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Anticlassical approximation in the problem of electromagnetiv wave reflection from an inhomogeneous medium

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Expressions are derived for the reflection coefficients of electromagnetic waves with p and s type polarizations from a semi-infinite dielectric medium having an inhomogeneous layer. The influence of the layer was taken into account by the method of perturbation theory in a quadratic approximation of the layer thickness. A method is proposed for converting expressions derived using perturbation theory into other expressions that give more accurate values of the reflection coefficient. The angular dependences of the reflection coefficient obtained by the developed method are compared with those obtained by the numerical solution of electrodynamic equations. Requirements for the layer characteristics are formulated to minimize the error of the analytical solution.

Keywords: dielectric medium, inhomogeneous layer, electromagnetic wave reflection coefficient, analytical calculation, perturbation theory.

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Introduction

The interest in reflection of electromagnetic waves off dielectric materials and transmission of waves through these materials is stimulated by the progress in such fields as optoelectronics, radio engineering systems, semiconductor optics, and polymer optics. Metamaterials (artificially structured media with such electromagnetic properties that are not found in naturally occurring media) also attract much research attention. Inhomogeneous dielectric layers may be used as component parts of a metamaterial.

The problem of reflection of an electromagnetic wave off a plane boundary of a medium with an inhomogeneous permittivity profile has no general analytical solution. The wave equation, which is the basis for calculating the reflection coefficient, may be transformed into the Helmholtz equation, and an exact solution of the equation for an inhomogeneous medium is known only for particular cases. The substitution of variables and functions, which allows one to transform the initial equation into a certain standard one, is a classical method for finding the exact solution. This method was used to find the solutions for linear, parabolic, and exponential permittivity profiles. Methods for reducing the electrodynamics equations, wherein the substitution of variables is supplemented by the transformation of components of the wave field with the use of auxiliary functions [1–3], have already been developed. The conditions of such transformations define broad classes of continuous distributions of the permittivity profile that allow for exact analytical expressions of components of the electromagnetic field. Either numerical or approximate analytical methods [4–10] are used for a medium with an arbitrary coordinate dependence of material parameters. The quasiclassical approximation, which is also known as the WKB (Wentzel–Kramers–Brillouin) method, is efficient if the refraction index varies slowly over the distance of a wavelength. The opposite case is a medium with its parameters varying considerably over the distance of a wavelength (anticlassical approximation). Specifically, the reflection of an electromagnetic wave off an inhomogeneous structure composed of layers of a subwavelength thickness has already been examined theoretically and experimentally [11,12]. It turned out that a model based on the substitution of this structure by an effective medium with homogeneous properties does not produce correct reflection parameters.

In the present study, we discuss another problem within the anticlassical approximation: the problem of reflection of an electromagnetic wave off a medium with an inhomogeneous layer of a subwavelength thickness. It was assumed that the reflective properties of this medium are characterized by permittivity and that the phase thickness of the region of inhomogeneity is small. The latter requirement allowed us to apply the methods of perturbation theory and obtain approximate analytical expressions for the reflection coefficient. As in [1–3], the components of the wave field were expressed in terms of auxiliary functions, and a method for calculation of these functions was developed. In addition, a model based on the substitution of an inhomogeneous layer by a homogeneous one with effective parameters was proposed. This substitution extended the potential of the analytical expression of the reflection coefficient with respect to thicker layers. The mathematical framework of the developed theory is also of interest with regard to the analysis of wave fields in other fields of physics of continuous media.

1. Mathematical model

Fig. 1 illustrates the geometric aspects of the model. An inhomogeneous layer is modeled by a coordinate dependence of permittivity $\varepsilon(z)$. The geometry of the layer is defined by two parameters characterizing the depth of its occurrence (Δ) and its thickness (δ). It was assumed that function $\varepsilon(z)$ at the layer boundaries approaches asymptotic value $\varepsilon(+\infty)$ no slower than an exponential function. This requirement ensures the convergence of integrals that are found in the final formulae and contain function $\varepsilon(z)$.

The wave polarization is of great importance in the problem of reflection of an electromagnetic wave. In the present study, linear p- and s-type polarizations of waves are considered. Detailed calculations are presented for p-polarized waves, and only the resultant expressions are given for s-polarized waves.

The magnetic-field vector of a p-polarized wave in the adopted system of coordinates (Fig. 1) may be written as

$$\mathbf{H} = \begin{bmatrix} 0 \\ h(z) \\ 0 \end{bmatrix} \cdot \exp i(\omega t - k_x x),$$

where ω is the circular wave frequency, t is time, $k_x = \omega \sin\theta / c$, c is the speed of light in vacuum, and i is the imaginary unit. The differential equation defining

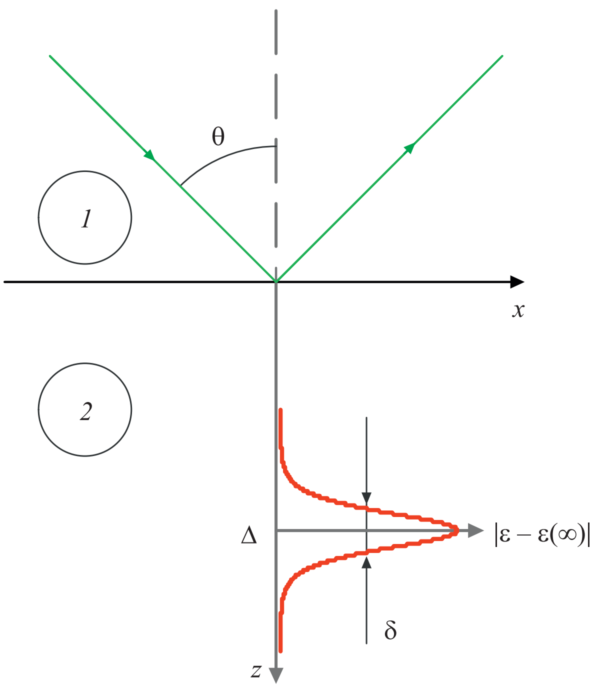

Figure 1. Geometric parameters of the model of reflection of an electromagnetic wave off a medium with an inhomogeneous layer. Coordinate axes x and z lie within the plane of beam incidence. The interface between vacuum *1* and reflecting medium *2* is the plane $z = 0$. The deviation of permittivity in the inhomogeneous layer from the asymptotic value is shown schematically. The layer geometry is characterized by distance Δ to the interface and thickness δ.

function $h(z)$ (in Gaussian units) follows from the electrodynamics equations

$$\frac{1}{\varepsilon(z)} \frac{d}{dz} \left[\frac{1}{\varepsilon(z)} \frac{dh(z)}{dz} \right] + k_p^2(z) h(z) = 0. \quad (1)$$

Function $k_p(z)$ is defined as (index „p“ denotes the polarization type)

$$k_p^2(z) = \frac{1}{\varepsilon^2(z)} \left[\frac{\omega^2}{c^2} \varepsilon(z) - k_x^2 \right]. \quad (2)$$

Let us divide coordinate axis z into three regions and define function $h(z)$ in each region in the following way:

$$h(z) = \begin{cases} a_1 \exp(-ik_z z) + b_1 \exp(ik_z z), & z < 0, \\ a_2(z) \exp\left[-ik_p \int\limits_0^z \varepsilon(\xi) d\xi\right] \\ \quad + b_2(z) \exp\left[ik_p \int\limits_0^z \varepsilon(\xi) d\xi\right], & 0 < z < \Delta, \\ a_3(z) \exp\left[-ik_p \int\limits_\Delta^z \varepsilon(\xi) d\xi\right] \\ \quad + b_3(z) \exp\left[ik_p \int\limits_\Delta^z \varepsilon(\xi) d\xi\right], & \Delta < z, \end{cases} \quad (3)$$

where $k_z = \omega \cos\theta / c$ and $k_p = k_p(+\infty)$.

The coefficient of reflection of a wave off a medium is expressed as

$$R_p = \left| \frac{b_1}{a_1} \right|^2.$$

Let us transform this formula using the condition for continuity of tangential components of intensities of the magnetic and electric field of the wave (components H_y and E_x, respectively) at boundary $z = 0$. Since the electrodynamics equations allow one to express E_x in terms of H_y, the condition for continuity of the ratio of components may be used

$$\frac{H_y}{E_x} = -\frac{i\omega}{c} \frac{h(z)}{[1/\varepsilon(z)][dh(z)/dz]}. \quad (4)$$

Inserting the expressions for $h(z)$ from Eq. (3), we obtain the following relation:

$$\frac{b_1}{a_1} = \frac{r_p + V_p}{1 + r_p V_p}.$$

Here, $V_p = b_2(0)/a_2(0)$ and

$$r_p = \frac{k_z - k_p}{k_z + k_p}. \quad (5)$$

Thus, the problem of calculation of the reflection coefficient may be reduced to calculating the value of parameter V_p and inserting this value into

$$R_p = \left| \frac{r_p + V_p}{1 + r_p V_p} \right|^2. \quad (6)$$

Since two functions ($a_2(z)$ and $b_2(z)$) are introduced instead of a single one ($h(z)$) within the $0 < z < \Delta$ interval, an additional condition may be imposed on these functions. Let us choose the following condition:

$$\frac{da_2(z)}{dz} \exp\left[-ik_p \int_0^z \varepsilon(\varsigma) d\varsigma\right] + \frac{db_2(z)}{dz} \exp\left[ik_p \int_0^z \varepsilon(\varsigma) d\varsigma\right] = 0. \quad (7)$$

It then follows from Eq. (1) that

$$\frac{ik_p}{\varepsilon(z)} \frac{da_2(z)}{dz} \exp\left[-ik_p \int_0^z \varepsilon(\varsigma) d\varsigma\right] - \frac{ik_p}{\varepsilon(z)} \frac{db_2(z)}{dz} \times \exp\left[ik_p \int_0^z \varepsilon(\varsigma) d\varsigma\right] = \left[k_p^2(z) - k_p^2\right] h(z). \quad (8)$$

Using algebraic transformations, we may transform Eqs. (7) and (8) into the following form:

$$\frac{da_2(z)}{dz} = \frac{\varepsilon(z)}{2ik_p} \left[k_p^2(z) - k_p^2\right] h(z) \exp\left[ik_p \int_0^z \varepsilon(\varsigma) d\varsigma\right], \quad (9)$$

$$\frac{db_2(z)}{dz} = -\frac{\varepsilon(z)}{2ik_p} \left[k_p^2(z) - k_p^2\right] h(z) \exp\left[-ik_p \int_0^z \varepsilon(\varsigma) d\varsigma\right]. \quad (10)$$

Function $h(z)$ in these equations is defined by Eq. (3) within the $0 < z < \Delta$ interval.

The interrelations between functions $a_3(z)$, $b_3(z)$, and $h(z)$ in the $\Delta < z$ region may be determined in a similar manner:

$$\frac{da_3(z)}{dz} = \frac{\varepsilon(z)}{2ik_p} \left[k_p^2(z) - k_p^2\right] h(z) \exp\left[ik_p \int_\Delta^z \varepsilon(\varsigma) d\varsigma\right], \quad (11)$$

$$\frac{db_3(z)}{dz} = -\frac{\varepsilon(z)}{2ik_p} \left[k_p^2(z) - k_p^2\right] h(z) \exp\left[-ik_p \int_\Delta^z \varepsilon(\varsigma) d\varsigma\right]. \quad (12)$$

Equations (9)–(12) will be used to determine ratio $b(0)/a_2(0)$ that is needed to calculate the reflection coefficient.

2. Perturbation theory

Let us transform Eqs. (9)–(12) into an integral form by inserting the expressions for $h(z)$ from Eq. (3) in each region:

$$a_2(z) - a_2(0) = \frac{1}{2ik_p} \int_0^z \varepsilon(\varsigma) \left[k_p^2(\varsigma) - k_p^2\right] \left\{ a_2(\varsigma) + b_2(\varsigma) \exp\left[2ik_p \int_0^\Delta \varepsilon(\eta) d\eta\right] \exp\left[2ik_p \int_\Delta^\varsigma \varepsilon(\eta) d\eta\right] \right\} d\varsigma, \quad (13)$$

$$b_2(z) - b_2(0) = -\frac{1}{2ik_p} \int_0^z \varepsilon(\varsigma) \left[k_p^2(\varsigma) - k_p^2\right] \left\{ a_2(\varsigma) \times \exp\left[-2ik_p \int_0^\Delta \varepsilon(\eta) d\eta\right] \exp\left[-2ik_p \int_\Delta^\varsigma \varepsilon(\eta) d\eta\right] + b_2(\varsigma) \right\} d\varsigma, \quad (14)$$

$$a_3(+\infty) - a_3(z) = \frac{1}{2ik_p} \int_z^{+\infty} \varepsilon(\varsigma) \left[k_p^2(\varsigma) - k_p^2\right] \times \left\{ a_3(\varsigma) + b_3(\varsigma) \exp\left[2ik_p \int_\Delta^\varsigma \varepsilon(\eta) d\eta\right] \right\} d\varsigma, \quad (15)$$

$$b_3(+\infty) - b_3(z) = -\frac{1}{2ik_p} \int_z^{+\infty} \varepsilon(\varsigma) \left[k_p^2(\varsigma) - k_p^2\right] \times \left\{ a_3(\varsigma) \exp\left[-2ik_p \int_\Delta^\varsigma \varepsilon(\eta) d\eta\right] + b_3(\varsigma) \right\} d\varsigma. \quad (16)$$

The lack of a reflected wave at $z \to +\infty$ yields $b_3(+\infty) = 0$.

A small parameter is needed to apply perturbation theory to the problem. Let us impose the following condition on layer thickness δ (the condition of small phase thickness):

$$|k_p \max\{\varepsilon(z)\}|\delta \ll 1. \quad (17)$$

The zero approximation of perturbation theory corresponds to a homogeneous semi-infinite medium with $\varepsilon(z) = \varepsilon(+\infty)$ for all $z > 0$. Within this approximation, $a_2(z) = a_3(z) = a_2(0)$ and $b_2(z) = b_3(z) = b_2(0) = 0$. If condition (17) is fulfilled, the right-hand parts of Eqs. (13)–(16) may be regarded as perturbations that define the coordinate dependence of functions $a_2(z)$, $a_3(z)$, $b_2(z)$, and $b_3(z)$. In the first approximation, these functions may be obtained by inserting the zero-approximation constants into the right-hand parts of equations. It may also be assumed in this approximation that

$\exp\left[\pm 2ik_p \int_\Delta^\varsigma \varepsilon(\eta)d\eta\right] \approx 1$. Equations (13)–(16) then take the form

$$a_2(z) - a_2(0) = I_2(z)\frac{a_2(0)}{2ik_p},$$

$$b_2(z) - b_2(0) = -I_2(z)\frac{a_2(0)}{2ik_p}\exp\left[-2ik_p\int_0^\Delta \varepsilon(\eta)d\eta\right],$$

$$a_3(+\infty) - a_3(z) = I_3(z)\frac{a_2(0)}{2ik},$$

$$b_3 = I_3(z)\frac{a_2(0)}{2ik_p}.$$

Here, the following functions with their values being of the first order of smallness with respect to the thickness of an inhomogeneous layer ($\sim \delta$) were introduced:

$$I_2(z) = \int_0^z \varepsilon(\varsigma)\left[k_p^2(\varsigma) - k_p^2\right]d\varsigma,$$

$$I_3(z) = \int_z^{+\infty} \varepsilon(\varsigma)\left[k_p^2(\varsigma) - k_p^2\right]d\varsigma.$$

The second-order corrections may be obtained by inserting the first-approximation coordinate dependences into the right-hand parts of Eqs. (13)–(16). In addition, refinement substitutions $\exp\left[\pm 2ik_p \int_\Delta^\varsigma \varepsilon(\eta)d\eta\right] \approx 1 \pm 2ik_p \int_\Delta^\varsigma \varepsilon(\eta)d\eta$ need to be used in this approximation, and the terms of the third order of smallness ($\sim \delta^3$) should be dropped in the resultant expressions. The following equalities are then obtained in the second approximation

$$a_2(\Delta) = a_2(0) + \frac{I_2(\Delta)}{2ik_p} \times \left\{a_2(0) + b_2(0)\exp\left[2ik_p\int_0^\Delta \varepsilon(\eta)d\eta\right]\right\} + G_2(\Delta)b_2(0)\exp\left[2ik_p\int_0^\Delta \varepsilon(\eta)d\eta\right], \quad (18)$$

$$b_2(\Delta) = b_2(0) - \frac{I_2(\Delta)}{2ik_p} \times \left\{a_2(0)\exp\left[-2ik_p\int_0^\Delta \varepsilon(\eta)d\eta\right] + b_2(0)\right\} + G_2(\Delta)a_2(0)\exp\left[-2ik_p\int_0^\Delta \varepsilon(\eta)d\eta\right], \quad (19)$$

$$a_3(\Delta) = a_3(+\infty) - \frac{I_3(\Delta)}{2ik_p}a_3(+\infty), \quad (20)$$

$$b_3(\Delta) = \frac{I_3(\Delta)}{2ik_p}a_3(+\infty) - G_3(\Delta)a_3(+\infty). \quad (21)$$

Here, $G_2(\Delta)$ and $G_3(\Delta)$ are the values of functions

$$G_2(z) = \int_0^z \varepsilon(\varsigma)\left[k_p^2(\varsigma) - k_p^2\right]\left[\int_\Delta^\varsigma \varepsilon(\eta)d\eta\right]d\varsigma,$$

$$G_3(z) = \int_z^{+\infty} \varepsilon(\varsigma)\left[k_p^2(\varsigma) - k_p^2\right]\left[\int_\Delta^\varsigma \varepsilon(\eta)d\eta\right]d\varsigma.$$

Using the condition for continuity of relation (4) at $z = \Delta$, we obtain the following equation:

$$\frac{a_2(\Delta)\exp\left[-ik_p\int_0^\Delta \varepsilon(\eta)d\eta\right] + b_2(\Delta)\exp\left[k_p\int_0^\Delta \varepsilon(\eta)d\eta\right]}{-k_p a_2(\Delta)\exp\left[-ik_p\int_0^\Delta \varepsilon(\eta)d\eta\right] + k_p b_2(\Delta)\exp\left[k_p\int_0^\Delta \varepsilon(\eta)d\eta\right]} = \frac{a_3(\Delta) + g_3(\Delta)}{-k_p a_3(\Delta) + k_p b_3(\Delta)}.$$

Inserting Eqs. (18)–(21) and performing elementary transformations, we obtain the equation for finding $V_p = b_2(0)/a_2(0)$:

$$\frac{1 + G_2(\Delta) + [1 + G_2(\Delta)]V_p\exp\left[2ik_p\int_0^\Delta \varepsilon(\eta)d\eta\right]}{-k_p + iI_2(\Delta) + k_pG_2(\Delta) + \left[k_p + iI_2(\Delta) - k_pG_2(\Delta)\right]\times V_p\exp[2ik_p\int_0^\Delta \varepsilon(\eta)d\eta]} = \frac{G_3(\Delta) - 1}{k_p + iI_3(\Delta) + k_pG_3(\Delta)}.$$

This equation is algebraic, and its solution in the second order of perturbation theory may be transformed into the form

$$V_p = -\left(\frac{iI}{2k_p} + \frac{I^2}{4k_p^2} + G\right)\exp\left[-2ik_p\int_0^\Delta \varepsilon(\eta)d\eta\right], \quad (22)$$

where the following is introduced

$$I = I_2(\Delta) + I_3(\Delta) = \int_0^{+\infty} \varepsilon(\varsigma)\left[k_p^2(\varsigma) - k_p^2\right]d\varsigma,$$

$$G = G_2(\Delta) + G_3(\Delta) = \int_0^{+\infty} \varepsilon(\varsigma)\left[k_p^2(\varsigma) - k_p^2\right]\left[\int_\Delta^\varsigma \varepsilon(\eta)d\eta\right]d\varsigma.$$

Parameters I and G depend linearly ($I \sim \delta$) and quadratically ($G \sim \delta^2$) on the layer thickness.

Formula (22) has a significant drawback in that both function $\varepsilon(z)$ and parameter Δ, which is found in the exponent and in the expression for parameter G, are needed for calculations. Let us transform formula (22) into the form without parameter Δ. Staying within the second approximation, we present formula (22) in the following form:

$$V_p \approx -\frac{iI}{2k_p} \times \exp\left\{-i\left[2k_p\int_0^{\Delta}\varepsilon(\eta)d\eta + \frac{I}{2k_p} + 2k_p\frac{G}{I}\right]\right\}. \quad (23)$$

Let us introduce function

$$I(z) = \int_z^{+\infty}\varepsilon(\varsigma)\left[k_p^2(\varsigma) - k_p^2\right]d\varsigma.$$

Parameter G may be expressed in terms of this function

$$G = -\int_0^{+\infty}\frac{dI(\varsigma)}{d\varsigma}\left[\int_{\Delta}^{\varsigma}\varepsilon(\eta)d\eta\right]d\varsigma.$$

Integrating by parts and taking equalities $I(0) = I$ and $I(+\infty) = 0$ into account, we obtain equality

$$2k_p\int_0^{\Delta}\varepsilon(\eta)d\eta + \frac{I}{2k_p} + 2k_p\frac{G}{I} = \frac{1}{2k_pI}\left[I^2 + 4k_p^2\int_0^{+\infty}\varepsilon(\varsigma)I(\varsigma)d\varsigma\right]. \quad (24)$$

At the next phase of transformation, we introduce function

$$Y(z) = I^2(z) + 4k_p^2\int_z^{+\infty}\varepsilon(\varsigma)I(\varsigma)d\varsigma.$$

Differentiating it and performing the necessary substitutions, one may obtain the following equation:

$$\frac{dY(z)}{dz} = -2\varepsilon(z)\left[k_p^2(z) + k_p^2\right]I(z).$$

Integrating it and considering that $Y(+\infty) = 0$ and $Y(0)$ is the same as the expression in brackets in the right-hand part of formula (24), we obtain

$$I^2 + 4k_p^2\int_0^{+\infty}\varepsilon(\varsigma)I(\varsigma)d\varsigma = 2\int_0^{+\infty}\varepsilon(z)\left[k_p^2(z) + k_p^2\right]\left\{\int_z^{+\infty}\varepsilon(\varsigma)\left[k_p^2(\varsigma) - k_p^2\right]d\varsigma\right\}dz.$$

Inserting this expression into equality (24) and further into (23), we find the following relation as a substitution for (22)

$$V_p = -\frac{i}{2k_p}\int_0^{+\infty}\varepsilon(z)\left[k_p^2(z) - k_p^2\right]dz \cdot \exp(-2ik_p\Lambda_p), \quad (25)$$

where

$$\Lambda_p = \frac{\int_0^{+\infty}\varepsilon(z)\left[k_p^2(z) + k_p^2\right]\left\{\int_z^{+\infty}\varepsilon(\varsigma)\left[k_p^2(\varsigma) - k_p^2\right]d\varsigma\right\}dz}{2k_p^2\int_0^{+\infty}\varepsilon(z)\left[k_p^2(z) - k_p^2\right]dz}. \quad (26)$$

As a result, the reflection coefficient may be calculated using formula (6) with expressions (25), (26) inserted into it. This calculation is based only on the dependence and does not explicitly involve parameters Δ and δ.

3. Refinement of the formulae of perturbation theory

Models with an exact analytical solution of the problem of reflection of an electromagnetic wave are known. One of these is the model with a homogeneous layer. A method for extrapolating the formulae for this model to the model with an inhomogeneous layer is outlined in the present section. This method allows one to transform the derived formulae of perturbation theory into more exact ones.

The coefficient of reflection of a p-polarized wave off a medium with a homogeneous layer may be calculated using the following expression

$$R_p = \left|\frac{r_p + V_{pu}}{1 + r_pV_{pu}}\right|^2. \quad (27)$$

Parameter r_p is defined by relation (5), while parameter V_{pu} is defined by the following exact expression:

$$V_{pu} = \frac{1 - \exp(-i\varphi_{pu})}{1 - r_{pu}^2\exp(-i\varphi_{pu})}r_{pu}\exp(-i\Phi_{pu}), \quad (28)$$

where

$$r_{pu} = \frac{k_p - k_{pu}}{k_p + k_{pu}}, \qquad k_{pu}^2 = \frac{1}{\varepsilon_u^2}\left[\frac{\omega^2}{c^2}\varepsilon_u - k_x^2\right],$$

$$\varphi_{pu} = 2k_{pu}\varepsilon_u\delta, \qquad \Phi_{pu} = 2k_p\varepsilon_d\Delta_u.$$

Here, ε_u is the permittivity of a homogeneous layer with thickness δ located in a medium with permittivity ε_d between planes $z = \Delta_u$ and $z = \Delta_u + \delta$.

We obtain the following for a layer with a small phase thickness ($|\varphi_{pu} \ll 1|$) from expression (28) in the second order of perturbation theory

$$V_{pu} \approx i\varphi_{pu}\frac{k_p^2-k_{pu}^2}{4k_pk_{pu}}\exp\left[-i\left(\Phi_{pu}+\frac{k_p^2+k_{pu}^2}{4k_pk_{pu}}\varphi_{pu}\right)\right]. \quad (29)$$

Formula (29) may be transformed into form (25) by substituting parameters k_{pu}, φ_{pu}, and Φ_{pu} with suitable expressions. Equating the right-hand parts of formulae (25) and (29), which are complex quantities, we may construct two equations for the sought-for substitutions. However, two equations are not sufficient to obtain the expressions for substitution of three parameters. Therefore, an additional condition imposed on the putative substitutions is needed. In the present study, substitution $k_{pu} \to k_{pl}$ was chosen as such a condition. Parameter k_{pl} in it is defined as follows:

$$k_{pl}^2 = \frac{\int\limits_0^{+\infty} k_p^2(z)\left[k_p^2(z)-k_p^2\right]\varepsilon(z)dz}{\int\limits_0^{+\infty}\left[k_p^2(z)-k_p^2\right]\varepsilon(z)dz}. \quad (30)$$

It should be noted that parameter k_{pl} may be interpreted as the mean of function $k_p(z)$ in an inhomogeneous layer. Specifically, this formula yields equality $k_{pl}=k_{pu}$ when applied to a homogeneous layer. In other words, the proposed expression corresponds to the physical meaning of the substituted parameter.

The equality of pre-exponential factors in formulae (25), (29) with substitution $k_{pu} \to k_{pl}$ allows one to derive the expression for substitution $\varphi_{pu} \to \varphi_{pu}$:

$$\varphi_p = -\frac{2k_{pl}}{k_p^2-k_{pl}^2}\int\limits_0^{+\infty}\varepsilon(z)\left[k_p^2(z)-k_p^2\right]dz. \quad (31)$$

The equality of exponents in the context of the already performed substitutions gives rise to another substitution $\Phi_{pu} \to \Phi_p$:

$$\Phi_p = 2k_p\Lambda_p - \frac{k_p^2+k_{pl}^2}{4k_pk_{pl}}\varphi_p. \quad (32)$$

Similar to the transition from formula (29) of perturbation theory to exact formula (28), we transform formula (25) into form

$$V_{pl} = \frac{1-\exp(-i\varphi_p)}{1-r_{pl}^2\exp(-i\varphi_p)}r_{pl}\exp(-i\Phi_p), \quad (33)$$

$$r_{pl} = \frac{k_p-k_{pl}}{k_p+k_{pl}}. \quad (34)$$

The resultant expression for the reflection coefficient is

$$R_p = \left|\frac{r_p+V_{pl}}{1+r_pV_{pl}}\right|^2. \quad (35)$$

This formula has the same structure as formula (6), but features parameter V_{pl} instead of V_p.

To conclude this section, we provide a brief summary regarding the refinement of the expression for the reflection coefficient of an s-polarized wave. The calculations for this wave type relied on the same approach that was used for a p-polarized wave.

The electric-field vector of an s-polarized wave has the following form (Fig. 1):

$$\mathbf{E} = \begin{bmatrix} 0 \\ e(z) \\ 0 \end{bmatrix} \cdot \exp i(\omega t - k_x x).$$

The wave equation, which forms the basis of calculations, has the form

$$\frac{d^2e(z)}{dz^2} + k_s^2(z)e(z) = 0.$$

Function $k_s^2(z)$ is defined as

$$k_s^2(z) = \frac{\omega^2}{c^2}\varepsilon(z) - k_x^2. \quad (36)$$

Components E_y and H_x are subject to the condition for continuity. Their ratio may be calculated using the following formula:

$$\frac{E_y}{H_x} = \frac{i\omega}{c}\frac{e(z)}{[de(z)/dz]}.$$

The calculation of the reflection coefficient within perturbation theory yields the following relations:

$$R_s = \left|\frac{r_s+V_s}{1+r_sV_s}\right|^2, \quad (37)$$

$$V_s = -\frac{i}{2k_s}\int\limits_0^{+\infty}\left[k_s^2(z)-k_s^2\right]dz \cdot \exp(-2ik_s\Lambda_s), \quad (38)$$

$$\Lambda_s = \frac{\int\limits_0^{+\infty}\left[k_s^2(z)+k_s^2\right]\left\{\int\limits_z^{+\infty}\left[k_s^2(\varsigma)-k_s^2\right]d\varsigma\right\}dz}{2k_s^2\int\limits_0^{+\infty}\left[k_s^2(z)-k_s^2\right]dz}, \quad (39)$$

$$r_s = \frac{k_z-k_s}{k_z+k_s}, \quad (40)$$

$$k_z = \omega\cos\theta/c \quad \text{and} \quad k_s = k_s(+\infty).$$

The refined formula for the reflection coefficient takes the form

$$R_s = \left|\frac{r_s+V_{sl}}{1+r_sV_{sl}}\right|^2, \quad (41)$$

where

$$V_{sl} = \frac{1-\exp(-i\varpi_s)}{1-r_{sl}^2\exp(-i\varphi_s)}r_{sl}\exp(-i\Phi_s), \quad (42)$$

$$\varphi_s = -\frac{2k_{sl}}{k_s^2-k_{sl}^2}\int\limits_0^{+\infty}\left[k_s^2(z)-k_s^2\right]dz, \quad (43)$$

$$k_{sl}^2 = \frac{\int\limits_0^{+\infty} k_s^2(z)\left[k_s^2(z) - k_s^2\right] dz}{\int\limits_0^{+\infty} \left[k_s^2(z) - k_s^2\right] dz}, \quad (44)$$

$$r_{sl} = \frac{k_s - k_{sl}}{k_s + k_{sl}}, \quad (45)$$

$$\Phi_s = 2k_s\Lambda_s - \frac{k_s^2 + k_{sl}^2}{4k_s k_{sl}}\varphi_s. \quad (46)$$

Having inserted the corresponding expressions for the parameters found in (35) and (41) into these formulae, one may calculate the reflection coefficient using the $\varepsilon(z)$ dependence. Specifically, these formulae provide exact expressions for the reflection coefficient for a medium with a homogeneous layer of any thickness. In contrast, the perturbation-theory formulae ((6), (25), and (26) for a p-polarized wave and (37)–(39) for an s-polarized wave) even in this simple case are only approximate.

4. Test calculations and discussion

Test calculations were performed for inhomogeneous media with the coordinate dependence of permittivity characterized by the Epstein formula [13]:

$$\varepsilon(z) = \varepsilon_\infty + 4\varepsilon_l \frac{\exp[(z - z_l)/\delta_l]}{\left\{1 + \exp[(z - z_l)/\delta_l]\right\}^2}, \quad (47)$$

where ε_∞ is the asymptotic permittivity value at $z \to \infty$, $\varepsilon_l = \varepsilon(z_l)$, z_l is the depth of occurrence of the layer, and δ_l characterizes the layer thickness.

In order to estimate the accuracy of the analytical solution, the angular dependences of the reflection coefficient calculated using the derived formulae were compared to the dependences obtained numerically using the electrodynamics equations.

Various interpretations of the recursive method and the transfer matrix method are used widely in numerical methods. In both methods, an inhomogeneous medium is substituted with a structure with thin homogeneous layers. The electrodynamics equations in each layer have solutions in the form of transmitted and reflected waves. The requirement of continuity of tangential components of intensities of the magnetic and electric field gives rise to a system of algebraic equations that interrelate the amplitudes of waves in neighboring layers. This system forms the basis for calculation of the reflection coefficient in both methods. In the present study, the recursive method was used for numerical calculations. Each layer is characterized in this method by its own amplitude reflection coefficient (the ratio of amplitudes of the reflected wave and the transmitted one). Moving successively from the most distant layer to the surface layer, one may calculate all the local coefficients. It is assumed also that the medium becomes virtually homogeneous beyond the external boundary of the most distant layer. The electrodynamics boundary conditions allow one to express the overall reflection coefficient in terms of the local coefficient of the layer adjacent to the surface. Reducing the thickness of each layer and increasing their overall thickness, one may achieve the needed accuracy of calculation of the coefficient of reflection off the entire inhomogeneous structure.

Fig. 2 illustrates the results of calculations. The permittivity of the inner layer either had an imaginary part (Figs. 2 *a, b*) or was real with a minimum value $\min\{\varepsilon(z)\} = 1$ (Figs. 2, *c, d*). Calculations were also performed for a system of two layers (Figs. 2, *e, f*).

It follows from the analysis of the angular dependences of the reflection coefficient that the accuracy of calculations carried out using the refined formulae is significantly higher than the accuracy of calculations relying on the perturbation-theory formulae. The results of calculations for different values of the layer thickness demonstrate that the refined formulae characterize accurately the angular dependences for $\delta_l/\lambda \le 0.02$, while the perturbation-theory formulae are suitable only for $\delta_l/\lambda \le 0.01$ ($\lambda = 2\pi c/\omega$ is the length of an electromagnetic wave in vacuum). The errors of calculation with the perturbation-theory formulae in the $0.01 \le \delta_1/\lambda \le 0.02$ interval may be rather significant. At $\delta_l/\lambda \le 0.01$, the angular dependences calculated using the perturbation-theory formulae and the refined formulae are almost the same as those calculated numerically.

It should also be noted that the amplitude of oscillations of the reflection coefficient in the dependences in Fig. 2 is approximately an order of magnitude greater than the δ_l/λ ratio. This suggests that the inner layer exerts a strong influence on the electrodynamic parameters of the medium even if the layer is of a subwavelength thickness. In addition, the proposed analytical model was found to be highly sensitive to the choice of values of the free parameters of the layer.

Special consideration should be given to inhomogeneous layers with the $\varepsilon(z) - \varepsilon(\infty)$ difference assuming values of different sign within different intervals of z. Equalities $\int\limits_0^{+\infty} \varepsilon(z)\left[k_p^2(z) - k_p^2\right] dz = 0$ or $\int\limits_0^{+\infty} [k_s^2(z) - k_s^2] dz = 0$ may be fulfilled in this case. Since these integrals are in the denominators of expressions (26), (30), (39), and (44), they should not assume zero value. A medium with two Epstein layers was considered as an example (Figs. 2, *e, f*):

$$\varepsilon(z) = \varepsilon_\infty + 4\varepsilon_l \frac{\exp\lfloor(z - z_l + \delta_l)/\delta_l\rfloor}{\{1 + \exp[(z - z_l + \delta_l)/\delta_l]\}^2}$$
$$- \gamma \cdot 4\varepsilon_l \frac{\exp\lfloor(z - z_l - \delta_l)/\delta_l\rfloor}{\{1 + \exp[(z - z_l - \delta_l)/\delta_l]\}^2}. \quad (48)$$

The permittivity is higher than $\varepsilon(\infty)$ in one layer and lower than $\varepsilon(\infty)$ in the other. If the value of $\gamma = 0.3$ is chosen, the mentioned integrals assume nonzero values, and the analytical dependences agree closely with the numerical solution of the electrodynamics equations. The error

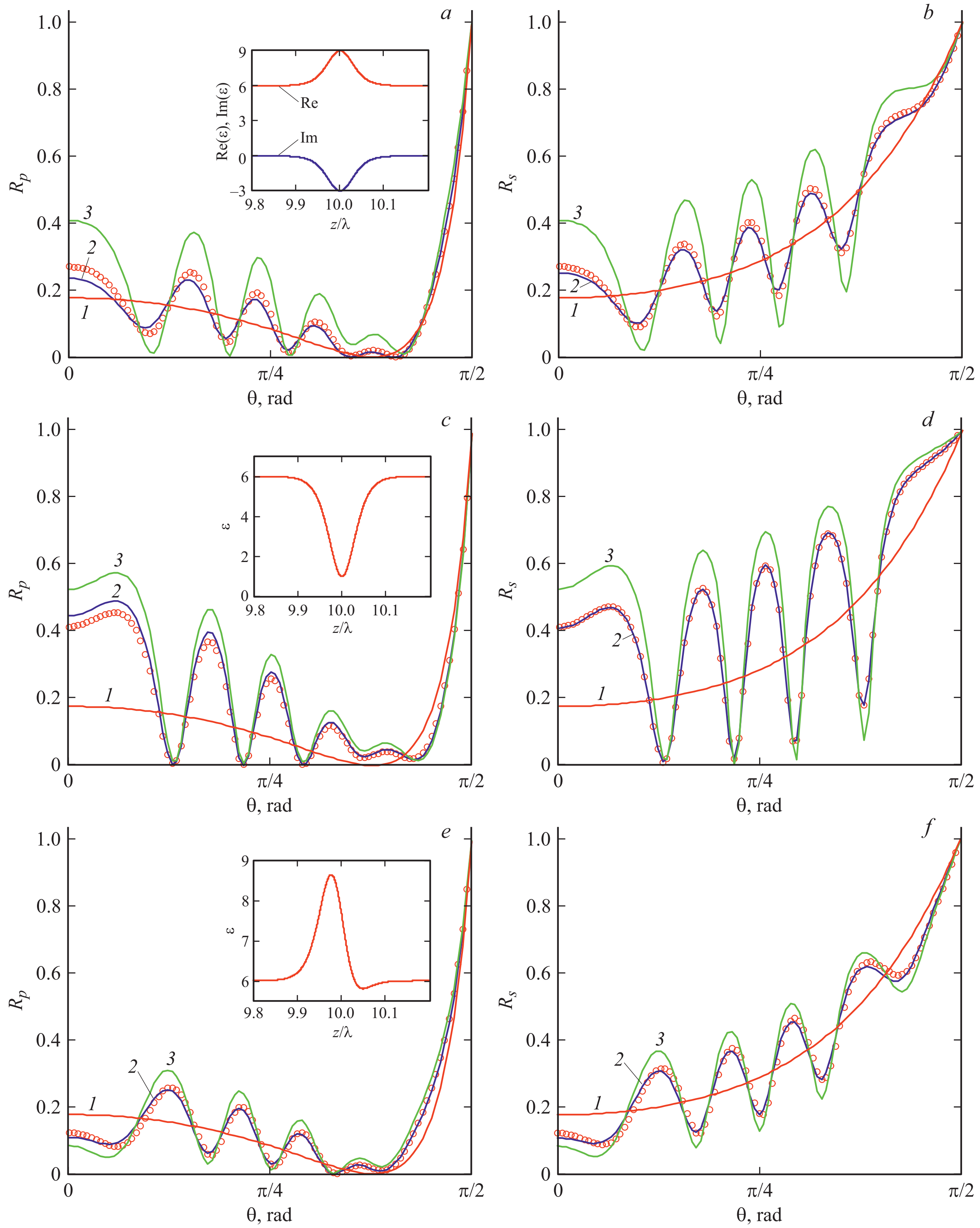

Figure 2. Angular dependences of the reflection coefficient for p- and s-polarized waves (a, c, e and b, d, f, respectively) obtained (*1*) for a homogeneous medium, (*2*) using the refined formulae for a medium with an inhomogeneous layer, and (*3*) using the perturbation-theory formulae for a medium with an inhomogeneous layer. Circles denote the results of numerical calculations utilizing the electrodynamics equations for a medium with an inhomogeneous layer. Coordinate dependences of permittivity $\varepsilon(z)$ in the inhomogeneous layer are shown in the insets. Dependences $\varepsilon(z)$ are defined by formula (47) (a–d) and formula (48) (e, f). The values of $\varepsilon_\infty = 6$, $z_l = 10\lambda$, and $\delta_l = 0.02\lambda$ were the same for all the calculated dependences. The other values were $\varepsilon_l = 3(1-i)$ (a, b); $\varepsilon_l = -5$ (c, d); $\varepsilon_l = 3$, $\gamma = 0.3$ (e, f).

increases as γ gets higher, and the analytical dependences are rendered completely invalid at $\gamma = 1$.

Conclusion

A method for analytical calculation of the coefficient of reflection off a medium with an inhomogeneous layer was developed. The method is based on perturbation theory and on extrapolation of the exact formulae of the model with a homogeneous layer onto the inhomogeneous model. It was proposed to characterize an inhomogeneous layer by parameters that are a generalization of parameters used in the model with a homogeneous layer. Formulae (35) and (41) with substitutions defined by analytical expressions were derived as a result. In contrast to numerical methods, the proposed method does not require layer-by-layer discretization of the medium. All expressions were derived for the coordinate dependence of permittivity of a general form $\varepsilon(z)$. This fact simplifies the computational aspects of the problem considerably and allows one to analyze multiparametric models of $\varepsilon(z)$. The results of test calculations demonstrated that extrapolation provides for a significant increase in the accuracy of calculation of the reflection coefficient. The angular dependences of the reflection coefficient calculated using the extrapolation formulae reproduce fairly accurately the interference effects of reflection. The derived formulae may be used to analyze and predict the properties of materials with modified layers. Since no restrictions regarding the depth of occurrence of an inhomogeneous layer were applied, the obtained results may be regarded as a generalization of the results presented in [9]. In addition, restrictions related to the choice of geometry of reflection near the Brewster angle and the lack of absorption in the medium were substantial in [9]. In the present study, the reflection coefficient was not limited to small values, and the permittivity could contain a coordinate-dependent imaginary part.

Conflict of interest

The author declares that he has no conflict of interest.

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