11.1

Analytical solution of the problem of synthesis of three-link stepped Chebyshev's microwave filter

A.S. Aref'ev

Povolzhsky State University of Telecommunications and Informatics, Samara, Russia
E-mail: arefyev.as@inbox.ru

Received December 7, 2021
Revised December 14, 2021
Accepted December 16, 2021

The problem of synthesis of the three-link stepped Chebyshev's microwave filter is reduced to two independent fourth-degree equations, including a single link wave impedance as unknown. The solution of Descartes –Euler is applied to these equations. It is proved that, in case wave impedances of extreme links are equal, the problem of the filter synthesis has two solutions. Identical phase-frequency responses correspond to these solutions. It is proved that for each link a product of the wave impedances relating to these solutions is equal to a square of the wave impedance of the transmission line including the filter.

Keywords: Microwave filter, stepped microwave filter, Chebyshev's microwave filter, synthesis of microwave filter.

DOI: 10.21883/TPL.2022.03.52882.19101

The stepped microwave filter is a cascade structure of regular transmission line sections (links) with different wave impedances [1–4]. There is a numerical technique providing an exact solution of the task of synthesizing such a device with an arbitrary number of links having equal electrical lengths [5]. Wave impedances of the filter links may be expressed via the parameters of the prototype stepped waveguide junction. Paper [1] presents the fourth-degree equation to which the synthesis of the three-link junction with the Chebyshev's frequency response may be reduced. The analytical solution of the synthesis task described in this paper allows rigorous substantiation of a number of properties of stepped Chebyshev's microwave filters as exemplified by a simplest three-link structure.

The figure demonstrates a schematic diagram of the three-stepped microwave filter. Here it is assumed that the link wave impedances ρ_1, ρ_2, ρ_3 and the wave impedance ρ_0 of the transmission line comprising the filter are frequency independent, while the link electrical lengths θ are equal to each other. Element T_{11} of the filter transmission wave matrix (transmission factor) is defined as follows [1]:

$$T_{11} = (1/2)[A_{11} + (1/\rho_0)A_{12} + \rho_0 A_{21} + A_{22}]. \quad (1)$$

Here A_{ij} are the filter transmission matrix elements defined as

$$A_{11} = \cos(\theta)\left[\cos^2(\theta) - \left(\frac{\rho_1}{\rho_2} + \frac{\rho_1}{\rho_3} + \frac{\rho_2}{\rho_3}\right)\sin^2(\theta)\right],$$

$$A_{12} = i\sin(\theta)\left[(\rho_1 + \rho_2 + \rho_3)\cos^2(\theta) - \frac{\rho_1\rho_3}{\rho_2}\sin^2(\theta)\right],$$

$$A_{21} = i\sin(\theta)\left[\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3}\right)\cos^2(\theta) - \frac{\rho_2}{\rho_1\rho_3}\sin^2(\theta)\right],$$

$$A_{22} = \cos(\theta)\left[\cos^2(\theta) - \left(\frac{\rho_2}{\rho_1} + \frac{\rho_3}{\rho_1} + \frac{\rho_3}{\rho_2}\right)\sin^2(\theta)\right], \quad (2)$$

where i is the imaginary unit. The working attenuation function of the filter is $L = |T_{11}|^2$. Parameter L may be represented in the form of a cubic polynomial in degrees of $\sin^2(\theta)$

$$L = 1 + \sum_{j=1}^{3} C_j \sin^{2j}(\theta) \quad (3)$$

with coefficients

$$C_1 = \frac{1}{4}\left[-6 - 2\Omega^{[3]}\left(\frac{\rho_1}{\rho_2}, \frac{\rho_1}{\rho_3}, \frac{\rho_2}{\rho_3}\right) + \Omega^{[3]}\left(\frac{\rho_0^2}{\rho_1^2}, \frac{\rho_0^2}{\rho_2^2}, \frac{\rho_0^2}{\rho_3^2}\right)\right.$$
$$\left. + 2\Omega^{[3]}\left(\frac{\rho_0^2}{\rho_1\rho_2}, \frac{\rho_0^2}{\rho_1\rho_3}, \frac{\rho_0^2}{\rho_2\rho_3}\right)\right], \quad (4)$$

$$C_2 = \frac{1}{4}\left[6 + 4\Omega^{[2]}\left(\frac{\rho_1}{\rho_2}, \frac{\rho_2}{\rho_3}\right) + 6\Omega^{[1]}\left(\frac{\rho_1}{\rho_3}\right)\right.$$
$$- 2\Omega^{[3]}\left(\frac{\rho_0^2}{\rho_1^2}, \frac{\rho_0^2}{\rho_2^2}, \frac{\rho_0^2}{\rho_3^2}\right) + \Omega^{[3]}\left(\frac{\rho_1^2}{\rho_2^2}, \frac{\rho_1^2}{\rho_3^2}, \frac{\rho_2^2}{\rho_3^2}\right)$$
$$- 6\Omega^{[1]}\left(\frac{\rho_0^2}{\rho_1\rho_3}\right) - 4\Omega^{[2]}\left(\frac{\rho_0^2}{\rho_1\rho_2}, \frac{\rho_0^2}{\rho_2\rho_3}\right)$$
$$\left. + 2\Omega^{[2]}\left(\frac{\rho_1^2}{\rho_2\rho_3}, \frac{\rho_3^2}{\rho_1\rho_2}\right) - 2\Omega^{[2]}\left(\frac{\rho_0^2\rho_2}{\rho_1^2\rho_3}, \frac{\rho_0^2\rho_2}{\rho_1\rho_3^2}\right)\right], \quad (5)$$

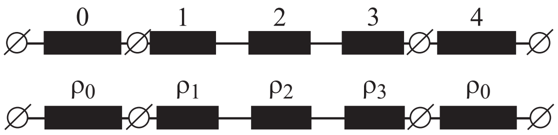
$$C_3 = \frac{1}{4}\left[\Omega^{[1]}\left(\frac{\rho_1^2\rho_3^2}{\rho_0^2\rho_2^2}\right) - 2\right] - C_1 - C_2. \quad (6)$$

Here $\Omega^{[m]}(x_1, \ldots, x_m) = \sum_{k=1}^{m}(x_k + 1/x_k) \quad (m = \overline{1,3})$.

The working attenuation function of the three-link Chebyshev's filter [1] is:

$$L = 1 + h^2 T_3^2[\sin(\theta)/S], \quad (7)$$

where h and S are the amplitude and scale factors ($h > 0$, $0 < S < 1$), $T_3(x)$ is the Chebyshev's first-kind cubic polynomial. It follows from relation (7) that the average link



The equivalent circuit diagram of the filter.

electrical length in the first attenuation band is $\theta_0 = \pi/2$. Setting equal the factors at the same $\sin^2(\theta)$ degrees in relations (3) and (7), obtain a set of nonlinear equations in the link wave impedances:

$$C_1 = 9h^2/S^2, \quad C_2 = -24h^2/S^4, \quad C_3 = 16h^2/S^6. \quad (8)$$

As per (3) and (7),

$$L(\theta = \pi/2) = 1 + \sum_{j=1}^{3} C_j = 1 + h^2 T_3^2(1/S).$$

Taking into account relation (6), obtain

$$(\psi^2 - 2 + 1/\psi^2)/4 = h^2 T_3^2(1/S), \quad (9)$$

where $\psi = \rho_1\rho_3/(\rho_0\rho_2)$. Solutions of equation (9) may be represented as follows:

$$\psi^{(k)} = \sqrt{1 + h^2 T_3^2(1/S)} + (-1)^k h T_3(1/S) \ (k = 1, 2). \quad (10)$$

The number of unknowns in equations (8) may be reduced to two by assuming that

$$\rho_2^{(k)} = \rho_1^{(k)}\rho_3^{(k)}/(\psi^{(k)}\rho_0) \ (k = 1, 2). \quad (11)$$

Relations (4)–(6), (11) are invariant to mutual substitution $\rho_1^{(k)} \leftrightarrow \rho_3^{(k)}$. Thus, any of equations (8) may be written in two ways. For example,

$$C_1^{(k)}\left(\rho_1^{(k)}, \rho_3^{(k)}\right) = 9h^2/S^2,$$

$$C_1^{(k)}\left(\rho_3^{(k)}, \rho_1^{(k)}\right) = 9h^2/S^2 \ (k = 1, 2).$$

Each of these equations defines the first argument $C_1^{(k)}$ as an implicit function of the second argument. Thus, obtain two identical functions $\rho_1^{(k)} = r^{(k)}\left(\rho_3^{(k)}\right)$ and $\rho_3^{(k)} = r^{(k)}\left(\rho_1^{(k)}\right)$ whose plots are mirror-symmetric about line $\rho_1^{(k)} = \rho_3^{(k)}$. Intersection points of these curves relate to the solutions of the filter synthesis task. If such solutions exist, then the abscissa and ordinate of one of them meet the following condition:

$$\rho_1^{(k)} = \rho_3^{(k)} \ (k = 1, 2). \quad (12)$$

Taking into account (12), relation

$$2C_1^{(k)} + C_2^{(k)} = -6(h^2/S)T_3(1/S) \ (k = 1, 2),$$

following from (8) may be reduced to

$$\sum_{j=0}^{3} \alpha_j^{(k)}\left(\rho_1^{(k)}\right)^j + \left(\rho_1^{(k)}\right)^4 = 0 \quad (k = 1, 2), \quad (13)$$

where

$$\alpha_0^{(k)} = -(\psi^{(k)})^2\rho_0^4, \quad \alpha_1^{(k)} = -2\psi^{(k)}\rho_0^3,$$

$$\alpha_2^{(k)} = -(-1)^k 6\psi^{(k)}\frac{h}{S}\rho_0^2, \quad \alpha_3^{(k)} = 2\psi^{(k)}\rho_0 \ (k = 1, 2). \quad (14)$$

Applying the Descartes–Euler solution [6] to the first of the fourth-degree equations (13), obtain

$$\rho_{1,m}^{(1)} = \sqrt{y_1} - (-1)^m\sqrt{y_2} + (-1)^m \operatorname{sign}(\beta_1)\sqrt{y_3} - \frac{1}{4}\alpha_3^{(1)}$$

$$(m = 1, 2),$$

$$\rho_{1,n}^{(1)} = -\sqrt{y_1} - (-1)^n\sqrt{y_2} - (-1)^n \operatorname{sign}(\beta_1)\sqrt{y_3} - \frac{1}{4}\alpha_3^{(1)}$$

$$(n = 3, 4). \quad (15)$$

Here

$$\beta_0 = \alpha_0^{(1)} - (1/4)\alpha_1^{(1)}\alpha_3^{(1)} + (1/16)\alpha_2^{(1)}\left(\alpha_3^{(1)}\right)^2 - (3/256)\left(\alpha_3^{(1)}\right)^4,$$

$$\beta_1 = \alpha_1^{(1)} - (1/2)\alpha_2^{(1)}\alpha_3^{(1)} + (1/8)\left(\alpha_3^{(1)}\right)^3,$$

$$\beta_2 = \alpha_2^{(1)} - (3/8)\left(\alpha_3^{(1)}\right)^2,$$

y_m $(m = \overline{1, 3})$ are the solutions of equation

$$\sum_{j=0}^{2} \delta_j y^j + y^3 = 0 \quad (16)$$

with coefficients $\delta_0 = -(1/64)\beta_1^2$, $\delta_1 = -(1/4)\beta_0 + (1/16)\beta_2^2$, $\delta_2 = (1/2)\beta_2$.

To determine y_m, use the Cardano's solution for a cubic equation [6]:

$$y_1 = \nu_1 + \nu_2 - \frac{\delta_2}{3},$$

$$y_m = -\frac{\nu_1 + \nu_2}{2} + (-1)^m i\sqrt{3}\frac{\nu_1 - \nu_2}{2} - \frac{\delta_2}{3} \ (m = 2, 3),$$

$$\nu_n = \sqrt[3]{-(1/2)\xi_0 - (-1)^n\sqrt{\eta}} \quad (n = 1, 2),$$

$$\eta = (1/4)\xi_0^2 + (1/27)\xi_1^3,$$

$$\xi_0 = \delta_0 - (1/3)\delta_1\delta_2 + (2/27)\delta_2^3, \quad \xi_1 = \delta_1 - (1/3)\delta_2^2.$$

It is easy to verify that quantities y_m are interrelated as follows:

$$\prod_{m=1}^{3} y_m = (1/64)\beta_1^2. \quad (17)$$

Based on (14), find

$$\beta_0 = \frac{3}{2}(\psi^{(1)})^3\left(\frac{h}{S} - \frac{1}{8}\psi^{(1)}\right)\rho_0^4,$$

$$\beta_1 = -\psi^{(1)}\left(8\frac{h}{S^3}\psi^{(1)} + 1\right)\rho_0^3,$$

$$\beta_2 = 3\psi^{(1)}\left(2\frac{h}{S} - \frac{1}{2}\psi^{(1)}\right)\rho_0^2,$$

$$\xi_0 = -\frac{1}{4}\left(\psi^{(1)}\right)^3\frac{h}{S^3}(2+h^2)\rho_0^6,$$

$$\xi_1 = -\frac{3}{4}\left(\psi^{(1)}\right)^2\frac{h^2}{S^2}\rho_0^4,$$

$$\eta = \frac{1}{16}\left(\psi^{(1)}\right)^6\frac{h^2}{S^6}(1+h^2)\rho_0^{12},$$

$$\nu_{1,2} = \frac{1}{2}\psi^{(1)}\frac{\sqrt[3]{h}}{S}\left(\sqrt{1+h^2}\pm 1\right)^{2/3}\rho_0^2.$$

Thus the equation (16) solutions are

$$y_1 = \rho_0^2\psi^{(1)}\left(\frac{1}{4}\psi^{(1)} + \frac{1}{2}\frac{\sqrt[3]{h}}{S}\xi\right),$$

$$y_m = \rho_0^2\psi^{(1)}\left[\frac{1}{4}\psi^{(1)} - \left(\frac{3}{2}\frac{h}{S} + \frac{1}{4}\frac{\sqrt[3]{h}}{S}\xi\right)\right.$$

$$\left. +(-1)^m i\frac{\sqrt{3}}{4}\frac{\sqrt[3]{h}}{S}(\sigma_2-\sigma_1)\right] \quad (m=2,3), \qquad (18)$$

where

$$\xi = \left(\sqrt[3]{\sqrt{1+h^2}+1} - \sqrt[3]{\sqrt{1+h^2}-1}\right)^2,$$

$$\sigma_{1,2} = \left(\sqrt{1+h^2}\mp 1\right)^{2/3}.$$

The y_2 and y_3 quantities are complex-conjugate:

$$y_2 = y_3^*, \quad y_2y_3 = |y_2|^2 > 0. \qquad (19)$$

The y_2 absolute value does not vanish to zero since $\mathrm{Im}(y_2) > 0$. Therefore, taking into account (17), obtain $y_1 > 0$. From (19) it also follows that $\sqrt{y_2} = (\sqrt{y_3})^*$. Thus, among expressions (15) real ones are

$$\rho_{1,m}^{(1)} = \sqrt{y_1} - (-1)^m 2\mathrm{Re}\left(\sqrt{y_2}\right) - \frac{1}{2}\psi^{(1)}\rho_0 \quad (m=1,2). \qquad (20)$$

Using relation

$$\mathrm{Re}(\sqrt{y_2}) = \frac{1}{\sqrt{2}}\sqrt{\mathrm{Re}(y_2)+|y_2|},$$

find that

$$\mathrm{Re}\left(\sqrt{y_2}\right) = \rho_0\frac{1}{2}\sqrt{\psi^{(1)}}\left[\frac{\psi^{(1)}}{2} - \left(3\frac{h}{S} + \frac{\sqrt[3]{h}}{S}\frac{\xi}{2}\right)\right.$$

$$\left. + \sqrt{\left(3\frac{h}{S} - \frac{\psi^{(1)}}{2}\right)^2 + \left(6\frac{h}{S} - \frac{\psi^{(1)}}{2}\right)\frac{\sqrt[3]{h}}{S}\xi + \frac{h^{2/3}}{S^2}\xi^2}\,\right]^{1/2}. \qquad (21)$$

From (18), (20), (21), inequality $\rho_{1,1}^{(1)} > 0$ follows. Taking into account relation

$$\rho_{1,1}^{(1)}\rho_{1,2}^{(1)} = \left(\sqrt{y_1} - \frac{\psi^{(1)}}{2}\rho_0\right)^2 - 4\left[\mathrm{Re}\left(\sqrt{y_2}\right)\right]^2 = \rho_0^2\psi^{(1)}$$

$$\times\left[3\frac{h}{S} + \frac{\sqrt[3]{h}}{S}\xi + \sqrt{\psi^{(1)}\left(\frac{\psi^{(1)}}{4} + \frac{\sqrt[3]{h}}{S}\frac{\xi}{2}\right)}\right.$$

$$\left. + \sqrt{\left(3\frac{h}{S} - \frac{\psi^{(1)}}{2}\right)^2 + \left(6\frac{h}{S} - \frac{\psi^{(1)}}{2}\right)\frac{\sqrt[3]{h}}{S}\xi + \frac{h^{2/3}}{S^2}\xi^2}\,\right]^{-1}$$

$$\times\left[\psi^{(1)}\left(3\frac{h}{S} - \frac{\psi^{(1)}}{2}\right)\right.$$

$$\left. - \sqrt{(\psi^{(1)})^2\left(3\frac{h}{S} - \frac{\psi^{(1)}}{2}\right)^2 + 2\psi^{(1)}\frac{h}{S^3}\xi(3h^{2/3}+\xi)^2}\,\right] < 0,$$

obtain $\rho_{1,2}^{(1)} < 0$. Therefore, the first equation of (13) has a unique real positive solution

$$\rho_1^{(1)} = \rho_0\sqrt{\psi^{(1)}}\left\{\sqrt{\frac{\psi^{(1)}}{4} + \frac{\sqrt[3]{h}}{S}\frac{\xi}{2}} - \frac{1}{2}\sqrt{\psi^{(1)}}\right.$$

$$+ \left[\sqrt{\left(3\frac{h}{S} - \frac{\psi^{(1)}}{2}\right)^2 + \left(6\frac{h}{S} - \frac{\psi^{(1)}}{2}\right)\frac{\sqrt[3]{h}}{S}\xi + \frac{h^{2/3}}{S^2}\xi^2}\right.$$

$$\left.\left. + \frac{\psi^{(1)}}{2} - 3\frac{h}{S} - \frac{\sqrt[3]{h}}{S}\frac{\xi}{2}\right]^{1/2}\right\}. \qquad (22)$$

As per (10), $\psi^{(1)}\psi^{(2)} = 1$ with $\psi^{(1)} < 1$, $\psi^{(2)} > 1$. Substitutions $\psi^{(2)} = 1/\psi^{(1)}$, $\rho_1^{(2)} = \rho_0^2/\rho_1^{(1)}$ transform the second equation of (13) to the first one. Thus, solutions of the filter synthesis task are bound by condition $\rho_1^{(1)}\rho_1^{(2)} = \rho_0^2$. Equations (11), (12) allow extending it to wave impedances of all the filter links by writing

$$\rho_j^{(2)} = \rho_0^2/\rho_j^{(1)} \quad (j = \overline{1,3}). \qquad (23)$$

The phase shift between the direct voltage waves at the filter input is $\varphi = \arg(T_{11})$. As it follows from (1), (2), transmission factor T_{11} is invariant to substitution $\rho_j^{(1)} = \rho_0^2/\rho_j^{(2)}$ or $\rho_j^{(2)} = \rho_0^2/\rho_j^{(1)}$ $(j = \overline{1,3})$. Therefore, phase-frequency responses related to solutions (11), (12), (22), (23) are identical.

The analysis performed shows that, under the assumption that wave impedances of extreme links are equal, the problem of synthesizing the three-link stepped Chebyshev's microwave filter has two solutions. Filters consistent with these solutions have identical phase-frequency responses. For each link, the product of wave impedances corresponding to these solutions is the squared wave impedance of the transmission line comprising the filter.

Conflict of interests

The authors declare that they have no conflict of interests.

References

[1] A.L. Fel'dshtein, L.R. Yavich, *Sintez chetyrekhpolyusnikov i vos'mipolyusnikov na SVCh* (Svyaz', M., 1971) (in Russian).
[2] *Spravochnik po elementam poloskovoy tekhniki*, pod red. A.L. Fel'dshteina (Svyaz', M., 1979) (in Russian).
[3] A. Das, A. Roy, K. Roy, A. Bhattacharyya, D. Chowdhury, S. Paul, in *2015 IEEE Int. Conf. on electrical, computer and communication technologies* (ICECCT) (IEEE, 2015), p. 1−3. DOI: 10.1109/ICECCT.2015.7226196
[4] W. Zhou, W. Xia, J. Zhang, D. He, C. Liu, Z. Wu, in *2018 IEEE Int. Conf. on computational electromagnetics (ICCEM)* (IEEE, 2018), p. 1–3. DOI: 10.1109/COMPEM.2018.8496690
[5] A.L. Fel'dshtein, L.R. Yavich, V.P. Smirnov, *Spravochnik po elementam volnovodnoy tekhniki*, 1-e izd. (Gosenergoizdat, M., 1963) (in Russian).
[6] G. Korn, T. Korn, *Spravochnik po matematike dlya nauchnykh rabotnikov i inzhenerov* (Nauka, M., 1984) (in Russian).