Viscous flow of two-component electron fluid in magnetic field

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In pure conductors with a low density of defects, frequent electron-electron collisions can lead to the formation of a viscous fluid consisting of conduction electrons. In this work, is studied magnetotransport in a viscous fluid consisting of two types of electrons, for which some of their parameters are different. The difference between such system and the one-component electron fluid is as follows. The scattering of electrons with their transitions from one component to another can lead to an imbalance in flows and concentrations, which affects the flow as a whole. In this work, the balance transport equations for such a system are constructed and solved for the case of a long sample with rough edges. The equation for the flow of the unbalance value towards the edges contains the bulk viscosity term. It is shown that in sufficiently wide samples, the transformation of particles into each other during scattering leads to the formation of a single viscous fluid flowing as a whole, while in narrow samples the two components flow as two independent fluids. The width of the sample at which this transition occurs is determined by the internal parameters of the fluid and the magnitude of magnetic field. The distributions of the flow of the fluid components over a sample cross section and the magnetoresistance of a sample are calculated. The latter turns out to be positive and saturating, corresponding to the transition with increasing of magnetic field from independent Poiseuille flows of the two components to the Poiseuille flow of a uniform fluid.

Keywords: electron fluid, viscosity, two-component system, nanostructures, magnetoresistance.

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1. Introduction

Electron–electron collisions in two- and three-dimensional conductors with a low density of defects may induce the formation of a viscous electron fluid and establish the hydrodynamic transport regime. Although the theory of a viscous electron fluid in solids has been developed extensively for a long period of time [1–3], unambiguous experimental evidence of the formation of such a fluid have been obtained only recently in high-quality graphene, Weyl semimetals, and quantum wells based on GaAs [4–27]. The conclusion regarding the presence of hydrodynamic flow in these experiments was made based on observations of the dependence of the average resistance of a sample on its width [4]; nonlocal negative resistance [5–7]; giant negative magnetoresistance [12–20]; and magnetic resonance at a doubled cyclotron frequency [21–27]. The spatial distribution of the current density and the Hall electric field in a flow of two-dimensional electrons in graphene strips has recently been measured directly in [10,11]. The experimental observation of hydrodynamic transport facilitated the development of its theory in various new directions (see, e.g., [28–42]).

Other types of hydrodynamic phenomena related to nonuniform distributions of particle flows throughout a sample may also be manifested in Ohmic samples, where the scattering of carriers off defects produces the dominant contribution to momentum relaxation. A consistent microscopic calculation of shear and bulk viscosities for two-dimensional electrons in a sample with defects of a small radius was performed in [39,40]. Hydrodynamic transport in two-component electron–hole systems was studied both in Ohmic samples with defects and in pure samples in [43–47]. It was demonstrated that near-edge flow layers with active carrier diffusion and recombination are formed in such systems. Specifically, the current through the sample may flow primarily along these layers in two-component systems with equal densities of electrons and holes, inducing a strong linear magnetoresistance. Such layers may form both in Ohmic samples with defects and in pure samples [43,46,47]. In the latter case, balance in near-edge layers is maintained by shear viscosity and recombination effects.

It follows from the results of analysis of data on the giant negative magnetoresistance that (see, e.g., [19,20]), although viscous flows of a one-component electron fluid are established in samples with a low defect density, the hydrodynamic transport regime is retained at low temperatures when electron–electron collisions become very rare. Apparently, the interelectron interaction in this regime remains important for the formation of a viscous fluid, but the relaxation of shear stresses in it, which defines the shear viscosity, proceeds [19,20] via the scattering of electron fluid quasiparticles off defects. The viscosity coefficients of electrons (or, in more exact terms, quasiparticles of an electron Fermi fluid) due to their scattering off defects are the quantities that govern hydrodynamic transport in this regime. The authors of [19,20] obtained estimates of this

„residual shear viscosity" by analyzing the experimental data from [14–18].

In the present study, a theory of magnetotransport in a viscous two-component fluid consisting of two types of electrons, which differ in their parameters and may transform into each other in the event of scattering, is developed. Such systems are found, e.g., in two-dimensional conductors in a strong tilted magnetic field, which induces spin splitting of the electron spectrum and the formation of two Zeeman-split subbands with different parameters. The intensity of scattering of electrons within each subband off each other is significantly higher than the intensity of scattering of electron pairs from different subbands.

Transport phenomena in such systems may be assigned to the spin Hall effect group. Related effects of the group of the spin Hall effect in a viscous electron fluid have been studied recently in [41,42] in zero and weak magnetic fields. In contrast to these studies, the parameters of two types of electrons (e.g., electrons in two strongly split Zeeman subbands) are assumed to differ greatly in the present research; therefore, these electrons have almost independent dynamics, with rare transformations into each other in the event of scattering.

The flow of a two-component electron fluid in a long sample with rough edges, where the momentum relaxation is effected via particle scattering off edge irregularities, is considered. Following [43–47], we construct balance equations with allowance for the emergence of a Hall electric field, shear viscosity, and weak scattering with particle type change (e.g., spin-flip transitions). It is demonstrated that independent Poiseuille flows of each fluid component form in sufficiently narrow samples. Owing to the transformation of fluid component flows into each other, the flow along a sufficiently wide sample is accompanied by emerging flows of particles of each type toward opposite sample edges. A deficiency or excess of fluid components is established near opposite edges, and the Poiseuille flow of a uniform two-component fluid forms in the bulk of a sample. The near-edge flow is governed by the balance between, on one side, the shear viscosity effect and, on the other side, transitions with particle type change and diffusion toward the edges. The obtained hydrodynamic equations demonstrate that the last two effects constitute a microscopic bulk viscosity mechanism for transport of a density imbalance of two fluid components.

Analytical formulae for the distributions of flows of fluid components and the net current are derived. The obtained formula for current yields a positive saturating magnetoresistance in the region of small fields where the dependence of viscosity coefficients on the magnetic field is insignificant. It is noted that the predicted positive magnetoresistance may constitute a mechanism (or one element of a mechanism) of suppression of the giant negative magnetoresistance by a tilted magnetic field and thus provide an explanation for the observations made in [14,23,27] in experiments with pure GaAs quantum wells.

2. Model

Let us consider the flow of a two-dimensional two-component electron fluid in a sufficiently narrow sample where the disorder scattering of electrons is relatively insignificant. Two types of electrons, $\alpha = 1,\ 2$, have different parameters: viscosity coefficients η_α, Fermi velocities $v_{\mathrm{F},\alpha}$, and concentrations $n_{0,\alpha}$. Magnetic field $\mathbf{B}$ is applied perpendicularly to the fluid layer, and electric field $\mathbf{E}_0$ is applied along the sample (see Fig. 1).

Such a system may be implemented, e.g., in a two-dimensional electron Fermi liquid of Fermi gas in a quantum well, which has both perpendicular magnetic field $\mathbf{B}$ and strong magnetic field $\mathbf{B}_\parallel$ in the layer plane applied to it. The latter filed splits the electron spectrum in spin by $\Delta \gg T$, where $T \ll \varepsilon_\mathrm{F}$ is temperature. The value of Δ may be much lower than Fermi energy ε_F or comparable to it. Therefore, index $\alpha =\uparrow, \downarrow$ here characterizes electrons with opposite spin directions (oriented along the quantization axis aligned with field $\mathbf{B}_\parallel$ or in the opposite direction). At low temperatures, electrons on two emerging Fermi surfaces have different Fermi velocities and, consequently, different concentrations, rates of electron-electron collisions, and viscosity coefficients. A quantum well with two filled size quantization levels or a double quantum well with a filled split ground level are another examples of a two-component electron system with rare scattering of two types of electrons off each other.

The fluid flow is characterized by flows $\mathbf{j}_\alpha(\mathbf{r}, t)$. Transitions with particle type change induce perturbations of densities of two types of particles $\delta n(\mathbf{r}, t)$ in this flow.

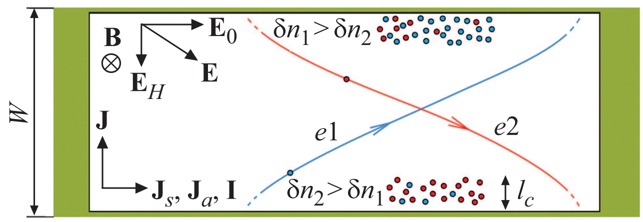

Figure 1. Long sample in magnetic field $\mathbf{B}$ with a two-component electron fluid. Current flow lines of fluid components $e1$ and $e2$ are shown. Longitudinal electric field $\mathbf{E}_0$ is applied, and Hall field $\mathbf{E}_\mathrm{H}(y)$ emerges due to the redistribution of densities $\delta n_{1,2}(y)$ of two fluid components. Electric current $\mathbf{I}$ and its constituent flows $\mathbf{J}_s$ and $\mathbf{J}_a$ are directed along the sample, while flow $\mathbf{J}$, which characterizes the transport of imbalance of densities of electrons $e1$ and $e2$, is perpendicular to the sample. An excess or deficient concentration of components $e1$ and $e2$ is established at longitudinal edges; therefore, intense particle diffusion and scattering (with particle type change) proceed there. These processes govern the bulk viscosity effect for flows along the normal to edges.

Following [43–47], we write the balance equations for the number and momentum of particles in the following way:

$$\frac{\partial \delta n_\alpha}{\partial t} + \operatorname{div} \mathbf{j}_\alpha = -(\Gamma_\alpha \delta n_\alpha - \Gamma_{\bar{\alpha}} \delta n_{\bar{\alpha}}),$$

$$\frac{\partial \mathbf{j}_\alpha}{\partial t} = \frac{e n_{0,\alpha}}{m} \mathbf{E} + \omega_c [\mathbf{j}_\alpha \times \mathbf{e}_z] - d_\alpha \nabla \delta n_\alpha + \eta_\alpha \Delta \mathbf{j}_\alpha. \quad (1)$$

Here, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator in layer plane xy; the bar over α denotes the opposite type of a particle: $\bar{e} = h$ and $\bar{h} = e$; m are the effective masses of two types of electrons, which are considered to be equal; $\omega_c = eB/(mc)$ are their cyclotron frequencies; compressibility coefficients $d_\alpha \sim v_{\mathrm{F},\alpha}^2$ define the amplitude of hydrostatic force ∇P_α expressed in terms of perturbation of particle densities δn_α (P_α are the contributions of fluid components to pressure); and $\eta_\alpha = \eta_{xx,\alpha}$ are the diagonal viscosity coefficients that depend on magnetic field [19] in accordance with formula $\eta_\alpha = \eta_{0,\alpha}/[1 + (2\omega_c \tau_{2,\alpha})^2]$, where $\eta_{0,\alpha}$ are the viscosity coefficients in zero field and $\tau_{2,\alpha}$ are the times of relaxation of shear stresses die to electron–electron collisions.

The rates of transitions with particle type change Γ_α in Eqs. (1) are proportional to the equilibrium concentrations of particles of the opposite type: $\Gamma_\alpha = \gamma n_{0,\bar{\alpha}}$, and coefficient γ is defined by the microscopic mechanism of such transitions. For example, scattering with „spin flip“ due to the spin-orbit interaction of electrons is such a mechanism for spin-split states. The terms with Hall viscosity coefficient $\eta_{xy,\alpha}[\Delta \mathbf{j}_\alpha \times \mathbf{e}_z]$, which yields only minor corrections in the Hall electric field in low-frequency flows [47], are neglected in Eqs. (1). Electric field $\mathbf{E}$ in Eqs. (1) contains the contribution from external applied field $\mathbf{E}_0$ and Hall contribution $\mathbf{E}_{\mathrm{H}}(\mathbf{r}, t)$, which emerges due to the redistribution of densities δn_α for compensation of the magnetic Lorentz force.

The criterion of applicability of Eqs. (1) is the smallness of spatial variation of all quantities relative to microscopic lengths $l_{2,\alpha} = v_{\mathrm{F},\alpha} \tau_{2,\alpha}$ and (or) cyclotron radii $R_{c,\alpha} = v_{\mathrm{F},\alpha}/\omega_{2,\alpha}$ (hydrodynamic approximation) and the slowness of collisions with particle type change. Specifically, times $\tau_s \sim 1/\Gamma_\alpha$ should be much longer than times $\tau_{2,\alpha}$ characterizing the shear viscosity [44].

It is also assumed that the scattering of one type of electrons off the other type is very weak. Therefore, the momentum relaxation in the bulk of the sample in collisions of electrons of different types, which leads to relaxation of flow difference $\mathbf{j}_1 - \mathbf{j}_2$, in neglected in Eqs. (1). It was demonstrated in [47] for an electron–hole system that, in order for the latter processes to be negligible in transport equations similar to (1), the following conditions regarding characteristic time τ_1 of these processes and the sample width need to be satisfied:

$$\tau_{2,\alpha} \ll \tau_s \ll \tau_1^2/\tau_{2,\alpha}, \quad W \ll l_{\mathrm{G}0}, \quad (2)$$

where $l_{\mathrm{G}0} \sim \sqrt{\eta_\alpha \tau_1}$ is the Gurzhi length that characterizes the relative intensities of momentum relaxation in collisions of different types of particles and in momentum diffusion due to shear viscosity and scattering off rough sample edges. Apparently, similar criteria apply to a system of two types of electrons with sufficiently different characteristics: $|n_{0,1} - n_{0,2}| \sim n_{0,\alpha}$, $|\eta_{0,1} - \eta_{0,2}| \sim \eta_{0,\alpha}$. Thus, only sufficiently narrow samples ($W \ll l_{\mathrm{G}0}$) are considered below, and the momentum relaxation in the bulk of the sample is neglected.

Note that the relaxation of quantity $\mathbf{j}_1 - \mathbf{j}_2$, which is the z component of the spin current tensor, for two-dimensional electrons in zero field or a sufficiently weak in-plane magnetic field (when Zeeman splitting Δ of subbands is smaller than temperature T or the quantum width of levels) is a fast process. Since characteristic time τ_1 of such a process is in the last case close in the order of magnitude to the quantum lifetime $\tau_{q,ee,\alpha} \ll \tau_{2,\alpha}$ [48–51], inequality (2) cannot be satisfied. However, the electrons of the types 1 and 2 interact only weakly with each other in the system considered here, and their states are not coherent. Therefore, the characteristic times of relaxation of their momentum and transformation into each other may be close in this system at rare collisions of these two types of particles: $\tau_s \sim \tau_1 \gg \tau_{2,\alpha}$, and inequality (2) is satisfied.

Let us consider the flow in a long sample with rough edges (see Fig. 1). The applied electric field is directed along the sample ($E_x = E_0$), while the Hall field is perpendicular to it: $E_y = E_{\mathrm{H}}(y)$. Frequent interelectron collisions, which define the shear viscosity effect, and rare events of scattering of electrons off each other with change of a particle type occur in the bulk of the sample. The scattering of electrons at edges, $y = \pm W/2$, is assumed to be purely diffusive: the momentum of reflected electrons is distributed isotropically. Such scattering may be characterized roughly by the following simplest boundary condition:

$$\mathbf{j}_\alpha \big|_{y=\pm W/2} = 0. \quad (3)$$

Let us examine a sample with a metallic gate at distance d from the layer with an electron fluid. The Hall electric field in such a system is related to charge density perturbation $e\delta n$ by a simple formula for a plane capacitor:

$$E_{\mathrm{H}}(y) = -\frac{4\pi d e}{\kappa} \frac{\partial \delta n}{\partial y}. \quad (4)$$

Here, $\delta n = \delta n_1 + \delta n_2$ is the total perturbation of the electron concentration and κ is the permittivity of a dielectric between the gate and the layer with an electron fluid. The derivative with respect to coordinate y is denoted hereinafter by a prime mark.

The balance equations for a steady-state flow in this geometry take the form

$$j'_{y,\alpha} = -\gamma (n_{0,\bar{\alpha}} \delta n_\alpha - n_{0,\alpha} \delta n_{\bar{\alpha}}),$$

$$\frac{e n_{0,\alpha}}{m} E_0 - \omega_c j_{y,\alpha} + \eta_\alpha j''_{x,\alpha} = 0,$$

$$\frac{e n_{0,\alpha}}{m} E_{\mathrm{H}}(y) + \omega_c j_{x,\alpha} - d_\alpha \delta n'_\alpha + \eta_\alpha j''_{y,\alpha} = 0. \quad (5)$$

In the case of degenerate electron statistics with a quadratic dispersion law, $\varepsilon_p = p^2/(2m)$, the coefficients in Eqs. (5) have the normal form: $d_\alpha = v_{\mathrm{F},\alpha}^2/2$, $n_{0,\alpha} = k_{\mathrm{F},\alpha}^2/(4\pi)$, $v_{\mathrm{F},\alpha} = \hbar k_{\mathrm{F},\alpha}/m$. Relaxation time $\tau_{2,\alpha}$ of shear stresses, which enters the formulae for viscosity coefficients $\eta_\alpha = (v_{\mathrm{F},\alpha}^2 \tau_{2,\alpha}/4)/[1+(2\omega_c\tau_{2,\alpha})^2]$, depends on temperature and the Fermi energy in the following way if the electron gas (or the electron liquid) are degenerate: $\tau_{2,\alpha} = A\varepsilon_{\mathrm{F},\alpha}/T^2$, where $\varepsilon_{\mathrm{F},\alpha} = mv_{\mathrm{F},\alpha}^2/2$ — are the Fermi energies of components and $A = A(\varepsilon_{\mathrm{F},\alpha}, T)$ is a function of temperature and interelectron interaction strength parameter $r_s = r_s(\varepsilon_{\mathrm{F},\alpha})$. This function takes values on the order of unity in typical GaAs quantum wells at temperatures $T \ll \varepsilon_{\mathrm{F},\alpha}$ [20].

3. Flow controlled by shear and bulk viscosities

Let us solve Eqs. (5) describing viscous flows of a two-component electron fluid in a long sample.

It follows from the balance equations for the particle number (the first line of (5)) that derivatives of flows $j'_{y,1}$ and $j'_{y,2}$ are opposite. In view of boundary conditions (3), we find that the following condition is satisfied within the entire sample:

$$j_{y,1}(y) = -j_{y,2}(y). \tag{6}$$

The physical meaning of this relation comes down to the assertion that electric current $e(j_{y,1}+j_{y,2})$ cannot flow in the cross section of the sample (along axis y). However, the „imbalance flow“ $J = j_{y,1}(y) - j_{y,2}(y)$, which amounts to transport of perturbed concentrations of fluid components, may be directed this way. This flow J relaxes in the vicinity of sample edges $y = \pm W/2$ due to the transition of particles from one type to the other (see Fig. 1). The imbalance flow for a two-component electron system formed by spin-split subbands is a component of the spin current of particles that corresponds to the spin component along field $\mathbf{B}_\parallel$ in the well plane in direction y. A detailed description of processes of this type for a two-component electron–hole system was given in [43–47].

To simplify Eqs. (5), we introduce the following quantities: $\delta\rho = (n_{0,2}\delta n_1 - n_{0,1}\delta n_2)/n_0$ is a quantity characterizing the imbalance of density perturbation; $1/\tau_s = 2\gamma n_0$ is the averaged rate of transitions between two fluid components; $n_0 = n_{0,1} + n_{0,2}$ is the overall equilibrium electron concentration; and $\rho_0 = n_{0,1} - n_{0,2}$ is the equilibrium difference between the concentrations of two types of electrons.

Condition $d \gg a_\mathrm{B}$, where a_B is the Bohr radius of two-dimensional electrons, needs to be satisfied in order for electrostatic relation (4) („smooth channel approximation“) to be applicable. It is easy to demonstrate (see, e.g., [26]) that this condition yields the following relation between the coefficients of Eq. (5): $e^2 n_{0,\alpha} d/(m\kappa) \ll d_\alpha$. Using this inequality and the notation introduced in the previous paragraph, we may rewrite Eqs. (5) in the following from:

$$J' = -\delta\rho/\tau_s,$$

$$en_0E_0/m + \eta_1 j''_{x,1} + \eta_{12} j''_{x,2} = 0,$$

$$e\varrho_0E_0/m - \omega_c J + \eta_1 j''_{x,1} - \eta_{12} j''_{x,2} = 0,$$

$$M\left(\frac{n_{0,1}}{\eta_1} + \frac{n_{0,2}}{\eta_2}\right)\delta n' + \omega_c\left(\frac{j_{1,x}}{\eta_1} + \frac{j_{2,x}}{\eta_2}\right) = 0,$$

$$M\left(\frac{n_{0,1}}{\eta_1} - \frac{n_{0,2}}{\eta_2}\right)\delta n' + \omega_c\left(\frac{j_{1,x}}{\eta_1} - \frac{j_{2,x}}{\eta_2}\right) = \left(\frac{d_1}{\eta_1}\delta n'_1 - \frac{d_2}{\eta_2}\delta n'_2\right) + J''. \tag{7}$$

Here, $M = 4\pi e^2 d/(m\kappa)$.

The first of these equations characterizes the relation between imbalance current J and concentration perturbation $\delta\rho$ imbalance. The second and the third equations characterize the transport of momentum of two fluid components along the sample (in direction x) due to the shear viscosity effect and the contribution of imbalance current J (in direction y) to this transport. The latter contribution emerges due to the magnetic Lorentz force and the difference in parameters of two fluid components. The fourth equation in (7) characterizes the balance of the magnetic Lorentz force and the electric force from the Hall field in direction y. The last equation in (7) defines the balance of flows in direction y driven by the electric, $\sim \delta n'$, and magnetic, $\sim j_{\alpha,x}$, forces; spatially nonuniform perturbations of concentrations $\delta n_\alpha(y)$, which lead to diffusion; and the influence of shear viscosity on the flows along y.

Let us derive the expressions for overall density perturbation δn and asymmetrized density perturbation $\delta\rho$ from the fourth and the first equations in (7), respectively, and insert the obtained quantities into the last (fifth) equation. The resulting equation characterizes the balance of the asymmetrized magnetic Lorentz force and the friction force for flow J from the bulk viscosity effect:

$$\omega_c \frac{n_{0,2}j_{1,x} - n_{0,1}j_{2,x}}{n_0} + \xi J'' = 0. \tag{8}$$

It was taken into account here that $\tau_s \gg \tau_{2,\alpha}$, and the „overall“ coefficient of bulk viscosity of a two-component fluid, which is controlled by the scattering with particle type change and the diffusion of concentration perturbation, was introduced:

$$\xi = \frac{\xi_1\eta_1 - \xi_2\eta_2}{\eta_1\eta_2}\,\frac{n_{0,1}\eta_2 - n_{0,2}\eta_1}{n_0}, \tag{9}$$

where „partial“ coefficients of bulk viscosity of each particle type take the form of $\xi_\alpha = d_\alpha\tau_s/2$. Since the criterion of applicability of the initial equations is $\tau_s \gg \tau_{2,\alpha}$, both ξ_α and overall bulk viscosity ξ are large comparted to shear viscosities η_α.

It follows directly from the second equation in (7) that the symmetrized density of flow along the sample

$J_s = (\eta_1 j_{x,1} + \eta_2 j_{x,2})/\eta$ is calculated independently of the other quantities and has the Poiseuille profile:

$$J_s(y) = \frac{en_0}{m}\frac{E_0}{2\eta}\left[\left(\frac{W}{2}\right)^2 - y^2\right], \qquad (10)$$

where $\eta = \eta_1 + \eta_2$.

The obtained Eq. (8) and the third equation in (7) need to be used to find some asymmetric linear combination, J_a, of flows j_1 and j_2 and imbalance current J. It is convenient to choose $J_a = (\eta_1 j_{x,1} - \eta_2 j_{x,2})/\eta$, which appears in the third equation in (7), as such a combination. The third equation in (7) and Eqs. (8) and (10) then yield the following final equation for the imbalance current:

$$\frac{e\Delta n_0}{m}E_0 - \omega_c J - \frac{\xi\eta}{a_s\omega_c}J'''' = 0, \qquad (11)$$

and boundary conditions $J|_{y=\pm W/2} = 0$ and $J''|_{y=\pm W/2} = 0$. Additional notation was introduced here:

$$\Delta n_0 = \frac{2n_{0,1}n_{0,2}(\eta_2 - \eta_1)}{n_{0,1}\eta_1 + n_{0,2}\eta_2},\ a_s = \frac{\eta}{n_0}\left(\frac{n_{0,2}}{\eta_1} + \frac{n_{0,1}}{\eta_2}\right). \qquad (12)$$

The solution of the formulated boundary problem for function $J(y)$ takes the form

$$J(y) = \frac{e\Delta n_0 E_0}{m\omega_c} \times \left[1 - \frac{\cosh(\sqrt{i}\lambda y)}{2\cosh(\sqrt{i}\lambda W/2)} - \frac{\cosh(\sqrt{-i}\lambda y)}{2\cosh(\sqrt{-i}\lambda W/2)}\right]. \qquad (13)$$

Number λ in this formula is the modulus of eigenvalues of Eq. (11). It depends on the magnetic field and the overall coefficients of bulk and shear viscosities:

$$\lambda = \sqrt{\omega_c}\sqrt[4]{\frac{a_s}{\xi\eta}}. \qquad (14)$$

This number specifies the widths of the near-edge regions, $l_c = 1/\lambda$, where intense diffusion and particle type transformations, which define the bulk viscosity, and diffuse transport of x-components of momenta of fluid components in direction y due to the shear viscosity effect occur.

Note that intrinsic length l_c is estimated as $\sqrt{R_{c,\alpha}}\sqrt[4]{l_{s,\alpha}l_{2,\alpha}}$ in a weak magnetic field, when $R_{c,\alpha} \gg l_{2,\alpha}$, at the parameters of two types of particles being of the same order of magnitude (here, $l_{2,\alpha} = v_{\mathrm{F},\alpha}\tau_{2,\alpha}$ and $l_{s,\alpha} = v_{\mathrm{F},\alpha}\tau_s$ are the relaxation length of shear stress and the relaxation length with respect to transitions $1 \leftrightarrow 2$). Since $\tau_s \gg \tau_{2,\alpha}$, length l_c is much greater than length $l_{2,\alpha}$; this is exactly the relation needed for the hydrodynamic equations of the model to be applicable.

Asymmetric combination of longitudinal flows J_a is expressed in terms of symmetric combination J_s and imbalance flow J in the following way:

$$J_a = -(a_a/a_s)J_s - [\xi/(a_s\omega_c)]J'', \qquad (15)$$

where $a_a = \eta/n_0[(n_{0,2}/\eta_1) - (n_{0,1}/\eta_2)]$. The maximum amplitude of flow J_s at the channel center and the amplitudes of flows J and J_a near the channel center are related each to others (in order of magnitude) as the following quantities: W^2/η, $1/\omega_c$, $\sqrt{\tau_s/\tau_{2,\alpha}}/\omega_c$ (under the condition that $\Delta n_0 \sim n_0$ and $a_a \sim a_s$).

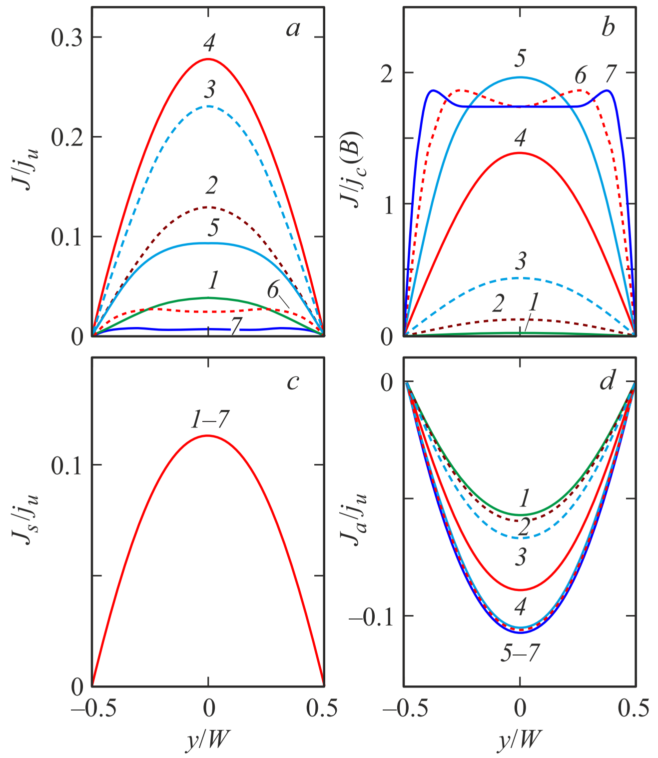

Figure 2. Distributions of imbalance flow J (a, b), symmetric J_s (c) and asymmetric J_a (d) flow components over the sample cross section for several values of magnetic field B corresponding to different ratios between the width l_c of the near-wall layers and the sample width W. These quantities are plotted in characteristic scales of flows $j_u = en_uE_0/(m\omega_u)$ (panels $(a), (c), (d)$) and $j_c(B) = j_u\omega_u/\omega_c$ (panel (b)). Here, $\omega_u = \eta_u/W^2$, n_u, and η_u are the units of measurement of frequency, concentration, and viscosity, the specific values of which do not affect the profile shape. The curves are plotted for the following parameters of two components of an electron fluid: $n_1/n_u = 1$, $n_2/n_u = 3$, $\eta_1/\eta_u = 0.4$, $\eta_1/\eta_u = 4$, $\xi/\eta_u = 29$. Curves *1–7* in each panel correspond to the following ratios ω_c/ω_u, which set the magnitudes of magnetic field: $\omega_c/\omega_u = 2.5$, 9, 19, 50, 210, 700, 2700. The following widths of the near-edge layers correspond to these field values: $l_c/W = 1.2$, 0.66, 0.45, 0.28, 0,14, 0.074, 0.038. Curves *1–7* in panel (c) are all matching.

In what follows, we consider the range of magnetic fields where the field dependence of coefficients of shear viscosity η_α is insignificant, which corresponds to the condition $\omega_c\tau_{2,\alpha} \ll 1$.

Figure 2 shows the distributions of flows J, J_s, J_a over the sample profile at several values of magnetic field B and certain values of shear viscosities η_α, concentrations $n_{0,\alpha}$, and the overall bulk viscosity ξ. It can be seen that the imbalance flow for narrow samples and weak fields, $W \lesssim l_c$, has a parabolic profile and increases with magnetic field.

The imbalance current for wide samples and sufficiently strong fields, $W \gg l_c$, decreases with increasing field and assumes a flat profile in the bulk of the sample; at the same time, it oscillates in near-edge layers with a thickness on the order of l_c and turns to zero at $y = \pm W/2$ (see Figs. 2, *a, b*). Thus, there exists a sample width $W \sim l_c$ at which amplitude J reaches its maximum (see the inset of Fig. 3). The profiles of flow J_s are exactly parabolic and independent of the magnetic field. The profile of J_a are almost parabolic, and their amplitude depends strongly on the magnetic field (see Figs. 2, *c, d*).

With the flows J, J_s, J_a determined, one may calculate the contributions of each fluid component to the flow density: $j_1 = (\eta/\eta_1)(J_s + J_a)/2$ and $j_2 = (\eta/\eta_2)(J_s - J_a)/2$. These values were, in turn, used to calculate the net electric current, $I = e\int_{-W/2}^{W/2} dy[j_1(y) + j_2(y)]$, and the sample resistance per unit width, $R = E_0 W/I$. The following expression was obtained:

$$I = \frac{e n_0 E_0/m}{(n_{0,1}/n_0)\eta_1 + (n_{0,2}/n_0)\eta_2}\left\{\frac{W^3}{12} + \frac{2\xi}{\omega_c^2} \times \frac{n_{0,1}n_{0,2}(\eta_1-\eta_2)^2}{n_0(n_{0,1}\eta_1 + n_{0,2}\eta_2)}\,\mathrm{Re}\left[\sqrt{i}\tanh\left(\sqrt{i}\,\frac{\lambda W}{2}\right)\right]\right\}. \quad (16)$$

It can be seen that the term with hyperbolic tangent is small compared to the first term in curly brackets in samples with their width being much greater than the characteristic near-edge layer width: $W \gg l_c$, $l_c = 1/\lambda$. Formula (16) thus yields the following expression:

$$I_w = \frac{eE_0}{M}\frac{W^3}{12}\frac{n_0}{(n_{0,1}/n_0)\eta_1 + (n_{0,2}/n_0)\eta_2}, \quad (17)$$

which corresponds to the Poiseuille flow of a fluid with shear viscosity $\eta_{\rm tot} = (n_{0,1}/n_0)\eta_1 + (n_{0,2}/n_0)\eta_2$ (viscosity of a uniform two-component fluid with partial contributions from two components). Frequent transformations of particles type 1 and 2 into each other occur in this regime near the sample edges, and imbalance flux $J(y) \approx \mathrm{const}$ emerges in the bulk: particles of the first type move toward one edge, while particles of the second type move toward the opposite edge (see Eqs. (6) and (13) and Figs. 1 and 2).

It has been demonstrated that formula (16) for narrow samples, $W \ll l_c$, yields the following expression for current I, which consists of a sum of contributions from two independent Poiseuille flows of particles type 1 and 2:

$$I_{\rm sh} = \frac{eE_0}{M}\frac{W^3}{12}\left(\frac{n_{0,1}}{\eta_1} + \frac{n_{0,2}}{\eta_2}\right). \quad (18)$$

It follows from analysis that the value of I_w is always lower than the value of $I_{\rm sh}$. Therefore, the averaged sample resistivity R increases from $E_0W/I_{\rm sh}$ to E_0W/I_w as magnetic field B grows stronger and width $l_c(B) \sim 1/B^{1/2}$ of the near-edge region decreases from the case of $W \ll l_c$ to $W \gg l_c$. We remind that only the limit of weak magnetic fields, $\omega_c\tau_{2,\alpha} \ll 1$, with small variations of the diagonal shear viscosity coefficient η_1 are considered here. It can be demonstrated that the latter condition is congruent with inequality $W \gg l_c$ at sample width $\tau_{2,\alpha}^{3/4}\tau_s^{1/4} \ll W/v_{{\rm F},\alpha} \ll \tau_{2,\alpha}^{1/2}\tau_1^{1/2}$ (it is assumed here that the difference between the parameters of two types of electrons is on the order of these parameters themselves: $|\eta_1 - \eta_2| \sim \eta_1 \sim \eta_2$ and $|n_{0,1} - n_{0,2}| \sim n_{0,1} \sim n_{0,2}$).

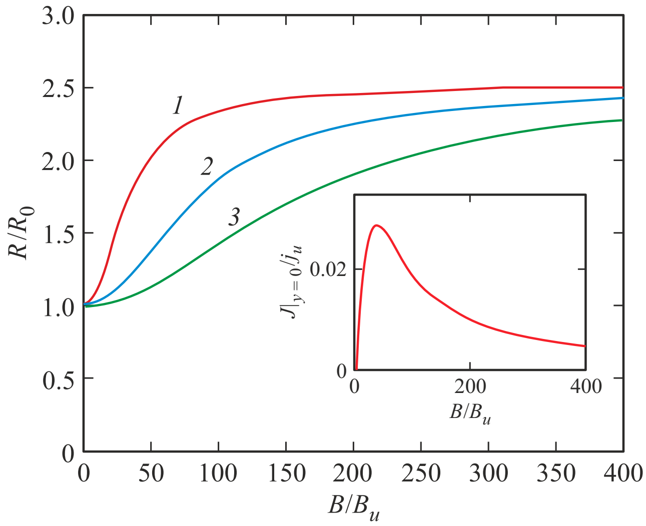

Figure 3. Dependence of averaged sample reistivity R on magnetic field B for several values of bulk viscosity ξ. Magnetic field B_0 of transition from the flow regime with independent fluid components *1* and *2* to the regime of unification of components due to scattering with particle type change is characterized by equality $l_c(B,\xi) = W$ and formula (19). Resistance R is plotted in units of $R_0 = E_0W/I_{\rm sh}$. The magnetic field is plotted in units of $B_u = mc\omega_u/e$. Curves *1–3* correspond to the values of the bulk viscosity $\xi/\eta_u = 8.7$, 58, 203; the values of other parameters are the same as in Fig. 2. The field dependence of imbalance current $J|_{y=0}$ at the sample center is shown in the inset. This plot demonstrates that the absolute value of the imbalance current induced by scattering with electron type change is small in the limit of both strong, $B \gg B_0$, and weak, $B \ll B_0$, magnetic fields.

Thus, a transition between the regimes described above (the regime of the independent flows of two fluid components and the regime with the imbalance flow J between the near-edge regions where $1 \leftrightarrow 2$ transformations proceed) occurs near the magnetic field at which the quantity $l_c = l_c(B)$ assumes a value of W with the increase of magnetic field. Figure 3 presents the dependences of the averaged resistivity R on magnetic field B at several values of the bulk viscosity ξ. The transition between the mentioned regimes is a smooth crossover that occurs at the magnetic field $B_0(W) = mc\omega_{c,0}/e$ corresponding to the borderline cyclotron frequency:

$$\omega_{c,0} = \sqrt{\frac{\xi\eta}{a_s}}\,\frac{1}{W^2}. \quad (19)$$

The obtained magnetoresistance is similar in nature to the positive saturating magnetoresistance of two-component systems in bulk Ohmic samples with defects (see discussion in [44]). A transition from independent uniform Ohmic flows of two carrier types (e.g., electrons and holes) to the flow of two types of recombinating carries occurs in such systems as the magnetic field intensifies. Owing to recombination and diffusion, the magnetic field affects flows in the central part of a wide sample and in the near-edge layers in different ways; this induces spatial flow nonuniformity and a positive saturating magnetoresistance in wide samples.

4. Discussion of results

Let us, first of all, point out the following uncommon feature of the obtained result. As magnetic field B grows from values being much lower than transition field B_0 to values much higher than it, longitudinal current I undergoes a significant variation from I_{sh} to I_w, although transverse imbalance current J and the scattering with electron type change in the central part of a sample, which produces the dominant contribution to the change of the longitudinal current, are relatively weak within the indicated range. The reasons behind this are as follows.

The scattering with electron type change in zero magnetic field is much less probable than collisions of electrons of one and the same type (it is worth reminding that sufficiently narrow samples, $W \ll l_{G0}$, with insignificant momentum relaxation in collisions of electrons of different types are considered here). Therefore, the processes of shear momentum transport of two types of electrons do not merge into a single process at $B = 0$, and each component forms an independent Poiseuille flow with its own viscosity η_α.

The characteristic distance traveled by an electron of a certain type before transforming into another particle in diffusion due to collisions with electrons of the same type is estimated in zero field as $v_{\text{F}}\sqrt{\tau_2\tau_s}$ (here and elsewhere in the text below, index α is omitted for brevity).

A magnetic field is needed to mix longitudinal and transverse flow components. The above analysis and the solution of hydrodynamic equations (7) suggest that the characteristic distance sufficient to complete all processes (including field-induced flow turning) needed to obtain a strong effect of scattering with type change on the net current is estimated as $l_c \sim \sqrt{R_c v_{\text{F}}}\,\sqrt[4]{\tau_2\tau_s}$. If length l_c is comparable to (or smaller than) sample width W, the time interval of electron diffusion between edges is sufficiently long for electrons to undergo scattering with type change. Herewith longitudinal and transverse flows also are mixed under the influence of the magnetic Lorentz force.

In moderate fields with $l_c \sim W$, the mixing of longitudinal and transverse flows is significant within the entire sample, and the contribution of the discussed processes to current I is large. The case of strong magnetic fields with $l_c \ll W$ is more complicated. The mixing of x and y flow components and the bulk viscosity effect are significant only in near-edge regions $W/2-|y| \lesssim l_c$. Although in the central region, $W/2-|y| \gg l_c$, electrons manage to undergo transitions with type change before reaching the edges, these processes do not contribute to flows J_s and J_a (owing to the spatial uniformity of flow J). Indeed, the values of flows J_s, J_a, and J in this region are derived from the balance condition for the Hall field force, for the Lorenz force, and for the forces of shear friction and the longitudinal electric field (this follows from Eqs. (7) at $J = \text{const}$). Thus, J_s, J_a, and J in the central region are not related to transitions with electron type change and the bulk viscosity. The amplitude of J is finite, but small compared to those of J_s and J_a (see formula (15) and explanation below it).

Thus, transitions with particle type change and the bulk viscosity in this regime may be called a „catalysts“ of the unification of electrons into a uniform fluid in wide samples, $W \gg l_c$. The characterized nature of flow is reflected in the fact that current I is independent of the bulk viscosity coefficient in the limit of both weak and strong magnetic fields. The discussed effect defines only the position and the width of the transition region (see formulae (17), (18) and Fig. 3). Note that catalysts in chemical reactions behave in a similar way: they affect the reaction rate, but are not present in the initial and final substances.

Let us now turn our attention to the probable experimental manifestations of the discussed effects.

Magnetotransport in high-purity GaAs quantum wells with a magnetic field tilted by different angles applied to them was examined experimentally in [14,23,27]. A strong negative („giant“) magnetoresistance in a moderate magnetic field perpendicular to the two-dimensional electron plane was observed. This effect was attributed in [19] to the formation of a viscous electron fluid of two-dimensional electrons in a well and to the field dependence of their shear viscosity. The application of a magnetic field component in the well plane in experiments [14,23,27] resulted in a rather fast suppression of the giant negative magnetoresistance and the emergence of a positive magnetoresistance. This suppression is considered to be an important characteristic property of the giant negative magnetoresistance, but remained unexplained (see, e.g., discussion in [52]).

Our hypothesis is that the emergence of Zeeman splitting of states of two-dimensional electrons and the formation of two weakly coupled components of an electron fluid with opposite spins and differing parameters may contribute to the observed suppression of the negative magnetoresistance in the presence of an in-plane magnetic field component.

Indeed, the Fermi velocities, densities, relaxation times, and viscosity coefficients of electrons in spin-split subbands may differ considerably at low temperatures in sufficiently strong in-plane fields. Thus, a one-component electron fluid in quantum wells becomes a two-component one that may sustain various types of flows (including the ones considered in the present study).

The obtained results demonstrate that the magnetoresistance of a two-component fluid in sufficiently narrow samples, $\tau_2^{3/4}\tau_s^{1/4} \ll W/v_F \ll \tau_2^{1/2}\tau_1^{1/2}$, and in relatively weak magnetic fields, $\omega_c\tau_2 \ll 1$, becomes positive due to the effect of shear viscosity and rare spin-flip scattering events, which give rise to bulk viscosity for the imbalance flow. The amplitude of this magnetoresistance is specified by the difference in parameters of electrons in two Zeeman-split subbands (thus, the strength of magnetoresistance ultimately defined by the magnetic field component $B_\parallel$ in the well plane).

In order to apply this reasoning to the experimental results reported in [14,23,27], one needs to investigate the issue of coexistence of the giant negative magnetoresistance due to the dependence of shear viscosities η_α on perpendicular field component B_z [19] and the positive magnetoresistance examined in the present study. Apparently, this requires calculating the influence of Zeeman splitting and rearrangement of the electron spectrum in a quantum well with increasing $B_\parallel$ on relaxation times τ_2 and τ_s. It appear to be a promising avenue for further research to perform such a calculation and compare its results with experimental data [14,23,27].

5. Conclusion

Flows of a viscous two-component electron fluid in a magnetic field in long samples with rough edges were examined. It was demonstrated that the transition of electrons from one fluid component to the other at scattering results in the formation of a uniform viscous fluid, which flows as a whole, in sufficiently wide samples. In narrow samples, two fluid components flow independently. The sample width corresponding to this transition is defined by the magnetic field and internal fluid parameters (specifically, bulk viscosity). The distributions of flows of fluid components and the sample magnetoresistance were calculated. The magnetoresistance was found to be positive and saturating. The section of a rapid increase in magnetoresistance corresponds to the transition between two flow regimes characterized above.

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Conflict of interest

The authors declares that he has no conflict of interest.

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