relation was previously found in [5]).

It is shown in the Appendix that the fulfillment of (11) is not related to the scalar nature of the wave equation, and can be obtained from equation (7) in general form using simple operator transformations. Relation (11) is the operator form of GOT. In it, the operator on the left side is directly related to the extinction of radiation, the first term in the right side is due to radiative losses, and the last term is due to absorption in the scatterer. Let us show how other forms of OT used in the literature follow from this relation.

To pass from (11) to expressions for powers, we multiply both parts of (11) in the right side by the source function q, and in the left side by $q^\dagger$

$$q^\dagger G^{0\dagger} T^a G^0 q = q^\dagger G^{0\dagger} T^\dagger G^{0a} T G^0 q + q^\dagger G^\dagger V^a G q$$

or taking into account (3) and (8)

$$u^{0\dagger} T^a u^0 = u^{0\dagger} T^\dagger G^{0a} T u^0 + u^\dagger V^a u. \quad (12)$$

This relation can be written as

$$P_{\mathrm{ext}} = P_{\mathrm{sc}} + P_{abs}, \quad (13)$$

where

$$P_{\mathrm{ext}} = u^{0\dagger} T^a u^0, \quad P_{\mathrm{sc}} = u^{0\dagger} T^\dagger G^{0a} T u^0, \quad P_{abs} = u^\dagger V^a u. \quad (14)$$

Relation (13) is an extension of ordinary OT to the case of a non-plane incident wave u^0. In it, the extinction power P_{ext}, equal to the sum of radiative and heat losses, according to (14), is represented as corresponding to u^0 diagonal matrix element of the dissipative part T^a of the operator T. This element replaces the imaginary part of the „forward"scattering amplitude, and passes into it for a plane incident wave. The value P_{abs} expresses the time-averaged power of radiation absorbed by the scatterer and is given by the diagonal matrix element V^a for the total field u. The defined (14) scattering power P_{sc} is also represented by the diagonal matrix element over the field u^0 of the scattering operator $T^\dagger G^{0a} T$, which contains operator G^{0a} describing radiative losses. Let us consider these conclusions, as well as some consequences of the operator optical theorem (11) in more details.

4. Scattering of plane electromagnetic wave by isolated scatterer

Let us consider the incidence of the plane wave

$$u^0 = \mathbf{E}^0 \equiv \mathbf{e}_0 e^{ik_0\mathbf{n}_0\mathbf{r}} \tag{15}$$

on the isolated scatterer in free space. Here $\mathbf{n}_0 = \mathbf{k}_0/|\mathbf{k}_0|$ and $\mathbf{e}_0$ are unit vectors, respectively, of the direction and polarization of the incident wave. In the case under consideration, the scattered field $u^{\text{sc}} \equiv \mathbf{E}^{\text{sc}} = G^0 T\mathbf{E}^0$ away from scatterer has the form of a diverging spherical wave, so that in the wave zone at $k_0\mathbf{r} \gg 1$ the scattered field is expressed as

$$E^{\text{sc}} = G^0 T\mathbf{E}^0 \sim \frac{e^{ik_\varepsilon r}}{r}\,\mathbf{F}(\mathbf{n}, \mathbf{n}_0),$$

where $\mathbf{n} = \mathbf{r}/|\mathbf{r}|$ is the unit vector of the scattered wave direction. The scattering vector amplitude $\mathbf{F}(\mathbf{n}, \mathbf{n}_0)$ is proportional to the polarization vector of the incident wave,

$$\mathbf{F}(\mathbf{n}, \mathbf{n}_0) = f(\mathbf{n}, \mathbf{n}_0)\mathbf{e}_0.$$

In this case, the polarization transformation tensor $f(\mathbf{n}, \mathbf{n}_0)$ is expressed in terms of the „T-operator on the energy surface" by the relation

$$f(\mathbf{n}, \mathbf{n}_0) = \frac{1}{4\pi}\,\rho(\mathbf{n})T(k_0\mathbf{n}, k_0\mathbf{n}_0)\rho(\mathbf{n}_0). \tag{16}$$

Here

$$T(\mathbf{k}, \mathbf{k}_0) = \int e^{-ik\mathbf{r}}T(\mathbf{r}, \mathbf{r}_0)e^{ik_0\mathbf{r}}d\mathbf{r}d\mathbf{r}_0$$

is tensor kernel of the T- operator in the wave vectors representation, and $\rho(n) = 1 - \mathbf{n}\times\mathbf{n}$ is projection onto the plane with the normal $\mathbf{n}$ (for the scalar model the relation similar to (16) was derived in [5]). If $\{\mathbf{e}_\alpha(\mathbf{n})\}$ is an arbitrarily chosen orthogonal basis in the plane with the normal $\mathbf{n}(\alpha, \beta = 1, 2, \mathbf{e}_\alpha(\mathbf{n})^\dagger\mathbf{e}_\beta(\mathbf{n}) = \delta_{\alpha\beta})$, then $\rho(\mathbf{n})$ can be written as

$$\rho(\mathbf{n}) = \sum_{\delta=1,2} \mathbf{e}_\delta(\mathbf{n})\mathbf{e}_\delta(\mathbf{n})^\dagger.$$

According to (16), the tensor $f(\mathbf{n}, \mathbf{n}_0)$ is transverse with respect to the direction of propagation of the incident and scattered waves, so that $\mathbf{n}^\dagger\mathbf{f}(\mathbf{n}, \mathbf{n}_0) = \mathbf{f}(\mathbf{n}, \mathbf{n}_0)\mathbf{n}_0 = 0$. This allows us to reduce the 3×3 tensor $\mathbf{f}(\mathbf{n}, \mathbf{n}_0)$ to 2×2 matrix $f_{\alpha,\beta}$ relating polarization vectors of the incident and scattered waves [15]:

$$\begin{aligned} f_{\alpha,\beta}(\mathbf{n}, \mathbf{n}_0) &= \mathbf{e}_\alpha(\mathbf{n})^\dagger f(\mathbf{n}, \mathbf{n}_0)e_\beta(\mathbf{n}_0) \\ &= \frac{1}{4\pi}\,\mathbf{e}_\alpha(\mathbf{n})^\dagger T(\mathbf{k}, \mathbf{k}_0)e_\beta(\mathbf{n}_0). \end{aligned} \tag{17}$$

In the case considered here, it is easy to pass from the operator OT (11) to the usual generalized OT for the electromagnetic field, restricting ourselves to the description of fields in the far zone. To do this, it sufficient to take into account that in the representation of wave vectors the action of the operator G^{0a}, included into (11), is reduced to multiplication by the delta-function:

$$G^{0a}_{\mathbf{k}} = \rho(\mathbf{n})\,\frac{\pi}{2k_0}\,\delta(|\mathbf{k}| - k_0), \tag{18}$$

so that

$$G^{0a}(\mathbf{r}, \mathbf{r}_0) = \int G^{0a}_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_0)}\,\frac{d\mathbf{k}}{(2\pi)^3}, \tag{19}$$

and calculate the off-diagonal matrix elements of (11) corresponding to the incidence and scattering of plane waves.

Note that, in accordance with (18) and (19), the part G^{0a} of the free propagation operator G^0 describing the radiative losses in the space of wave vectors is localized on the „energy surface" $|\mathbf{k}| = k_0$ and is „transverse" with respect to the direction $\mathbf{n}$. Speaking nonstrictly, we can say that the operator G^{0a} „remembers"that radiative losses occur in the far zone, where the wave field approximately has the structure of plane waves, for which these conditions are satisfied.

Using (17)–(19) and integrating over modulus k, we can transform the optical theorem (11) to the form

$$\begin{aligned} &\frac{1}{2i}\,\big((f_{\alpha,\beta}(\mathbf{n}, \mathbf{n}_0) - (f^*_{\beta,\alpha}(\mathbf{n}_0, \mathbf{n})\big) \\ &= \frac{k_0}{4\pi}\sum_{\delta=1,2}\oint_{4\pi} f^*_{\delta,\alpha}(\mathbf{n}', \mathbf{n}))f_{\delta,\beta}(\mathbf{n}', \mathbf{n}_0)d\mathbf{n}' + \sum_{abs}, \end{aligned} \tag{20}$$

where the matrix $\sum_{abs}$ is related to absorption in the scatterer and is not written explicitly. This relation is the usual form of the generalized optical theorem for the electromagnetic field [15]. When the directions and polarizations of the scattered and incident waves coincide, $\mathbf{n} = \mathbf{n}_0$, $\alpha = \beta$ (20) passes into the „classical" optical theorem relating the extinction cross-section with the imaginary part of the forward scattering amplitude. In this case, $\sum_{abs}$ passes into the absorption cross-section, and the extinction cross-section is expressed by the relation

$$\sigma_{\text{ext}} = P_{\text{ext}} = \frac{4\pi}{k_0}\,\mathrm{Im} f_{\alpha,\alpha}(\mathbf{n}_0, \mathbf{n}_0).$$

5. Optical theorem for incident and scattered waves and case of scatterer near lossless half-space

One more form of the optical theorem widely cited in the literature follows directly from the general expressions for powers (14). Using the relations $A^{+}B^{a}A = (A^{+}BA)^{a}$ and $u^{0\dagger}T^{a}u^{0} = \mathrm{Im}\,u^{0\dagger}Tu^{0}$, and also (5) from (14) it is easy to obtain expressions (see Appendix)

$$P_{\text{ext}} = \mathrm{Im}\, u^{0\dagger}Vu,$$

$$P_{\text{sc}} = -\mathrm{Im}\, u^{s\dagger}Vu,$$

$$P_{abs} = \mathrm{Im}\, u^{\dagger}Vu. \tag{21}$$

Since the total field is expressed as the sum of the incident and scattered components, $u = u^{0} + u^{s}$, this, in particular, immediately implies the fulfillment of the optical theorem (13).

Equivalent to (21) relations were obtained in [16] for the scalar model, and in [17] for the electromagnetic radiation model (see also [8]). In these papers, the role of G^{0} belongs not to the „operator of free propagation", but to similar operators for a transparent half-space, explicit expressions for which are given, in particular, in [16,17]. Relations (21) were used in the indicated papers to obtain the optical theorem for scatterer near the transparent half-space, when instead of the „forward" scattering amplitude there is a weighted sum of the scattering amplitudes in the direction of the reflected and transmitted waves. The latter reflects the interference nature of the optical theorem for this problem, which is related to the in-phase nature of the scattered wave with the incident wave only for the directions of reflection and refraction.

Conclusions

In this paper, using the example of electromagnetic radiation scattering, we consider the operator form derivation of the generalized optical theorem related to the fulfillment of the law of conservation of energy in scattering theory problems. In contrast to the traditional approach to obtaining OT, this derivation is not directly related to the calculation of energy flows through closed surfaces and asymptotic estimates of rapidly oscillating integrals, and uses only simple general properties of linear operators acting in a unitary space. The main condition leading to this form of GOT is formulated. It requires the possibility of separating conservative and dissipative terms in the original differential formulation of the problem. In particular, it follows from the operator form of GOT that in the general case of a non-plane incident wave and observation points near the scatterer the value of the imaginary part of the „forward" scattering amplitude, which appears in the usual optical theorem, passes into the diagonal matrix element of scattering T-operator corresponding to the incident wave. The transition from the operator to the usual form of GOT for the electromagnetic field is traced, as well as to the case of scatterer near a lossless half-space described in the literature.

The proposed scheme is illustrated by the example of electromagnetic radiation scattering; however, the general results obtained cover the case of waves of an arbitrary nature that satisfy the conditions formulated above.

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Conflict of interest

The author declares that he has no conflict of interest.

Appendix

Let A and B be two linear invertible, generally non-commuting operators ($AB \neq BA$). For further it is sufficient to use the well-known simple properties of inversion and Hermitian conjugation $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$, as well as $(A+B)^{a} = A^{a} + B^{a}$, and $(A^{\dagger})^{a} = -A^{a}$, where the superscript „a" means the calculation of the dissipative part of the corresponding operator in accordance with (9). These relations are valid both for ordinary matrices and for operators. Using them and (9), it is also easy to check that

$$(A^{-1})^{a} = -(A^{\dagger})^{-1}A^{a}(A)^{-1} \tag{A1}$$

and

$$(A^{\dagger}BA)^{a} = A^{\dagger}B^{a}A. \tag{A2}$$

Let us take the Hermitian conjugation from the equation (6)

$$G^{\dagger} = G^{\dagger} + G^{0\dagger}T^{\dagger}G^{0\dagger}.$$

Multiply this relation in the right side by TG^{0}, replacing TG^{0} in the left side of the resulting expression by VG in accordance with (5):

$$G^{\dagger}VG = G^{0\dagger}TG^{0} + G^{0\dagger}T^{\dagger}G^{0\dagger}TG^{0}.$$

Taking from here the dissipative part (a) considering (A2) and (A1), we obtain the required equation (11).

The transition from the operator form of GOT (11) to the power values (21) is equivalent to calculation of the diagonal matrix element from both parts (11) corresponding to the incident wave u_0. Taking into account relations (5) and (8), the values (14) can be easily transformed as follows:

$$P_{\text{ext}} = \mathrm{Im}\, u^{0\dagger}Tu^{0} = \mathrm{Im}\, u^{0\dagger}TG^{0}q = \mathrm{Im}\, u^{0\dagger}VGq$$

$$= \mathrm{Im}\, u^{0\dagger}Vu, \tag{A3}$$

$$P_{\text{sc}} = -\mathrm{Im}\, u^{0\dagger}T^{\dagger}G^{0\dagger}Tu^{0} = -\mathrm{Im}\, u^{0\dagger}T^{\dagger}G^{0\dagger}Vu$$

$$= -\mathrm{Im}\, u^{s\dagger}Vu, \tag{A4}$$

$$P_{abs} = u^{\dagger}V^{a}u = \mathrm{Im}\, u^{\dagger}Vu. \tag{A5}$$

The expressions for powers (A3)−(A5) are equivalent to relations (21).

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