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## Phonons and the Ioffe–Regel crossover in random lattices with arbitrary dimension

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In the frame of the random matrix theory, it was shown, that the relaxation of the projection of the initial plane wave with the wave vector  $\mathbf{q}$  is described by the equation of motion with the memory function which corresponds to the complex dynamical Young modulus  $E(\omega)$ . In the harmonic scalar model of displacements with the absence of energy dissipation, the Ioffe–Regel crossover arises universally in amorphous systems with the dimension  $d \geq 3$ . Vibrations above the Ioffe–Regel crossover are related to the diffusive nature and can be described by the diffusion equation with the damping  $\Gamma(\mathbf{q}) \propto q^2$ .

**Keywords:** Amorphous solids, diffusons, random matrices.

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### 1. Introduction

Amorphous dielectrics have a number of vibrational properties that fundamentally distinguish such disordered systems from crystalline ones. Vibration relaxation processes that affect these properties play an important role in the study of amorphous systems [1]. The low-frequency region of vibrations of amorphous media is expressed by phonons with a large mean free path. However, in a wide range of frequencies below the localization limit, vibrations propagate through diffusional energy transfer from atom to atom [2,3]. These vibrations are responsible for the secondary increase in the thermal conductivity of amorphous dielectrics at temperatures from 20 to 1000 K [1]. The crossover between low-frequency phonons and higher-frequency diffuse vibrations is known as the Ioffe–Regel crossover, in which the mean free path of a phonon becomes comparable to its wavelength [3].

Practically in all glasses, regardless of their chemical composition, there are universal vibrational features near the Ioffe–Regel crossover [4,5]. This is a bosonic peak that characterizes the excess density of vibrational states  $g(\omega)$  compared to Debye law, according to which the low-frequency density of states is proportional to  $\omega^{d-1}$ , where  $d$  is the dimension of the space [6]. The peak observed in the reduced density of states  $g(\omega)/\omega^{d-1}$  in many experiments [7–10] is called the bosonic peak. In addition, data on inelastic X-ray scattering in glasses show a diffusion feature of  $\Gamma(\mathbf{q}) \propto q^2$  damping above the Ioffe–Regel crossover compared to the Rayleigh dependence  $\Gamma(\mathbf{q}) \propto q^{d+1}$  below the crossover [11–13]. A linear correlation of the Ioffe–Regel crossover frequency  $\omega_{\text{IR}}$  with the elastic moduli of the amorphous medium [14,15] is also noted. In a number of experimental works devoted to the study of vibrations

in two-dimensional systems, all of the above features are observed [16–19]. Therefore, the description of the Ioffe–Regel crossover provides the key to understanding the processes of vibrations in disordered media.

There are different theoretical approaches used to describe these phenomena of amorphous media [20–25]. One of such approaches is the application of the theory of random matrices [26–29]. It can be assumed that in amorphous bodies, due to local disorder, the elements of the dynamic matrix of the system are random to some extent [30]. At the same time, the short-range order in glasses resembles the short-range order in the corresponding crystals, but there is no long-range order. This allows us to consider only the short-range interaction of the atoms of the system located at the lattice sites of the corresponding crystal [30]. Such a consideration of a highly sparse random dynamic matrix allows us to study the above features of vibrations in some two-dimensional and three-dimensional disordered systems, both from the numerical and analytical points of view [29,31–33].

The aim of this work is an analytical description of the diffusion of vibrations and the Ioffe–Regel crossover using the theory of random matrices. At the same time, the found expressions and conclusions are not tied to any particular disordered system (amorphous body), but are general in nature for amorphous media with an arbitrary local arrangement of atoms and dimension. In particular, the study of the oscillatory properties of disordered systems with different dimensions is of interest in the study of quasi-local vibrations. The contribution of quasi-local vibrations to the density of states  $g_{\text{qlv}} \propto \omega^\beta$ ,  $\beta \approx 4$  for most glasses [34,35] does not depend on the dimension of the system. The study of systems with dimensions  $d > 3$  allows to distinguish the modes of these vibrations

from the Rayleigh scattering of phonons by disorder, which contributes  $g \propto \omega^{d+1}$ .

## 2. Representation of the dynamic matrix $\hat{M}$ as a correlated Wishart ensemble

The dynamic matrix  $\hat{M}$  is constructed taking into account the main symmetry properties of a mechanical disordered system. The model under consideration describes the harmonic vibrations of a system consisting of a large number of  $N$  particles near their equilibrium position without energy dissipation. For simplicity, it is assumed that atoms have unit masses, and atomic displacements are scalar values [29]. In this case, the number of degrees of freedom of the system coincides with the number of atoms  $N$ . The dynamic matrix  $\hat{M}$  is related to the total interaction energy of particles of such a system  $U$  as follows [36]:

$$M_{ij} = \frac{\partial^2 U}{\partial u_i \partial u_j}, \quad (1)$$

where the indices  $i$  and  $j$  number the atoms (degrees of freedom),  $u_i$  is the scalar displacement of the  $i$ -th atom from the equilibrium position.

The presence of the system near a stable equilibrium means that the dynamic matrix  $\hat{M}$  [30] is positive definite, so it can be represented in the following form:

$$\hat{M} = \hat{A}\hat{A}^T, \quad (2)$$

where  $\hat{A}$  is an  $N$  by  $K$  matrix. Each row of the matrix  $\hat{A}$  corresponds to the degree of freedom of the system, and each of its columns corresponds to the relation of a certain number of degrees of freedom to each other [31]. Due to the local fluctuation of the connections of the system, we can assume that the elements of the matrix  $\hat{A}$  are random numbers with a Gaussian distribution [30]. Then (2) corresponds to the representation of the dynamic matrix  $\hat{M}$  as a Wishart ensemble of random matrices. Such a model describes many universal oscillatory properties of disordered systems [29,33]. In this case, the mechanical stability of the system does not impose restrictions on the form of the matrix  $\hat{A}$ .

The ratio of the number of bonds  $K$  and the number of degrees of freedom  $N$  plays an important role in the study of vibrations of mechanical systems [37,38]. In this model, the parameter  $\kappa = K/N - 1$  is considered, which shows a relative increase in the number of bonds over the number of degrees of freedom of the system. For  $\kappa \leq 0$ , the system does not have macroscopic rigidity in accordance with Maxwell's rule [39]. For  $\kappa > 0$ , the system has a finite macroscopic rigidity, as a result of which the low-frequency vibrations of the system exist in the form of plane waves and obey the Debye law [29]. The case  $\kappa = 0$  corresponds to an extremely soft system in which all vibrations propagate through some diffusional transfer of energy from atom to

atom [29]. Let us vary the system parameter  $\kappa$  within a wide range  $0 \leq \kappa < \infty$ .

Vibrations of a mechanical system do not depend on its shift as a whole, as a result of which a symmetric dynamic matrix must obey the sum rule:  $\sum_i M_{ij} = \sum_j M_{ij} = 0$  [36]. This leads to the rule  $\sum_i A_{ij} = 0$ , as a result of which the elements of the matrix  $\hat{A}$  are correlated. The statistical properties of the uncorrelated Wishart ensemble are well studied in the framework of the theory of random matrices [40,41]. However, the sum rule leads to a more difficult problem of finding the spectral properties of a correlated Wishart ensemble. Nevertheless, taking into account the sum rule is important, since it ensures the presence of low-frequency vibrations in the form of plane waves, obeying the Debye law [29].

To take into account these correlations, we will assume that all connections are statistically identical and independent, that is, individual columns of the matrix  $\hat{A}$  do not correlate with each other. Then pair correlations between matrix elements  $A_{ij}$  have the form

$$\langle A_{ik} A_{jl} \rangle = \frac{1}{N} C_{ij} \delta_{kl}, \quad (3)$$

where  $\hat{C}$  is the correlation matrix, the brackets  $\langle \cdot \rangle$  mean averaging over different realizations of the dynamic matrix  $\hat{M}$ . The ensemble-averaged dynamic matrix is related to the correlation matrix

$$\langle M_{ij} \rangle = \langle \sum_k A_{ik} A_{jk} \rangle = K \langle A_{ik} A_{jk} \rangle = \frac{K}{N} C_{ij}. \quad (4)$$

If the number of bonds  $K$  significantly exceeds the number of degrees of freedom  $N$ , then the dynamic matrix  $\hat{M}$  differs slightly from the averaged matrix  $\langle \hat{M} \rangle = (\kappa + 1)\hat{C}$ . Therefore, the case of a small difference  $K$  and  $N$  seems to be of interest, i.e., the case of  $\kappa \ll 1$ .

The statistical properties of the random matrix  $\hat{M}$  are related to the statistical properties of the correlation matrix  $\hat{C}$  [42]. It can be expected that the elements of the dynamic matrix averaged over different implementations have the same values, as a result of which the correlation matrix describes a regular system [42]. Then the wave vector of the system  $|\mathbf{q}\rangle$  is an eigenvector of the matrix  $\hat{C}$ , and its eigenvalues correspond to the squared vibration frequencies of the regular system  $\omega_{\text{cor}}^2(\mathbf{q})$ . In works [32,33] regular systems were considered corresponding to two-dimensional and three-dimensional cubic lattices, for which the dispersion dependence  $\omega_{\text{cor}}(\mathbf{q})$  is known. In this work, it is assumed that in the general case, for small wave vectors,  $\omega_{\text{cor}}(\mathbf{q}) \propto q$ , regardless of the particular type of correlation matrix and system dimension.

## 3. Dynamical Young's modulus

Let us show that the motion of the atomic displacement profile  $u_{\mathbf{q}}(t)$  can be described using the equation of motion with a retarded part. The constructed dynamic matrix  $\hat{M}$  (2)

describes small harmonic vibrations in a disordered system in the form of an equation of motion

$$|\ddot{u}(t)\rangle = -\hat{M}|u(t)\rangle, \quad (5)$$

where  $N$ -dimensional vector  $|u(t)\rangle$  describes the deviation of atoms from their equilibrium position at time  $t$ . The equation of motion (5) can be solved by setting the initial conditions  $|u(0)\rangle = 0$ ,  $|\dot{u}(0)\rangle = v_0|\mathbf{q}\rangle$ .

At the same time, due to the redistribution of energy between the internal degrees of freedom of the system, the projection  $\langle u(t)|\mathbf{q}\rangle$  of a plane wave with a wave vector  $\mathbf{q}$  can experience relaxation. To study this process, consider the average plane wave projection  $u_{\mathbf{q}}(t) = \langle u(t)|\mathbf{q}\rangle$ , which is related to the Green's function  $G_{\mathbf{q}}(z)$  of a dynamic matrix in the form of a Fourier transform

$$u_{\mathbf{q}}(t) = -\frac{v_0}{2\pi} \int_{-\infty}^{\infty} G_{\mathbf{q}}((\omega - i0)^2) e^{i\omega t} d\omega, \quad (6)$$

$$G_{\mathbf{q}}(z) = \sum_n \frac{|\langle n|\mathbf{q}\rangle|^2}{z - \omega_n^2}. \quad (7)$$

The frequency  $\omega_n$  corresponds to the eigenstate  $|n\rangle$ ,  $z$  is a complex variable.

In works [33,43] it is shown that in the thermodynamic limit  $N \rightarrow \infty$  the Green's function  $G_{\mathbf{q}}(z)$  has the following form:

$$G_{\mathbf{q}}(z) = \frac{1}{z - \omega_{\text{cor}}^2(\mathbf{q})E(z)}, \quad (8)$$

where the complex function  $E(z) = z/Z(z)$  in the scalar displacement model plays the role of the dynamic complex Young's modulus, the real part of which  $\text{Re} E(z)$  describes elasticity, and the imaginary part  $\text{Im} E(z)$  — viscosity. The complex parameters  $z$  and  $Z$  are related by a conformal transformation  $Z(z)$  of the following form [44]:

$$\kappa Z + \frac{Z^2}{N} \text{Tr} \hat{R}_{\text{cor}}(Z) = z, \quad (9)$$

where the resolution  $\hat{R}_{\text{cor}}(Z)$  corresponds to the resolution of the correlation matrix

$$\hat{R}_{\text{cor}}(Z) = \sum_{\mathbf{q}} \frac{|\mathbf{q}\rangle\langle\mathbf{q}|}{Z - \omega_{\text{cor}}^2(\mathbf{q})}. \quad (10)$$

This form of the Green's function (8) corresponds to the equation of motion of the average projection of a plane wave  $u_{\mathbf{q}}(t)$  with a retarded part

$$\ddot{u}_{\mathbf{q}}(t) + \omega_{\text{cor}}^2(\mathbf{q}) \int_{-\infty}^t \mathcal{K}(t - \tau) u_{\mathbf{q}}(\tau) d\tau = 0, \quad (11)$$

in which the retarded kernel is the Fourier transform of the dynamic Young's modulus

$$\mathcal{K}(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E((\omega - i0)^2) e^{i\omega(t - \tau)} d\omega. \quad (12)$$

The complex dynamic Young's modulus  $E(z)$  can be found from the conformal transformation (9) when the dependence  $Z(z)$  is found explicitly. This dependence, generally speaking, is special for a particular regular system described by the correlation matrix  $\hat{C}$ , with the corresponding variance  $\omega_{\text{cor}}^2(\mathbf{q})$ . In the next section, based on fairly general assumptions about the properties of most of these systems, an approximate general form  $E(z)$  is derived.

#### 4. Low frequency approximation

With the help of a series of approximations, in a universal way, it is possible to obtain the retarded part and the equation of motion of the displacement profile in an explicit analytical form. The normalized trace of the resolution (10) of the correlation matrix can depend on the complex parameter  $Z$  in a complicated way. This dependence is determined by the specific type of auxiliary regular system described by the correlation matrix. However, we will assume that the low-frequency region of vibrations of such a system, regardless of its specific form and dimension, is due to phonons with a linear dispersion law  $\omega_{\text{cor}}(\mathbf{q}) = \Omega q$  at  $q \rightarrow 0$ . The constant  $\Omega$  determines the characteristic vibration frequency of the system.

In two-dimensional systems  $d = 2$  with a low-frequency linear dispersion law, there is a logarithmic divergence  $N^{-1} \text{Tr} \hat{R}_{\text{cor}}(Z) \simeq \ln(-Z) + O(Z)$  for small  $Z \rightarrow 0$ . This feature of two-dimensional systems is associated with their constant density of distribution of squared frequencies near zero. However, in systems with dimension  $d \geq 3$ , the resolution expansion has the general form  $N^{-1} \text{Tr} \hat{R}_{\text{cor}}(Z) = -a^2 + f(Z)$ , where the constant  $a$  does not depend on  $Z$ . For example, for a simple three-dimensional cubic lattice, the constant  $a \approx 0.505462$  (Watson's constant) [45], and the function  $f(Z) \simeq \sqrt{-Z} + O(Z)$  has root singularity for small  $Z$  [46]. Therefore, in the first approximation, for  $Z \rightarrow 0$ , we restrict ourselves to a constant:  $N^{-1} \text{Tr} \hat{R}_{\text{cor}}(Z) \simeq -a^2$ . Such an approximation is valid for  $\kappa \ll 1$  and is generally suitable for a qualitative analysis of systems with dimensions  $d \geq 3$ , in which phonons with a linear dispersion law propagate at low frequencies. However, to reveal the features of specific systems, it is necessary to look for the form  $f(Z)$ .

Taking into account the described low-frequency approximations, the conformal transformation (9) is rewritten in the form

$$\kappa - a^2 \frac{z^2}{E(z)} = E(z). \quad (13)$$

In view of the dependence (6), for each complex  $z = (\omega - i0)^2$  from equation (13) one can find the explicit form of the dynamic Young's modulus  $E((\omega - i0)^2)$ . For simplicity, we omit  $-i0$  in the following equations and choose a complex solution that corresponds to a positive

density of vibrational states  $g(\omega) > 0$ :

$$E(\omega) = \frac{\kappa}{2} + \sqrt{\frac{\kappa^2}{4} - a^2\omega^2}. \quad (14)$$

The static Young’s modulus  $E(0) = \kappa$  determines the macroscopic rigidity of the medium. Then the equation of motion (11) takes the following form:

$$\ddot{u}_{\mathbf{q}}(t) + \frac{\Omega^2\kappa}{2}\Delta u_{\mathbf{q}}(t) + \Omega^2a\Delta\dot{u}_{\mathbf{q}}(t) + \int_{-\infty}^t F(\tau)\Delta u_{\mathbf{q}}(t-\tau)d\tau = 0, \quad (15)$$

$$F(\tau) = \frac{\kappa\Omega^2}{2\tau}J_1\left(\frac{\kappa\tau}{2a}\right), \quad (16)$$

where  $\Delta$  is a discrete Laplace operator acting on the projection of the original plane wave with the wave vector  $\mathbf{q}$ . This form of the retarded part (16) manifests itself in processes characterized by the behavior of a damped harmonic oscillator [47,48].

### 5. Ioffe–Regel crossover

The sign of the radical expression in (14) determines the behavior of the complex Young’s modulus  $E(\omega)$  and the relationship between elasticity and damping in the system. Frequency

$$\omega_c = \frac{\kappa}{2a} \quad (17)$$

divides vibrations into two regions:  $\omega < \omega_c$  and  $\omega > \omega_c$ . Let us analyze the characteristic features of vibrations in these frequency ranges.

In the  $\omega \ll \omega_c$  region, the dynamic Young’s modulus (14)  $E(\omega) \simeq \kappa$ , and the equation of motion (15) describes vibrations of elastic perturbation waves

$$\ddot{u}_{\mathbf{q}}(t) + k\Delta u_{\mathbf{q}}(t) = 0 \quad (18)$$

with stiffness coefficient  $k = \Omega^2\kappa$ . Such vibrations are phonons with the linear dispersion law  $\omega(q) = q\sqrt{k}$ .

To find the dispersion law in the frequency range  $\omega < \omega_c$ , let us assume that the phase velocity of phonons in this frequency range is determined by the dynamic Young’s modulus (14):  $v_{\text{ph}} = \omega/q = \sqrt{\Omega^2E(\omega)}$ . From this relation follows the relation between the frequency  $\omega$  and the wave vector  $q$ :

$$\omega(q) = \Omega^2aq\sqrt{2q_c^2 - q^2}, \quad (19)$$

where the wave vector  $q_c = \sqrt{\kappa/2\Omega^2a^2}$  corresponds to the frequency  $\omega_c$ . This law of dispersion, obtained in a higher approximation, describes elastic vibrations (phonons) well over the entire frequency range  $\omega < \omega_c$ .

In the  $\omega \gg \omega_c$  region, the dynamic Young’s modulus  $E(\omega) \simeq ia\omega$ , and the equation of motion (15) describes the diffusion of the atomic velocity profile

$$\ddot{u}_{\mathbf{q}}(t) + D\Delta\dot{u}_{\mathbf{q}}(t) = 0 \quad (20)$$

with diffusion coefficient  $D = \Omega^2a$ . At  $\kappa \ll 11$ , this diffusion feature of the vibrations is preserved in almost the entire frequency range  $\omega > \omega_c$ .

Thus, in the vibration region  $\omega < \omega_c$ , well-defined sound vibrations, phonons having a linear law of dispersion in the low-frequency region  $\omega \ll \omega_c$ , propagate. In the frequency range  $\omega > \omega_c$ , vibrations propagate due to some diffusional transfer of energy from atom to atom in the absence of an explicit mass transfer. The frequency  $\omega_c$  separates these two fundamentally different regions of vibrations. This allows us to assume that  $\omega_c$  corresponds to the Ioffe–Regel crossover frequency  $\omega_{\text{IR}}$  in the transition from long free path phonons to diffusion vibrations [2,3].

### 6. Density of vibrational states

To find an analytical expression for the density of vibrational states  $d$ -dimensional system  $g_{\text{ph}}(\omega)$  in the phonon frequency range  $\omega < \omega_c$ , we use the approximation  $\omega/q = \sqrt{\Omega^2E(\omega)}$ , in which the dynamic Young’s modulus is determined by relation (14):

$$g_{\text{ph}}(\omega) = \frac{s_d}{(2\pi)^d}q^{d-1}\frac{\partial q}{\partial \omega} = \frac{s_d\omega^{d-1}}{(2\pi\Omega\sqrt{a})^d\sqrt{(\omega_c^2 - \omega^2)(\omega_c + \sqrt{\omega_c^2 - \omega^2})^{\frac{d}{2}-1}}}, \quad (21)$$

where the coefficient  $s_d = d(\sqrt{\pi})^d/[(d/2)!]$  is equal to the area of the  $d$ -dimensional hypersphere of unit radius.

In the low-frequency region  $\omega \ll \omega_c$ , the density of vibrational states (21) has the Debye form

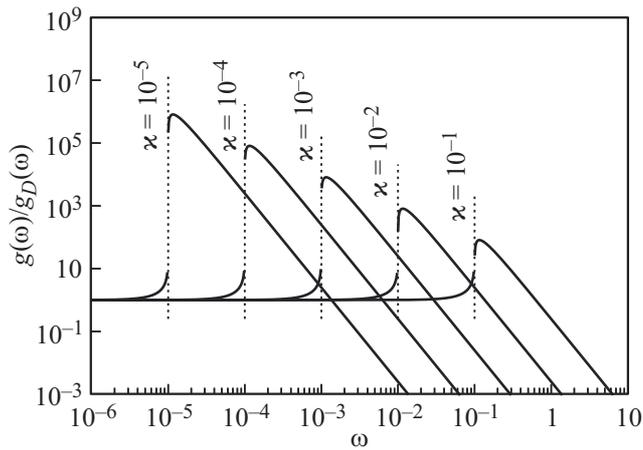
$$g_{\text{D}}(\omega) = \frac{s_d\omega^{d-1}}{(2\pi\Omega\sqrt{\kappa})^d}. \quad (22)$$

In the higher-frequency region  $\omega \lesssim \omega_c$  there is an excess over the Debye  $g_{\text{D}}(\omega) \propto \omega^{d-1}$  contribution to the density states

$$g_{\text{ph}}(\omega) - g_{\text{D}}(\omega) \sim \omega^{d+1}. \quad (23)$$

Such an increase in the density of vibrational states near the  $\omega_c$  Ioffe–Regel crossover is universal for systems with the dimension  $d \geq 3$ .

In the diffusion region of vibrations  $\omega > \omega_c$ , the dependence of the frequency on the wave vector is not a functional dependence similar to the dispersion relation for phonons (19). However, it follows from the statistical properties of the dynamic matrix and the correlation matrix that the density of vibrational states is determined by the imaginary part of the dynamic Young’s modulus [33]. Taking into account expression (14), the density of vibrational



The reduced density of vibrational states of the phonon region  $g_{\text{ph}}(\omega)/g_{\text{D}}(\omega)$  and the diffusion vibration region  $g_{\text{dif}}(\omega)/g_{\text{D}}(\omega)$  for different values of the  $\kappa$  parameter. The vertical dotted lines represent the crossover frequency  $\omega_c$ . Dimension of the system  $d = 4$ .

states in the region of diffusion of vibrations takes the following form:

$$g_{\text{dif}}(\omega) = \frac{2}{\pi\omega} \text{Im} E(\omega) = \frac{2a}{\pi\omega} \sqrt{\omega^2 - \omega_c^2}. \quad (24)$$

This equation corresponds to the Marchenko–Pastur law [41] for the classical uncorrelated Wishart ensemble. The sum rule is a necessary requirement for the existence of plane-wave phonons, so the density of vibrational states  $g_{\text{dif}}$  unconditionally describes diffusion vibrations.

The figure shows the density of vibrational states normalized to the Debye for the case  $d = 4$ . The resulting peak in the density of states is called the bosonic peak. As can be seen, with a decrease in the  $\kappa$  parameter, i.e., with an increase in disorder in the system, the height of the bosonic peak increases, and the crossover between the two vibration regions becomes sharper. There is a narrow smooth transition region between  $g_{\text{ph}}(\omega)$  and  $g_{\text{dif}}(\omega)$ , however it is small compared to  $\omega_c$  at  $\kappa \ll 1$ . For critical values of  $\omega = \omega_c$ , expressions (21) and (24) have asymptotic divergence.

The density of vibrational states determines, in particular, the damping  $\Gamma$ , in the following possible form [46]:

$$\Gamma(\mathbf{q}, \omega) = \frac{\pi}{2} \omega_{\text{cor}}^2(\mathbf{q}) g(\omega). \quad (25)$$

In the phonon region of vibrations, despite the absence of the imaginary part of the dynamic Young's modulus (14), damping  $\Gamma$  exists. Taking into account the expression for the density of states (21) and the dispersion relation (19), we can obtain the dependence of the vibration damping on the wave vector  $q$  in the phonon region  $q < q_c$ :

$$\Gamma(q) = \frac{\pi}{2} \Omega^2 q^2 g_{\text{ph}} = \frac{\pi s_d}{4a(2\pi)^d} q^{d+1} \frac{\sqrt{2q_c^2 - q^2}}{q_c^2 - q^2}. \quad (26)$$

It follows that in a wide range  $q \ll q_c$  the damping is determined by the Rayleigh scattering of phonons on disorder and contributes  $\Gamma \propto q^{d+1}$ . For critical values  $q = q_c$  there is an asymptotic divergence.

In the diffusion high-frequency vibration region  $q \gg q_c$  one can also find damping  $\Gamma$  depending on the wave vector  $q$ :

$$\Gamma(q) = \frac{\pi}{2} \Omega^2 q^2 g_{\text{dif}} \simeq Dq^2, \quad (27)$$

where the diffusion coefficient is  $D = \Omega^2 a$ . The obtained dependence  $\Gamma \propto q^2$  for diffusion vibrations is universal and does not depend on the dimension of the system. This diffusion feature is consistent with the results of experiments on inelastic X-ray scattering in glasses [11,12] and with the results of the molecular dynamics method for amorphous silicon [13].

## 7. Conclusion

By considering the statistical properties of a correlated Wishart ensemble of random matrices, which describes the vibrational properties of amorphous solids, taking into account their mechanical stability and translational invariance, in this work universal analytical expressions are obtained that describe the vibrational features of such systems, regardless of their dimension and the local arrangement of nearby atoms in them. Vibrations of disordered systems can be described using the equation of motion of the average projection of a plane wave  $u_{\mathbf{q}}(t)$  with a retarded part in the form of a dynamic complex Young's modulus  $E(\omega)$  (11). With this consideration, an important approximation is that the low-frequency vibrations of the medium under consideration are due to phonons with a linear dispersion law.

The work demonstrates the occurrence of the Ioffe–Regel crossover in amorphous systems with the dimension  $d \geq 3$  as a crossover between phonons described by a simple elasticity equation for the projection of atomic displacements (18) and diffusion vibrations described by the diffusion equation for the projection of velocities (20). The resulting crossover frequency  $\omega_c \cong \omega_{\text{IR}}$  (17) correlates linearly with the system parameter  $\kappa$ , which characterizes the macroscopic rigidity of the disordered system.

In the case of small values of  $\kappa \ll 1$ , the system has significant disorder and demonstrates an excess density of vibrational states  $g(\omega)$  compared to the Debye density of states  $g_{\text{D}}(\omega)$ . This leads to the universal appearance of a bosonic peak in the reduced density of vibrational states  $g(\omega)/g_{\text{D}}(\omega)$ . To illustrate the obtained formulas, a system with dimension  $d = 4$  was considered. The obtained oscillatory properties of a system with such a dimension are important for further research, for example, quasi-local vibrations. Within the framework of the random matrix approach, when considering the non-Gaussian statistics of the distribution of the elements of the matrix  $\hat{A}$ , quasi-local fluctuations can be obtained and studied [43].

The damping in the region of phonons demonstrates their Rayleigh scattering on disorder and contributes  $\Gamma \propto q^{d+1}$ . However, in the higher frequency region above the Ioffe–Regel crossover, the damping is  $\Gamma \propto q^2$ . Such a dependence is typical for diffusion hydrodynamic damping and has a universal character, regardless of the dimension of the system.

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### Conflict of interest

The author declares that there is no conflict of interest.

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