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Characteristic function of a self–similar random process

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A stochastic differential equation is proposed for a characteristic function whose inverse function describes a self–similar random process with a power–law behavior of power spectra in a wide frequency range and a power–law amplitude distribution function. Gaussian „tails" for the characteristic distribution make it possible to evaluate its stability according to the formulas of classical statistics using the maximum of the Gibbs–Shannon entropy and, therefore, the stability of a random process given by an inverse function.

Keywords: self–similar random processes, stochastic equations, power spectrum, $1/f$-noise, maximum entropy.

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Random processes with large fluctuations are self–similar and are characterized by power–law dependences of spectral density and amplitude distributions. Most of papers describe the self–similar random processes based on fractional integration of white noise: fractional diffusion equation, diffusion on fractional structures [1–4]. As a rule, random processes being obtained in such a simulation are non–stationary. Analysis of stability of complex physical systems with power–law distributions show that the statistical Gibbs–Shannon entropy does not ensure agreement with the principle of entropy maximum [5–7]. This paper proposes another approach to describing self–similar random processes with large fluctuations which is based on a set of nonlinear stochastic equations as in [8,9]; this approach makes the calculation procedure essentially shorter and simpler than that involving fractional integration. The set of stochastic equations looks as follows:

$$\frac{d\varphi}{dt} = -\varphi\psi^2 + \psi + \xi_1(t),$$

$$\frac{d\psi}{dt} = -\psi\varphi^2 + 2\varphi + \xi_2(t), \qquad (1)$$

where φ and ψ are the dynamic variables, ξ_1 and ξ_2 are the Gaussian δ-correlated noises with a zero mean and amplitudes σ_1 and σ_2, respectively. Equation set (1) describes interaction of large and small fluctuations in the critical region in the event of arising of wideband noise with low–frequency energy bursts. The second equation of set (1) is the master one, the first equation is the auxiliary one. Solution of the second equation of this set describes evolution of fluctuations obeying classical statistics, Gaussian distribution and exponential relaxation (variable ψ). Solution of the first equation gives a random function of large fluctuations with the power–law distribution and delayed relaxation (variable φ). Gaussian behavior of the master variable „tails" makes it possible to evaluate the random process stability via the classical statistics formulas with the use of the principle of Gibbs–Shannon entropy maximum.

In case of the critical value of noise intensity ($\sigma_1 = \sigma_2 \approx 1$), power spectrum of variable φ in equation set (1) takes the form $S_\varphi \sim 1/f$. At high frequencies, the variable ψ spectrum has the form $S_\psi = 1/f^2$. In numerical calculations with infinitely small time step Δt, the random process with the $1/f$-spectrum is nonstationary [9]. With increasing step Δt, the process becomes stationary, and power spectra begin exhibit a horizontal plateau at low frequencies. The less is the integration step, the wider is the stationary behavior frequency range [9]. Therefore, set (1) presented in the form of finite differences is applicable to stationary random processes with a finite high frequency.

Numerical calculations show that the spectrum of a random function $1/\psi(t)$ inverse to $\psi(t)$ is inversely proportional to the first degree of frequency $S_{1/\psi} \sim 1/f$ and coincides with spectrum S_φ of variable φ. Knowing properties of the set (1) numerical solutions, it is possible to obtaine based on it a master stochastic equation in classical variable ψ and to define variable φ as a quantity inverse to ψ [10]:

$$\varphi(t) = \frac{1}{\psi(t)} + \theta(t), \qquad (2)$$

where $\theta(t)$ is a certain random function with dispersion σ_θ^2. Substituting equation (2) into the second expression of set (1), it is possible to rewrite the equation for $\psi(t)$:

$$\frac{d\psi}{dt} = \frac{1}{\psi} - \theta^2\psi + \xi(t). \qquad (3)$$

Numerical calculations show that random function $\theta(t)$ is close to white noise; hence, it is possible to accept approximation in which standard deviation is expressed as $\langle\theta^2\rangle = \sigma_\theta^2$. Replacing squared function θ^2 in (3) whith σ_θ^2, obtain stochastic equation

$$\frac{d\psi}{dt} = \frac{1}{\psi} - \sigma_\theta^2\psi + \xi(t). \qquad (4)$$

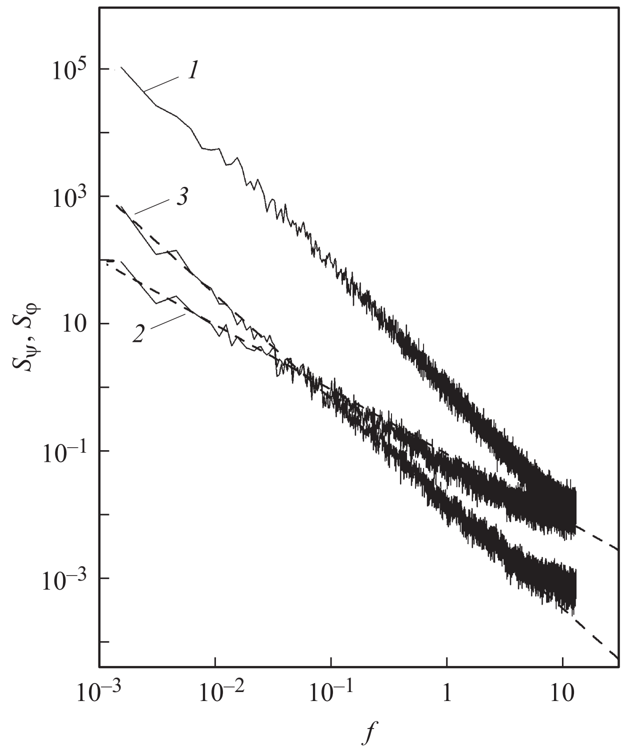

Figure 1. Power spectra of characteristic function $S_\psi \sim 1/f^2$ (*1*) and self–similar random processes $S_\varphi \sim 1/f$ (*2*) and $S_\varphi \sim 1/f^{5/3}$ (*3*).

Equation (4) describes random walk in a force field with potential

$$U = -\ln|\psi| + \sigma_\theta^2 \psi^2, \tag{5}$$

that is logarithmic at low ψ and parabolic at high ψ. The ψ function spectrum has the form $S_\psi = 1/f^2$. White noise was simulated by a sequence of Gaussian random numbers. Standard deviation $\sigma_\theta^2 = \langle\theta^2\rangle = \sigma^2\Delta t$ depends on the Δt subdivision [11]. Define function $\varphi(t)$ as $\varphi = \psi/(\varepsilon + \psi^2)$ where ε is a small constant preventing divergence of inverse function $1/\psi$ when $\psi(t)$ approaches zero in numerical calculations of random processes. Thus, it is possible to write the following set of equations:

$$\varphi = \frac{\psi}{\varepsilon + \psi^2} + \xi(t),$$

$$\frac{d\psi}{dt} = \frac{1}{\psi} - \sigma^2 \Delta t \psi + \xi(t). \tag{6}$$

The first equation describes a function inverse to the characteristic one to which white noise $\xi(t)$ is added. The function $\varphi(t)$ power spectrum is defined as $S_\varphi \sim 1/f$. If the slope of spectrum $S_\varphi \sim 1/f^\alpha$ frequency dependence is different, the inverse function power index will be different, and white noise intensity will be also different. For instance, for the Kolmogorov turbulence $S_\varphi \sim 1/f^{5/3}$ [12,13] it should be assumed that $\varphi = \psi/(\varepsilon + \psi^2)^{0.7}$ and $\xi(t) = 0$. Equations (6) are independent; as the master equation, the second one is regarded. The master equation of set (6) is in line with the Fokker–Planck equation whose stationary solution is as follows:

$$P(\psi) \sim \exp\left(-\frac{U(\psi)}{\sigma^2}\right) = \exp\left(-\frac{\ln|\psi|}{\sigma^2}\right)\exp\left(-\frac{\sigma_\theta^2\psi^2}{\sigma^2}\right)$$

$$= \psi^{\sigma^{-2}} \exp(-\psi^2\Delta t). \tag{7}$$

Distribution function of inverse quantity $\varphi = 1/\psi$ will be defined as

$$P(\varphi) \sim \frac{1}{\varphi^{\sigma^{-2}+2}} \exp\left(-\frac{\Delta t}{\varphi^2}\right). \tag{8}$$

Equations (7) and (8) show that the variable ψ distribution function decreases exponentially at large arguments, while the decrease of $P(\varphi)$ distribution function obeys at high φ the power law. Fig. 1 presents spectrum $S_\psi \sim 1/f^2$ and spectra $S_\varphi \sim 1/f$, $S_\varphi \sim 1/f^{5/3}$ obtained from the set (6) numerical solutions. The lower is Δt in numerical calculations, the more accurate is the white noise approximation with a sequence of Gaussian random numbers; this allows extending the frequency range of the $1/f$ and $1/f^2$ dependences in spectra S_φ and S_ψ. Fig. 2 presents in logarithmic coordinates the $P(\psi)$ and $P(\varphi)$ distribution functions obtained from the set (6) numerical solutions. At large φ, the $P(\varphi)$ distribution function obeys the power law. The behavior of power spectra and distribution functions obtained from the set (6) solutions almost fully coincides with that of the solution obtained from the stochastic equation set (1) [14].

The exponential decrease of function $P(\psi)$ makes it possible to use the Gibbs–Shannon entropy expression [7,15]

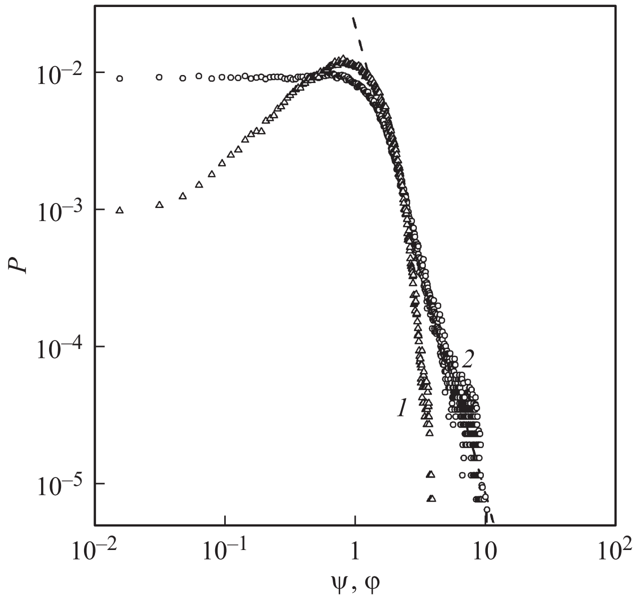

Figure 2. Distribution functions $P(\psi)$ (*1*) and $P(\varphi)$ (*2*) obtained from the set (6) numerical solutions. The dashed line represents the $P \sim \varphi^{-3}$ dependence.

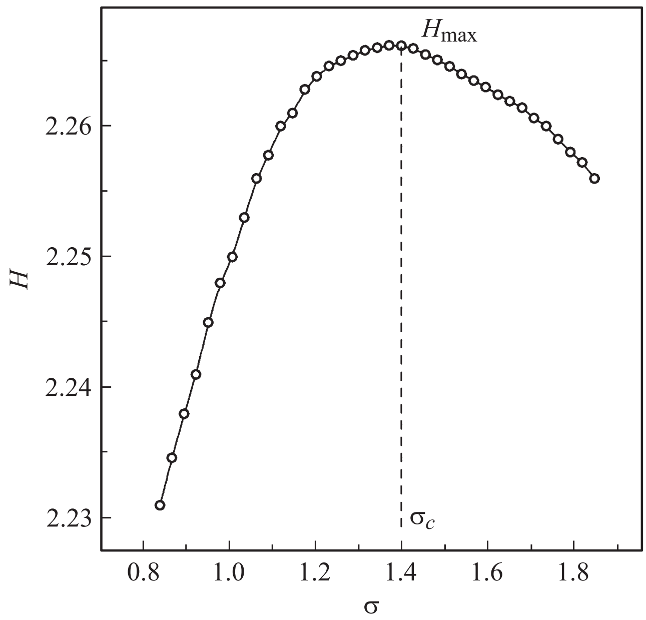

Figure 3. Gibbs–Shannon entropy H for squared characteristic function $\psi^2(t)$ versus the noise amplitude.

in evaluating the system (6) stability

$$H = -\sum_n P_n \lg P_n. \tag{9}$$

Fig. 3 presents the calculated dependence of entropy H for squared variable $\psi^2(t)$ on the noise amplitude. The entropy maximum corresponds to the critical noise amplitude ($\sigma_c \approx 1.4$) at which random process $\varphi(t)$ becomes maximally stable. The critical noise amplitude derived from the set (6) second equation is $\sqrt{2}$ times higher than in the case of modeling the self–similar process by two stochastic equations of set (1) and, respectively, than 2D white noise.

The paper proposes a stochastic differential equation whose solution is a characteristic function with the Gaussian distribution; its inverse quantity describes a self–similar random process with a power–law distribution. Gaussian „tail" of the characteristic function allows applying it in analyzing the self–similar process stability via the formulas of classical statistics.

Conflict of interests

The authors declare that they have no conflict of interests.

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