

Amplification of short-wavelength attosecond pulses in an electrostatic plasma undulator

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The amplification of a short-wavelength ultrashort pulse in an electrostatic undulator of a free-electron laser interacting with a relativistic electron bunch is considered. The system parameters that enable the amplification of attosecond X-ray pulses are determined.

Keywords: attopulses, free-electron laser, electrostatic undulator, stimulated emission.

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Introduction

Hard short electromagnetic impulses enable resolving the dynamics of various fast processes, including nuclear reactions at the attosecond scale of time [1]. It is commonly known that an ultrashort impulse (USI) with the minimum duration and energy of quantum of the kiloelectronvolt range for the modern laser intensities may be generated in the interaction of the intense laser impulse with foil [2], releasing the USI with a filter that cuts off „the substrate“ [3]. Since USI intensity decreases when it spreads and is filtered, further amplification is necessary for the practical application, which is possible, for example, in a free electron laser (FEL) [4–6]. In paper [7] we conducted the preliminary analysis of the possibility to generate and to further amplify the USI of photons of multikiloelectronvolt energy using high-energy electrons in a magnetic undulator of FEL. Such undulator has a limitation of the minimum period length and requires high energies of electrons in FEL to amplify the multikiloelectronvolt quanta. Contrary to the magnetic undulator, the electrostatic undulator may be developed on the basis of ionized filaments or clusters of submicron range and have small period length comparable to the Debye radius of a plasma filament. In this case, to amplify the hard quanta, the electron energy ~ 100 MeV is sufficient. A bunch of such electrons may be obtained currently using an intense laser impulse. And the laser impulse separated into two parts may be also used to generate USI and to accelerate the electrons. In the latter case the second part of the laser impulse moves in a gas target, creating behind an intense longitudinal plasma wave,

which accelerates electrons. As a result, an ultrarelativistic electron bunch is generated after the laser impulse, having the length and the diameter comparable to the length and the diameter of the laser impulse as such (Fig. 1). At laser energy ~ 1 J and intensity $\sim 10^{19}$ W/cm² the electron energy reaches hundreds of megaelectronvolts, and the total charge of the bunch is ~ 10 pQ [8]. The Rayleigh length of the laser impulse may amount to hundreds of microns, therefore, the relativistic intensity of the impulse is maintained at a considerable distance, which is sufficient for the ionization of the elements of the short undulator, and the electron bunch following the laser impulse is rejected by the electrostatic field of the undulator, radiating hard quanta.

In this paper, using analytical and numerical modeling, we study the amplification of USI of photons of multikiloelectronvolt energy and attosecond duration, which are generated by the second part of the laser impulse, with the help of electrons with the energy of several hundreds of megaelectronvolts (from the first part) as they spread via the electrostatic plasma undulator.

Free electron laser with an electrostatic undulator

An electrostatic undulator (periodical electrostatic field) with a micron period may be designed on the basis of a periodical structure from ionized filaments or clusters of submicron thicknesses with distances of λ_w , λ_{wy} , several microns between individual elements. A similar two-dimensional structure from filaments is shown in Fig. 2,

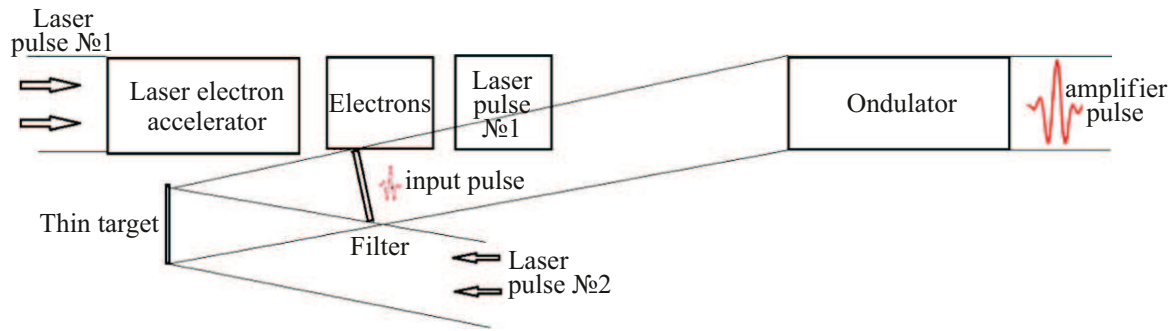


Figure 1. Diagram of the USI amplifier FEL using laser impulse separation into two parts.

and a three-dimensional structure may be organized in a cluster (dust) plasma [9]. With the submicron size of the element and micron distances between the elements, the undulator from ~ 100 links will have the length of less than a millimeter, i.e. of the order of the length of laser impulse absorption in it [10], and the transverse dimension of the undulator for the electron bunch movement therein will be around several dozens of microns. When radiated with the intense laser impulse traveling ahead of the electron bunch, the electrons of the ionized elements of the undulator are heated [11] to a specific temperature of $T_{eh} = m_e c^2 \sqrt{1 + (I_L \lambda_L^2) / (1.37 \cdot 10^{18})} - 1$, where I_L — intensity of laser radiation in W/cm^2 , λ_L — length of the laser wave in μm . Heated electrons leave the undulator element, and being exposed to the ambipolar field between the electrons and ions, the undulator element (filament) starts flying away. A density peak arises in the front of ion density [11], then the density profile for the filament with radius of R is described with function $n_i(r) = n_{i0} \theta(R - r) + (\delta n_i \delta r) \delta(r - R)$, where δr — shock wave front thickness, δn_i — ion density change in the front, n_{i0} — filament density, $\theta(x)$ — Heaviside stepped function, $\delta(x)$ — Dirac delta-function.

The solution to the Poisson equation for the potential of an individual element of the undulator (filament) in

cylindrical coordinates is

$$\begin{aligned} \frac{e\varphi(r)}{m_e c^2} \approx & \left(\frac{T_{eh}}{m_e c^2} \left(1 + \frac{R K_1(R/r_D)}{2 r_D K_0(R/r_D)} \left(1 - \frac{r^2}{R^2} \right) \right) \right. \\ & + \frac{T_{eh}}{m_e c^2} \left(\frac{\delta n_i \delta r}{n_{i0} R} \right) \frac{I_0(r/r_0)}{I_0(R/r_D)} \theta(R - r) \\ & \left. + \frac{T_{eh}}{m_e c^2} \left(1 + \frac{\delta n_i \delta r}{n_{i0} R} \right) \frac{K_0(r/r_D)}{K_0(R/r_D)} \theta(r - R) \right), \end{aligned} \quad (1)$$

where $\theta(\xi)$ — Heaviside function, $I_0(\xi)$, $K_0(\xi)$ — Bessel functions of the 1st and 2nd types, specific scale of the field (Debye radius) is $r_D = \sqrt{T_{eh} / 4\pi e^2 n_{eh}}$, and electron density is assessed as $n_{eh} \approx \pi m_e c^2 / e^2 \lambda_L^2$. The specific value of the filament potential is $e\varphi \sim T_{eh}$, and the specific electrostatic field near the surface of the undulator element will be $E_W \approx T_{eh} / e r_D = \sqrt{4\pi T_{eh} n_{eh}}$. The specific dimensionless field (undulator parameter) with account of the scale of its localization is conveniently managed as $K_E = e E_W r_D / m_e c^2 \approx T_{eh} / m_e c^2$. with C amplification K_E the angle of the electron trajectory deflection increases in the undulator. To implement the process of forced radiation, the deflection angle must not exceed the specific angle of radiation of the relativistic electron [5,12,13], which is achieved at $K_E \leq 1$. Accordingly, the parameters of numerical modeling were selected to obtain $K_E \sim 1$. Note that the field may fill both the entire period λ_w of the undulator in Fig. 2, and a part of the period — depending on the ratio between λ_w , r_D . Numerical modeling of the undulator field and comparison of the estimate (1) with the numerical data is given in Annex 1.

The specific energy of electric field of one filament of the undulator corresponds to the energy of hot electrons heated in a skin layer of the filament: $\varepsilon_1 \approx 2\pi R I_s D Z n_i T_{eh}$, where D — diameter of the laser impulse that heats the filament, $I_s = \lambda_L n_c \sqrt{(I_L \lambda_L^2) / (1.37 \cdot 10^{18})} / \pi \sqrt{2} Z n$ — length of the skin layer at relativistic intensity of the laser field ($n_c = \pi m_e c^2 / e^2 \lambda_L^2$ — critical concentration of the laser plasma, Z — plasma ion charge). Accordingly the length L_f of free path of the laser impulse, which heats the undulator, is estimated from the energy conservation law as $2\pi R I_s D Z n_i T_{eh} \frac{D}{\lambda_{wy}} \frac{L_f}{\lambda_w}$, where $\varepsilon_L = I_L \pi D^2 \tau_L / 4$ — laser

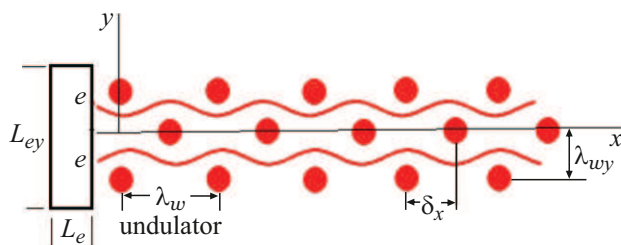


Figure 2. Undulator from nanofilaments. A rectangular electron bunch with volume of $L_{ey} \times L_e$ moves in a periodical (period λ_w) structure made of filaments (red circles). Adjacent rows of filaments are shifted along the axis x by value of $\delta x = \lambda_w / 2$ for formation of the only value of undulator period λ_w . Solid red lines show the electron trajectories.

impulse energy. Therefore, the laser impulse with intensity of I_L makes it possible to heat the undulator from plasma filaments to the length

$$L_f \leq \frac{\sqrt{2\pi c \tau_L \lambda_{Wy} \lambda_W} \sqrt{(I_L \lambda_L^2)/(1.37 \cdot 10^{18})}}{8R \lambda_L (\sqrt{1(I_L \lambda_L^2)/(1.37 \cdot 10^{18})} - 1)}$$

From this estimate you can see that the path length hardly depends on the laser intensity, but is inversely proportional to the filament radius. A similar estimate is provided in paper [14] for the cluster laser plasma. For the laser impulse with duration of 30 fs, interacting with the undulator with $\lambda_W = \lambda_{Wy} = 2 \mu\text{m}$ and filaments $R = 100 \text{ nm}$, we obtain $L_f \approx 240 \mu\text{m}$, therefore it is possible to create a plasma undulator with $N \approx 120$ periods, and increase of the filament radius up to 250 nm will cause $N \approx 50$.

The considered undulator is a dynamic plasma structure with a short lifetime. Ionization and electrostatic field arise for the time of action of the laser (pre)impulse ($\tau_L \sim 30 \text{ fs}$) and are maintained until the heated elements of the undulator fly away. The specific speed of flying away is $c_s \approx \sqrt{Z t_{eh} / A m_p}$, where A — atomic number of filament (cluster) ion, m_p — proton mass. The lifetime of the undulator element will be $\sim R/c_s = \frac{R}{c} \sqrt{\frac{A m_p}{Z m_e K_W}}$. We believe that the laser impulse heating the undulator filaments moves directly upstream the electron bunch and amplified USI, which is located inside the electron bunch. Time delay $\Delta\tau_{eL}$ between the end of the laser impulse and the start of the electron one is $\Delta\tau_{eL} \leq 30 \text{ fs}$, for this time the heated filaments of the undulator have no time to fly away.

The electron bunch moving in the undulator with a delay from the laser impulse may be produced by laser acceleration of the electrons in the plasma of subcritical density [6,10]. For our purposes, the powerful laser impulse may be divided into two parts, one of which is used to create the field in the undulator, and the second one — with delay of $\Delta\tau_{eL}$ — to accelerate the electrons. According to [10], the accelerated electron bunch has the length corresponding to the duration of the laser impulse ($L_e \approx 10 \mu\text{m}$ for the impulse with duration of 30 fs), the diameter of the order of the diameter D of the laser beam and the density $n_{b0} \leq 10^{19} \text{ cm}^{-3}$. The specific time of interaction between the electron bunch and the undulator filament is $\sim (L_e + 2R)/c$ and provided that

$$\frac{R}{c} \sqrt{\frac{A m_p}{Z m_e K_W}} g g \frac{(L_e + 2R)}{c} + \Delta\tau_{eL}$$

the bunch will have time to interact with the heated section of the undulator until it flies away. At optimal $K_E \approx 1$ and $A/Z \sim 2$ the inequation is $60R \gg L_e + c\Delta\tau_{eL}$, which is true, for example, for $R \sim 1$ and $L_e \sim 10 \mu\text{m}$. Therefore, the length of interaction of the electron bunch and USI with the undulator field is limited on top by the length of absorption of the laser impulse L_f , and the maximum quantity of the used periods of the undulator is $N_{\text{max}} \sim L_f/\lambda_W$.

Analytical model of USI amplification

The considered process is described using equations for the USI field and electron motion in the following setup: USI together with the electron bunch move in axis X , where the undulator occupies the interval $x \in [0; L_w]$. The Coulomb field of the bunch is presumed to be small compared to the undulator field, and the dynamics of electrons is described in a single-particle approximation. Hamiltonian of j th electron, moving in the electrostatic field of the lattice from filaments (in axis x period λ_w , in axis y — λ_{Wy} , N — number of filaments in axis x , N_y — in axis y) and in the field of the electromagnetic wave has the following appearance

$$\begin{aligned} H_j/mc^2 &= \varepsilon_j = \sqrt{1 + (\mathbf{p}_{j\perp} - \mathbf{a}(x_j, t))^2 + p_{jx}^2} \\ &+ K_E \sum_{s=-N_y/4}^{l=N} \tilde{\varphi}(x_j + l\lambda_w, y_j + 2s\lambda_{Wy}) \\ &+ K_E \sum_{s=-N_y/4}^{l=N} \tilde{\varphi}(x_j - l\lambda_w, y_j + (2s+1)\lambda_{Wy}) \\ &= \sqrt{1 + (\mathbf{p}_{j\perp} - \mathbf{a}(x_j, t))^2 + p_{jx}^2} + K_E \Phi(x_j, y_j), \quad (2) \end{aligned}$$

where $\mathbf{a}(x, t) = \mathbf{e}_y a(x, t)$, $a = eA/mc^2$ — dimensionless vector potential of the USI field, $p_{jx}(t)$, $\mathbf{p}_{j\perp}(t)$ — dimensionless (in units mc) longitudinal and transverse impulses of the j th electron, $x_j(t)$, $y_j(t)$ — Cartesian positions of the electron, $\tilde{\varphi} = (\varphi/T_{eh})$,

$$\Phi(x, y) = \sum_{s=-N_y/4}^{l=N} \tilde{\varphi}(x - \lambda_w/2 + l\lambda_w, y + (2s+1)\lambda_{Wy})$$

— total dimensionless potential of all undulator elements. At the inlet to the undulator the transverse impulse of the electron $\mathbf{p}_{j\perp}|_{x=0} \approx 0$, $p_{jx}|_{x=0} = \gamma_{0j}$. Note that for the undulator from foil plates the dependence on y in the potential is absent and (2) depends only on x , and for clusters the coordinate z is added.

At the inlet to the undulator, the electron distribution function $f(x_{j0}, y_{j0}, \gamma_{0j})$ will be selected as

$$f(x_{j0}, \gamma_{0j}) = n_{b0} \theta(L_p/2 - |x_{j0}|) \theta(L_{ev}/2 - |y_{j0}|) f_\varepsilon(\gamma_{0j}),$$

where L_e , L_{ey} — dimensions of the electron bunch in axes x , y ,

$$f_\varepsilon(\gamma_{0j}) = \pi^{-1/2} (\Delta\gamma)^{-1} \exp(-(\gamma_{0j} - \gamma_b)^2 / (\Delta\gamma)^2)$$

— Gaussian distribution of electrons by energies, n_{b0} — average density of the electron bunch, γ_b — average value

of Lorentzian factor of the electron bunch, $\Delta\gamma$ — spread of Lorentzian factors of electrons in the bunch.

Dimensionless (in units $m_e c^2$) energy of spontaneous radiation ε_{rad} (in disregard of $\mathbf{a}$ in (2)) of the electron in the electric field of the undulator $\mathbf{E}_W(\mathbf{r}) = -\nabla\varphi(\mathbf{r})$ is determined [5] by formula

$$\varepsilon_{rad} = \frac{2e^2}{3r_D m_e c^2} \times \int \frac{[\mathbf{K}_E(\mathbf{r}(t), t)]^2 - (\mathbf{v}\mathbf{K}_E(\mathbf{r}(t), t))^2/c^2}{(1 - \mathbf{v}^2/c^2)} d(ct/r_D). \quad (3)$$

Spontaneous radiation in the undulator with the transverse electric field corresponds to (at $K_E \sim K_B = eB\lambda_W/(m_e c^2)$) spontaneous radiation in the transverse field B of the magnetic undulator:

$$\varepsilon_{rad} = \frac{2e^2}{3\lambda_W m_e c^2} \int \frac{[\mathbf{v} \times \mathbf{K}_B(\mathbf{r}(t), t)]^2/c^2}{(1 - \mathbf{v}^2/c^2)} d(ct/\lambda_W).$$

The trajectories of electron motion in the transverse electric and magnetic undulators are also similar. Note that in case of additional grouping of electrons at the scales of the order of the radiated wavelength, the coherent radiation of the electron bunch is possible in the magnetic undulator [15,16].

Assessment of energy (3) for N of the filaments at $\lambda_{wy} \approx r_D$ depends on the conditions $K_E > 1$, $K_E < 1$. In case of $K_E > 1$ (synchrotronous mode of radiation) the assessment is proportionate to the number of filaments N :

$$\varepsilon_{\perp rad} \approx \begin{cases} 2Ne^2\gamma_b^2 K_E^2 / (3r_D m_e c^2 (1 + K_E^2/2)), & r_D \leq \lambda_w \\ 2Ne^2\gamma_b^2 K_E^2 \lambda_W / (3r_D m_e c^2 (1 + K_E^2/2)), & r_D > \lambda_w. \end{cases} \quad (4)$$

Note that the case $K_E > 1$ of spontaneous radiation of the electron in the undulator from the plasma nanofilaments was considered in paper [10]. Analytical estimates [10] of the energy radiated by the electron with a precision of up to the numerical factor coincide with the upper formula (4). At $K_E \leq 1$ the case of the undulator radiation considered in this paper is implemented. Undulator radiation is coherent (the intensity is proportionate to the square of the number of filaments) and has a local maximum of intensity at frequency $\omega_s = 4\pi\gamma_b^2 c/\lambda_W$, corresponding to the frequency of the process of forced radiation in FEL. For the undulator in the form of a set of parallel planes $\varepsilon_{||rad} \approx \gamma_b^{-2} \varepsilon_{\perp rad}$ due to the smaller value of the longitudinal acceleration for the relativistic electron compared to the transverse acceleration in the field of the filaments.

The equation of the electron motion in the transverse (in axis y) direction for (2) has the appearance of $\dot{p}_{j\perp} = cK_E \partial\Phi(x_j, y_j)/\partial y_j$. For implementation of the process of forced radiation in the undulator, the deflection angle of the trajectory of the j th electron must not exceed γ_{0j}^{-1} [5,12]. For $\gamma_{0j} \sim 10^3$ at the length of 1 mm (~ 100 periods) the transverse shift of the electron will not exceed $1\mu\text{m}$. Under such conditions

you can disregard the transverse shift and assume that $\dot{p}_{j\perp}(x_j) = cK_E \partial\Phi(x_j, y_{0j})/\partial y_{0j}$, where y_{0j} — electron coordinate at the inlet to the undulator. As a result

$$p_{j\perp}(t, y_{0j}) = cK_E \int_0^t dt (\partial\Phi(x_j(t), y_0)/\partial y_{0j}) \approx cK_E \int_0^t dt (\partial\Phi(x_0 + ct(1 - 0.5\gamma_{0j}^{-2}), y_{0j})/\partial y_{0j}). \quad (5)$$

Note that in process of undulator transit, the angle of electron deflection is $p_{j\perp}/p_{jx} < \gamma_{0j}^{-1}$. For $K_E \leq 1$ and for FEL operation it is also necessary that the relative spread of bunch electron energies $\Delta\gamma_j/\gamma_{0j}$ does not exceed N^{-1} [12]. Contrary to the magnetic one in the electrostatic undulator, the specific (average) electron energy is equal to γ_b , and the spread of electrons by energies $\Delta\gamma$ may vary in the case, when Debye shells of individual plasma filaments touch, and the field potential that is average in the space inside the undulator does not turn to zero. Formula (5) does not account for such change, which is fair at $\gamma_{0j} < N\sqrt{K_E}$.

The undulator potential $\Phi(x, y)$ is periodical in the plane (x, y) and is expanded into the Fourier series:

$$\Phi(x, y) = \sum_{l,n=0}^{\infty} C_{nl} \cos(2\pi nx/\lambda_W) \cos(2\pi ly/\lambda_{Wy}),$$

where the first harmonics $n = l = 1$ are determinant, and accordingly

$$p_{j\perp}(t) \approx K_E C_{11} \sin(2\pi ct/\lambda_W) \sin(2\pi y_{0j}/\lambda_{Wy}).$$

The presence of the integral (5) makes it possible to write the equations of electron motion in axis x for the dimensionless (in units $m_e c^2$) energy ε_j of the j th electron, where the initial transverse coordinate of the electron y_{0j} plays the role of the parameter:

$$\frac{d\varepsilon_j}{dx} = -\frac{p_{j\perp}(t, y_{0j}) - a(x, t)}{\varepsilon_j - K_E \Phi(x, y_{0j})} \frac{\partial}{\partial x} a(x, t) - K_E \frac{\partial}{\partial x} \Phi(x, y_{0j}),$$

$$\frac{cdt}{dx} = \frac{\varepsilon_j - K_E \Phi(x, y_{0j})}{\sqrt{(\varepsilon_j - K_E \Phi(x, y_{0j}))^2 - 1 - (p_{j\perp}(t, y_{0j}) - a(x, t))^2}},$$

$$\varepsilon_j|_{x=0} = \gamma_{0j}, \quad x|_{t=0} = x_{0j}. \quad (6)$$

Note that equations for the unidimensional undulator made of foil plates located perpendicularly to axis x , are identical (6), except for the fact that $\Phi(x, y_{0j}) \rightarrow \Phi(x)$ and $p_{j\perp}(t, y_{0j}) = p_{0j\perp} = 0$. In the mode of weak amplification in equations (6) $\varepsilon_j \gg K_E$, $\varepsilon_j \gg p$ and the right parts (6) are expanded in the Taylor series using small parameters K_E/ε_j , $p_{j\perp}/\varepsilon_j$. Equation for $d\varepsilon_j/dx$ describes the consistent change of the electron energy due to interaction with the combination wave of the USI field and the undulator field corresponding in the upper equation (6) to the summand $\sim p_{j\perp}(t, y_{0j})(\partial a(x, t)/\partial x) \sin(2\pi ct/\lambda_W) \sin(2\pi x/\lambda_x - \omega_x t)$

$\sim \cos(2\pi x/\lambda_x - (\omega_x - 2\pi c/\lambda_w)t)$. If the resonant condition is met

$$\omega_x = \gamma_{0j}^2 \frac{4\pi c/\lambda_w}{1 + K_E^2/2}$$

the phase speed $c(1 - \lambda_x/\lambda_w)$ of such wave coincides with the electron speed $c(1 - \gamma_{0j}^{-2}/2)$, and the electron gives energy to the amplified impulse [12]. The resonant condition also arises at the expense of the summand $\sim K_E \Phi(x, y_{0j}) \partial a^2(x, t)/\partial x$ with the expansion of the denominator $(\varepsilon_j - K_E \Phi(x, y_{0j}))^{-1} \approx \varepsilon_j^{-1} + \varepsilon_j^{-2} K_E \Phi(x, y_{0j})$. The summand $K_E \Phi(x, y_{0j}) \partial a^2(x, t)/\partial x$ remains in the unidimensional case, when the potential depends only on x and $p_{j\perp}(t, y_{0j}) = 0$. Amplification of the impulse energy is assessed by the expansion of the system (6) using the small parameters K_E/ε_j , $p_{j\perp}/\varepsilon_j$ through $\varepsilon_j(x)$ according to [12]. At the distance of $x = N\lambda_w$ the decrease in the energy of the separate electron will amount to $m_e c^2(\gamma_{0j} - \varepsilon_j(N\lambda_w))$, besides, only „resonant“ electrons change the energy, for which $\gamma_{0j} = \gamma(\omega_x) = \sqrt{\lambda_w(1 + K_E^2/2)/2\lambda_s}$. The change in the density of the energy flux of the resonant share in the bunch electrons will amount to $n_{e0} f_\varepsilon(\gamma(\omega_s)) m_e c^3(\gamma(\omega_s) - \varepsilon_j(N\lambda_w))$. Accordingly, the density of the energy flux of the amplified wave $a^2 m_e^2 c^3 \omega_s^2 / (8\pi e^2)$ will increase by the same value. As a result, the amplification coefficient for the wave energy with the frequency will amount to

$$\kappa_E(\omega_x, n_{e0}, y_0) = 8\pi e^2 n_{e0} f_\varepsilon(\gamma(\omega_x)) m_e c^3(\gamma(\omega_x) - \varepsilon_j(N\lambda_w)) / (a^2 m_e^2 c^3 \omega_s^2).$$

The amplification coefficient for the wave amplitude at the same time will amount to $\kappa_E(\omega_s, n_{e0}, y_{0j})/2$. The electron energy in the end of the undulator $\varepsilon_j(N\lambda_w)$ is found from equations (6) and the initial condition $\varepsilon_j|_{x=0} = \gamma(\omega_x)$, as a result of which the dependence on γ_{0j} disappears (only the resonant electrons are taken into account). Finally, for the weak amplification coefficient averaged by the initial transverse coordinate y_{0j} we receive the following

$$\begin{aligned} \bar{\kappa}_E(\omega_s, n_{b0}) &= \frac{1}{L_{ey}} \int_{-L_{ey}/2}^{L_{ey}/2} \kappa_E(\omega_s, n_{e0}, y_{0j}) dy_{0j} \\ &\approx \frac{\pi K_E^2}{L_{ey}} \left(\frac{4\pi e^2 n_{b0}}{m_e \omega_s^2} \right) f_\varepsilon(\gamma(\omega_s)) \gamma(\omega_x) N^3 \\ &\times \int_{-L_{ey}/2}^{L_{ey}/2} dy_{0j} \left(9 \left(\frac{\lambda_w \partial \varphi(0, y_{0j})}{T_{eh} \partial y_{0j}} \right)^2 + 5.3 a^2 \gamma^{-4}(\omega_x) \right). \end{aligned} \quad (7)$$

Two summands in the integral (7) correspond to the resonance of the electron with the combination wave $p_{j\perp}(t, y_{0j}) (\partial a(x, t)/\partial x)$ and wave $K_E \Phi(x, y_{0j}) \partial a^2(x, t)/\partial x$. In (6) the summand in brackets

is $a^2 \gamma^{-4}(\omega_x) \ll 1$, whereas $\frac{\lambda_w \partial \varphi(0, y_{0j})}{T_{eh} \partial y_{0j}} \sim 1$. Therefore, the unidimensional undulator, in which the potential does not depend on y (from foil plates), has a very low amplification coefficient compared to the two-dimensional one (from filaments). The physical reason for it is the longitudinal relativistic mass of the electron $\sim \gamma_{0j}^3$ contrary to the transverse mass $\sim \gamma_{0j}$, which complicates grouping of electrons into bunches and the process of the coherent radiation. For the undulator from the filaments we will disregard the second summand in brackets in (7). The integral for dy_{0j} in (7) due to the periodicity of the potential for y is transformed as follows:

$$\begin{aligned} &\frac{1}{L_{ey}} \int_{-L_{ey}/2}^{L_{ey}/2} \left(\frac{\lambda_w \partial \varphi(0, y_{0j})}{T_{eh} \partial y_{0j}} \right)^2 dy_{0j} \\ &= \frac{1}{\lambda_{wy}} \int_{-\lambda_{wy}/w}^{\lambda_{wy}/2} \left(\frac{\lambda_w \partial \varphi(0, y_{0j})}{T_{eh} \partial y_{0j}} \right)^2 dy_{0j} \\ &= \frac{\lambda_w^2}{\lambda_{wy} r_D^2} \int_{-\lambda_{wy}/2}^{\lambda_{wy}/2} \left(\left(\frac{y_{0j}}{R} \frac{K_1(R/r_D)}{K_0(R/r_D)} \right) \right. \\ &\quad \left. - \left(\frac{\delta n_i \delta r}{n_{i0} R} \right) \frac{I_1(y_0/r_D)}{I_0(R/r_D)} \right) \theta(R - |y_{0j}|) \text{sign}(y_{0j}) \\ &\quad + \left(1 + \frac{\delta n_i \delta r}{n_{i0} R} \right) \frac{K_1(y_{0j}/r_D)}{K_0(R/r_D)} \theta(|y_{0j}| - R) \text{sign}(y_{0j}) \Big) dy_{0j} \\ &\approx \frac{\lambda_w^2}{\lambda_{wy} r_D} \left(\left(1 - \frac{\delta n_i \delta r}{n_{i0} R} \right)^2 \frac{R^3}{3r_D^3} \right. \\ &\quad \left. + \left(1 + \frac{\delta n_i \delta r}{n_{i0} R} \right)^2 \left(1 - \frac{2R}{r_D} - \exp\left(-\frac{\lambda_{wy}}{r_D}\right) \right) \right), \quad R \ll r_D. \end{aligned} \quad (8)$$

Then

$$\begin{aligned} \bar{\kappa}_E(\omega_x, n_{b0}) &= 9\pi K_E^2 \left(\frac{4\pi e^2 n_{b0}}{m_e \omega_s^2} \right) f_\varepsilon(\gamma(\omega_s)) \gamma(\omega_x) N^3 \\ &\times \frac{\lambda_w^2}{\lambda_{wy} r_D} \left(\left(1 - \frac{\delta n_i \delta r}{n_{i0} R} \right)^2 \frac{R^3}{3r_D^3} + \left(1 + \frac{\delta n_i \delta r}{n_{i0} R} \right)^2 \right. \\ &\quad \left. \times \left(1 - \frac{2R}{r_D} - \exp\left(-\frac{\lambda_{wy}}{r_D}\right) \right) \right), \quad R \ll r_D, \quad N \leq N_{\max}. \end{aligned} \quad (9)$$

You can see that within the approximation (5) the additional peak of the ion density always results in the increase of the amplification coefficient (the mean square of the field intensity in the undulator increases). In the numerical calculations $\delta n_i \delta r / (n_{i0} R) \sim 1$, therefore, the change of the amplification coefficient due to the modification of the ion density profile in the filament that is flying away is around one. Also note that the amplification coefficient (7) of the undulator from the filaments corresponds to the amplifica-

tion coefficient of the magnetic undulator with the same period with a precision of up to the numerical factor. λ_W and effective magnetic field B_{eff} , such that $eB_{eff} \approx T_{eh}/r_D$ in case of $\lambda_{Wy} < r_D$ and $eB_{eff} \approx T_{eh}/\sqrt{r_D\lambda_{Wy}}$ for $\lambda_{Wy} > r_D$. The difference of the electric undulator from the magnetic one is the transverse heterogeneity of the deflecting field. Formula (8) corresponds to the averaging in this heterogeneity, having the scale of r_D . At $\lambda_{Wy} \ll r_D$ the field is practically homogeneous by y , and the electrostatic undulator is similar to the magnetic one, and the amplification coefficient (9) does not depend on λ_{Wy} . At $\lambda_{Wy} \gg r_D$ the heterogeneity is strong, the largest part of the period λ_{Wy} is not occupied by the field. The amplification at the same time drops $\sim \lambda_{Wy}^{-1}$.

We will specify as the example of the parameters of the electrostatic undulator that the undulator with $\lambda_W = 20 \mu\text{m}$ and $K_E \approx 1$ is tuned in resonance with the wavelength $\lambda_s = 2\pi/\omega_s = 0.2 \text{ nm}$ (period 0.7 as) at the electron energy of $\varepsilon_e = 200 \text{ MeV}$. For the amplification coefficient $\kappa_E \approx 1$, $N \approx 90$ is required for the electron bunch with the size of $10 \times 10 \times 10 \mu\text{m}$ and charge of 10 pQ. Such undulator will have the length of 1.8 mm, which is rather doable. The field intensity in such undulator will amount to 10^7 Gs at $r_D = 1.5 \mu\text{m}$, and the filament plasma temperature necessary for this will amount to $T_{eh} \approx 5 \text{ MeV}$ at plasma density between the filaments being $n_{eh} \approx 0.1\pi m_e c^2 / e^2 \lambda_L^2$. Note that it is not possible to currently design a magnetic undulator with the field intensity $\sim 10^7 \text{ Gs}$, besides, for the amplification of the considered impulse in the FEL with such undulator, an electron bunch will be required with the significantly higher energy ($\sim 1.5 \text{ GeV}$) than for the electrostatic undulator with the implemented intensity of the magnetic field. Therefore, the electrostatic undulator on the basis of the ionized microstructure makes it possible to substantially decrease the energy of the electron bunch in FEL and the undulator length. The amplification coefficient $\kappa_E(\omega_x, n_{b0}, y_0)$ of the monochromatic component of USI with frequency ω_x corresponds to the spatial increment of USI harmonic growth, therefore, to calculate the amplification of USI made of multiple harmonics, let us multiply the amplitude of each harmonic of the spectrum $a_s(\omega)$ of the dimensionless vector potential of USI by its amplification coefficient, after which using the inverse Fourier transform of the spectrum of the amplified USI and averaging by y_{0j} (for all electrons of the bunch) we will determine the time profile of the amplified impulse and assess its duration after amplification. Such method to assess the duration of the amplified USI is approximated. A stricter method based on the solution to the wave equation for the USI is described in Annex 2. Note that the dimensionless vector potential $a = eA/m_e c^2$ corresponds to the dimensional electric field $E = am_e c \omega_s / e$. Therefore, the time profile of USI after

amplification will be determined by the formula

$$a(t) = \frac{2\tau_a}{\lambda_{Wy}} \int_{-\lambda_{Wy}/2}^{\lambda_{Wy}/2} dy_{0j} \int_0^\infty \exp(\kappa(\omega_s, n_{b0}, y_{0j})/2) a_s(\omega_s t) d\omega_s, \quad (10)$$

The integral (10) may be assessed by the saddle method with the bell-shaped form of the inlet impulse spectrum [7] with duration of the envelope $2\pi/\omega^*$ and the period of the carrier $\sim \tau_a$:

$$a_s(t) = a_{s0} \exp(-(\omega^* t)^2) \sin \frac{\pi t}{\tau_a}, \quad \omega^* \tau_a \leq 1,$$

$$a_s(\omega) = \frac{ia_{s0} \exp(-(\pi^2 + (\omega\tau_0)^2)/4(\omega^* \tau_a)^2) \text{sh}(\pi\omega\tau_a/2(\omega^* \tau_a)^2)}{2\sqrt{\pi}\omega^*}. \quad (11)$$

The duration of the envelope depends on the boundary frequency ω^* of the filter that cuts off the harmonics with the frequencies below ω^* [17]. This cut-off makes the unipolar component [18] present in the USI reflected from the thin target disappear [4]. When (11) is substituted into (10), the assessment of the amplified impulse duration has the following appearance

$$\begin{aligned} \tau_N &\approx \tau_a + \int_{-\lambda_{Wy}/2}^{\lambda_{Wy}/2} dy_{0j} \frac{\omega_s}{8\lambda_{Wy}} \frac{\partial^2}{\partial \omega_s^2} \kappa(\omega_x, \gamma(\omega_x), n_{b0}, y_{j0}) \Big|_{\omega_s \approx \tau_a^{-1}} \\ &= \tau_a + \frac{\omega_x}{8} \frac{\partial^2}{\partial \omega_s^2} \bar{\kappa}(\omega_s, \gamma(\omega_s), n_{b0}) \Big|_{\omega_s \approx \tau_a^{-1}} \\ &\approx \tau_a \left[1 + \frac{15}{32} \bar{\kappa}(\omega_s, \gamma(\omega_x), n_{b0}) \Big|_{\omega_s \approx \tau_a^{-1}} \right]. \end{aligned} \quad (12)$$

Dimensionless increase of the impulse duration will be

$$\frac{\tau_N - \tau_a}{\tau_a} = \frac{\delta\tau_a}{\tau_a} \approx \frac{15}{32} \bar{\kappa}(\omega_x, \gamma(\omega_x), n_{b0}) \Big|_{\omega_s \approx \tau_a^{-1}},$$

where $\gamma(\omega_x = \tau_a^{-1}) = \gamma_b$,

$$\bar{\kappa}_E(\omega_x, n_{b0}) \approx 3\pi K_E^2 f_\varepsilon(\gamma) \gamma N^3 \frac{n_{b0}}{n_c} \omega_L^2 \tau_a^2 \frac{\lambda_2^2}{\lambda_{Wy} r_D} \frac{R^3}{r_D^3},$$

$$R \ll r_D, \quad n_{b0}/n_c \approx 10^{-2},$$

$$f_\varepsilon(\gamma) = \pi^{-1/2} (\Delta\gamma)^{-1} \exp(-(\gamma - \gamma_b)^2 / (\Delta\gamma)^2),$$

$$\gamma = \sqrt{(\lambda_W / 4\pi c \tau_a)(1 + K_E^2/2)}.$$

Under weak amplification of $\bar{\kappa} \sim 1$ the duration increases insignificantly: $\delta\tau_a/\tau_a \leq 1$.

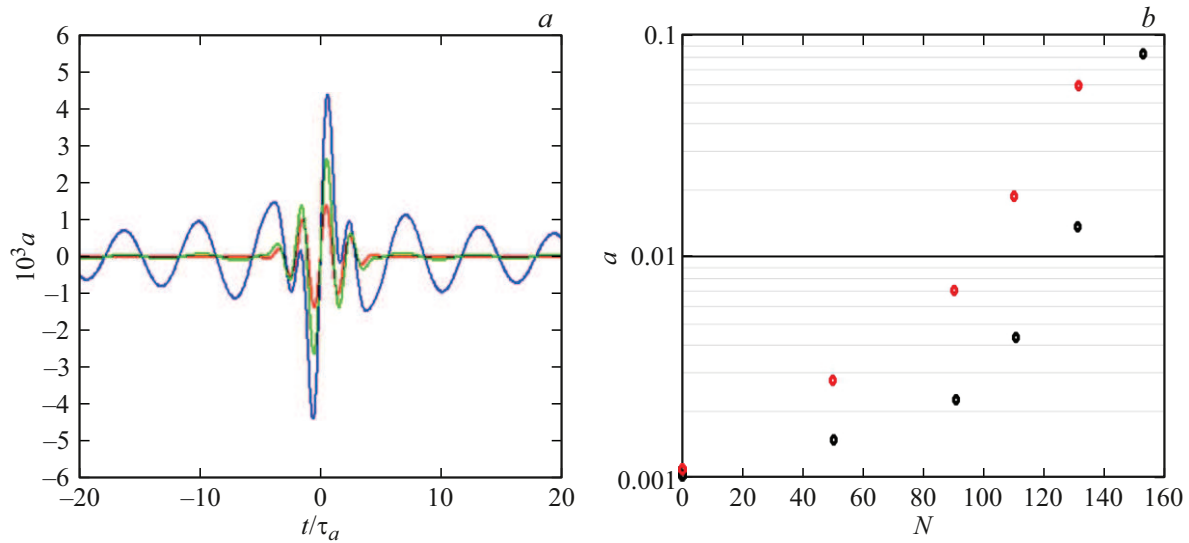


Figure 3. (a) The time profile (10) of the USI amplified in the undulator (blue and green curves) against the inlet impulse (red line): $K_E = 1$, $\Delta\gamma = \gamma_b/3$, $\omega^* = 0.4\tau_a^{-1}$, $N = 100$, $\lambda_W = 20\mu\text{m}$, $\lambda_s = 0.2\text{ nm}$ (period 0.7 as, the impulse contains 3 periods), resonance energy of electron is 200 MeV, $\lambda_{Wy} = 2r_D$ (blue curve), $\lambda_{Wy} = 3r_D$ (green curve). (b) Black circles — amplitude of the blue peak ($\lambda_{Wy} = 2r_D$) from the figure (a) as the function of the number of undulator periods; red circles — the transverse period of the undulator $\lambda_{Wy} = 0.5r_D$, $N \leq N_{\max} \sim 200$.

Calculation (10) by the saddle method and integration by y_{0j} yield the USI profile:

$$a(t) \approx a_{s0} \frac{2r_D \exp(\kappa_E(\tau_a^{-1}, n_{b0}, y_{0j}))}{\lambda_{Wy} \kappa_E(\tau_a^{-1}, n_{b0}, y_{0j} = 0)} \times sh\left(\frac{\lambda_{Wy} \kappa_E(\tau_a^{-1}, n_{b0}, y_{0j} = 0)}{2r_D}\right) \sin \frac{\pi t}{\tau_a} \exp\left(-\frac{t^2}{\tau_N^2}\right). \quad (13)$$

The profile $a(t)$ calculated from (10) together with the initial profile $a_s(t)$ (11) is provided in Fig. 3, a. Fig. 3, b shows the dependence of the maximum value of the amplified impulse field on the number of undulator periods N . From Fig. 3 you can see that the amplitude of the inlet signal increases significantly, and the duration of the amplified part is comparable to the duration of the inlet impulse. Besides, the amplification leads to „wings“ of the amplified impulse, which may be filtered. Amplification strongly depends on the parameter λ_{Wy}/r_D and decreases as it grows, since the part of the electron bunch gets into a very weak field and may not effectively amplify the USI. Note also that to amplify the entire spectrum of USI with $\omega^* = 0.4\tau_a^{-1}$, the spread of bunch electrons by energies $\Delta\gamma/\gamma \leq 0.3$ is sufficient. Amplification by amplitude that is higher by more than an order of magnitude requires $N > 100$ periods of the undulator. In paper [4] it is shown that the minimum duration of attosecond impulses generated upon reflection of the laser impulse from thin plates of foil is ~ 1 as. Therefore, the conducted assessments show the possibility to amplify the short impulses with the duration of around an attosecond.

Numerical modeling of USI amplification

To calibrate the developed analytical model, numerical modeling was used, in which the dynamics of the radiation impulses and the electron bunch in the undulator was conducted by the method of particles in the cell using EPOCH code [19]. Let us assess the parameters of calculations that are possible in our conditions: numerical modeling of the considered arrangement must be two-dimensional, and in solving the equations the calculation step in the undulator axis must be $\leq \lambda_x/5$, then $n_x \approx 5N\lambda_W/\lambda \approx 10\gamma^2 N$ steps will fit in its length, and in the transverse direction at least three rows of filaments are required, and the number of steps will be $n_y \approx 5\lambda_{Wy}/\lambda_x \approx 10\gamma^2$ at $\lambda_{Wy} \approx \lambda_W$. Then the total number of cells will be $\sim 10^2 \gamma^4 N$, and taking into account the fact that the number of the possible cells for the computer cluster $\leq 10^8$ used in the calculations, we will receive the limitation to the modeling parameters of the electrostatic FEL: $\gamma^4 N \leq 10^6$. Considering the undulator with $N \approx 100$, we receive the electron bunch with $\gamma \leq 100$, and at $\lambda_W = 2\mu\text{m}$ the minimum FEL wavelength available for the calculations will be $\lambda_x \sim 10\text{ nm}$ at calculation step of $\sim 1\text{ nm}$. I.e. the required precision of modeling enables considering the amplification of impulses > 30 as in our conditions. Note that at $\gamma \leq 10$ the initial function of electron bunch distribution in the undulator is distorted, therefore only a part of the undulator will be involved in the amplification. For the numerical modeling of the FEL operation process to produce the ambipolar field in the undulator, a laser impulse was selected with the duration of 30 fs and intensity of $3 \cdot 10^{20}\text{ W/cm}^2$. The undulator corresponded to Fig. 2 with periods $\lambda_W = \lambda_{Wy} = 2\mu\text{m}$ and three

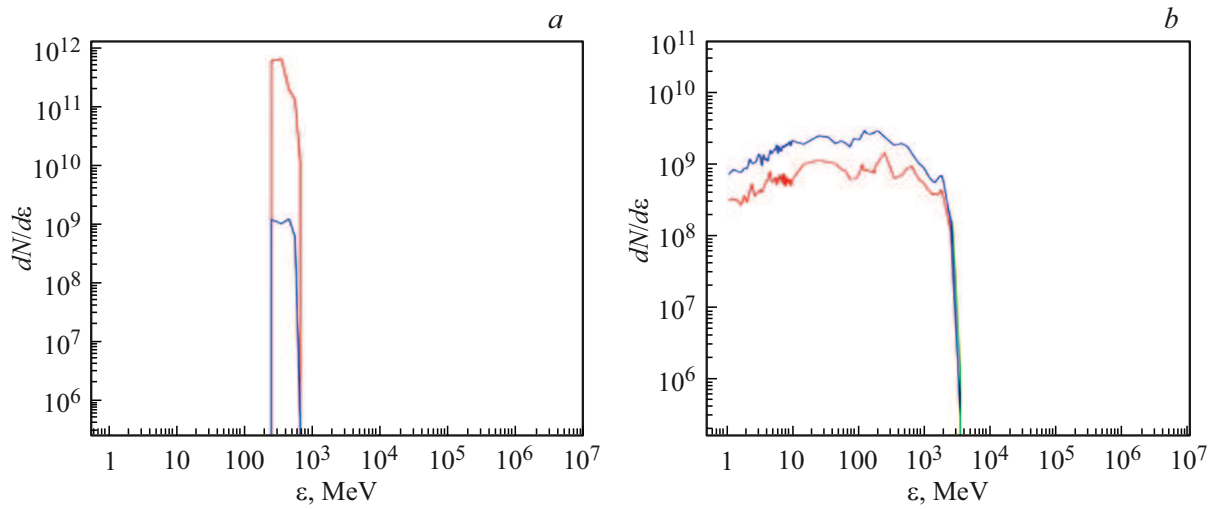


Figure 4. (a) Energy distribution $dN/d\varepsilon$ of electrons in the inlet to the modeling box; (b) energy distribution of electrons via $10\mu\text{m}$ of the electron path in the undulator. Red highlights the group that is necessary to amplify (flies to the angle $< 15^\circ$ towards the undulator axis), blue — flies to the angle from 15° to 30° .

rows of filaments with radius of 250 nm in the modeling box with size of $80 \times 8\mu\text{m}$ (the box length fit 40 undulator links). The undulator parameter when heated by the laser impulse in this calculation was $K_E = eE_{am}r_D/mc^2 \approx 0.7$. The modeling step due to limitations of computer resources and precision of calculation was 10 nm , therefore „the attopulse“ chosen was the impulse with the wavelength $\lambda_x = \lambda_L/5$ ($\sim 500\text{ as}$). According to the estimates (9), to amplify the second impulse with the wavelength $\lambda_x = \lambda_L/5$, the parameters of the electron bunch $n_{b0}/n_{cr} \approx 10^{-3}$ and $\gamma_b \approx 3$ are necessary. Besides, the amplification coefficient (9) at $N = 10$ periods of the undulator will be $\kappa_E \approx 2$. Accordingly, the second impulse was added to the modeling box with wavelength $\lambda_L/5$, intensity 10^{17} W/cm^2 , duration of 10 fs (duration depends on the noise-signal ratio in the Fourier analysis of electromagnetic fields in the modeling box) and the electron bunch with the parameters $n_{b0} = 5 \cdot 10^{18}\text{ cm}^{-3}$, $\gamma_b = 2.5$, $\Delta\gamma/\gamma_b = 0.3$ with duration of 15 fs . Compared to the optimal assessment, the initial density is increased to compensate for the distortion of the energy distribution of electrons when the bunch moves in the conductor. The second impulse lagged behind the first one by 40 fs and jointly with the bunch moved in the field of the undulator filaments formed by the laser impulse. In the transverse direction all impulses and the bunch occupy the entire modeling box, and the longitudinal profile of the impulses and the bunch were super-gaussian. Note that the first laser impulse also generates the fifth harmonic and the resonance group of the fast electrons, which spread together with it, but these factors are not sufficient for a noticeable amplification. In the calculations the impulses and the bunch passed through the entire undulator, and the amplification was diagnosed using the evolution of the amplitude of the fifth radiation harmonic. Additionally the function of electron bunch distribution in the angles and energies

was monitored. The bunch diagnostics demonstrated the change in the initial function of distribution by energies already in $10\mu\text{m}$ passes of the electron bunch through the undulator, as shown in Fig. 4. The reason for this is the electric field of filaments, causing blurred distribution by energies and angles. The condition $\gamma_b \geq N\sqrt{K_E}$ in this modeling is only met at the initial periods of the modulator, whereas for the selected parameters of the calculation it is $N_{\max} \sim 50$. The spatial step of modeling prevents increasing γ_b . Note that if the condition $\gamma_b \geq N_{\max}\sqrt{K_E}$ is not met, you can use the estimate (9) either with small N and initial γ_b , n_b , or for $N \leq N_{\max}$ and account for the change in the function of distribution of fast electrons shown in Fig. 4, b, and substitute parameters γ_b , n_b from the results of the numerical count. Modeling of the bunch motion showed that at ~ 100 it is slowed down significantly, and the energy component of the electrons necessary for amplification that is present in Fig. 4 decreases, b, and it is not possible to extend the path of the electrons in the undulator by increasing γ_b due to the limitations of the calculation accuracy. Fig. 5 shows the evolution of the amplitude of the fifth harmonic of USI as the function of time of laser impulse passage through the undulator. The second impulse enters the box at 70 fs and fully located in the undulator at the moment of 90 fs . The Fourier harmonic was calculated using the space (along axis x , at the fixed coordinate y) distribution in the box at the fixed moment of time. Red circles show the results of the calculation of the above arrangement with the intensity of the laser impulse $3 \cdot 10^{20}\text{ W/cm}^2$ and density of bunch electrons $5 \cdot 10^{18}\text{ cm}^{-3}$. In the interval of $0\text{--}70\text{ fs}$ (Fig. 5) there was no USI, and the fifth harmonic emerged due to the laser impulse passage in the undulator (this is „the noise level“ of the harmonic specified by the dashed horizontal line). In the interval $80\text{--}150\text{ fs}$ the electron bunch and USI move in the

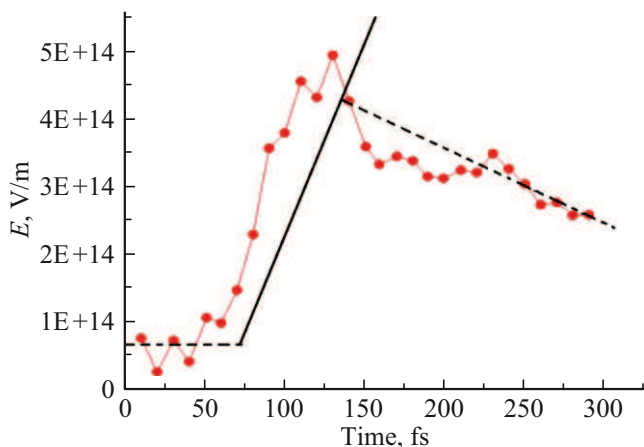


Figure 5. Amplitude (in V/m) of the fifth harmonic as the function of time (in fs). Red circles correspond to the FEL implementation with the specified parameters at intensity of the laser impulse heating the undulator, $3 \cdot 10^{20} \text{ W/cm}^2$ and the electron bunch with electron density $5 \cdot 10^{18} \text{ cm}^{-3}$ and energy $\gamma_b = 2.5$. The horizontal black dashed line corresponds to the level of noise of the fifth harmonic from the first laser impulse. The solid black inclined straight line — estimate of the amplitude of the amplified harmonic using formula (9). The dashed inclined straight line corresponds to the termination of amplification due to the end length of the undulator heating.

undulator, which causes the growth of its amplitude (fifth harmonic). The inclined black straight line corresponds to formula (9) of the fifth harmonic amplification coefficient, according to which

$$a_5(f > 70 \text{ fs}) = a_5(t = 70 \text{ fs})(1 + c(t - 70 \text{ fs})\kappa_E/2\lambda_W).$$

At times longer than 200 fs, the amplification stops, since the laser impulse is absorbed greatly in the undulator, and the field of the undulator elements decreases. To confirm this, the passage of two impulses in the undulator without the electron bunch was also calculated. This calculation showed attenuation of the fifth harmonic and absence of amplification. Note that the spontaneous radiation yields the level of amplitude of several orders of magnitude less than in Fig. 5. Also note that lower harmonics are not amplified, and their time evolution is the same with or without the electron bunch.

Therefore, the analytical and numerical modeling showed the principal possibility of FEL arrangement implementation with plasma electrostatic undulator and USI amplification, however, the limited precision of calculations prevented consideration of the dynamics of impulses with the duration of around an attosecond and to obtain the higher coefficients of radiation amplification predicted by the analytical estimates. The reason for it is related to the limited computer resources, which caused a correct calculation only at the scale of hundreds of attoseconds with the help of the used code, which we plan to modify in the future.

Conclusion

In this paper, using analytical and numerical modeling, the possibility of amplifying the ultrashort electromagnetic impulses was studied with the help of FEL built on a plasma electrostatic undulator produced when the laser impulse intensity passes through the periodical structure of nanofilaments. The period of this undulator is several microns, which makes it possible to use lower values of Lorentzian factor of the electron bunch in FEL compared to the magnetic undulator. It was found that the electrostatic undulator on the basis of the ionized microstructure makes it possible to substantially decrease the energy of the electron bunch and the undulator length. Theoretical estimates show the possibility of significant amplification of the short-wavelength attosecond impulses in the plasma undulator. In the conducted numerical modeling the limitations of choosing the spatial step of integration and the precision of calculations made it possible to implement small amplification coefficients and to consider only the impulses of subfemtosecond duration, the amplification of which corresponds to the produced analytical estimates. Further it is suggested to modify the numerical code to enable the calculations in the attosecond range of durations of the amplified impulses.

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Annexes

1. Parameters of laser plasma undulator

A density peak arises in the front of ion density [11], and the model profile of ion density of the filament takes the form of Fig. 6, *a*, corresponding to the data of numerical modeling of Fig. 6, *b*. In numerical modeling in Fig. 6 the laser impulse with intensity 10^{19} W/cm^2 and duration of 30 fs interacted with the ionized golden filament with radius of 100 nm. Distribution of the field for the potential (1) in the radius r of the filament has the form shown in Fig. 7, *a* compared to the numerical modeling in Fig. 7, *b*. The radial profile of the field depends on parameter $\delta n_i \delta r / (n_{i0} r_D)$. Comparison of Fig. 6, 7, *a* and Fig. 6, 7, *b* shows the adequate description with the analytical model of calculations.

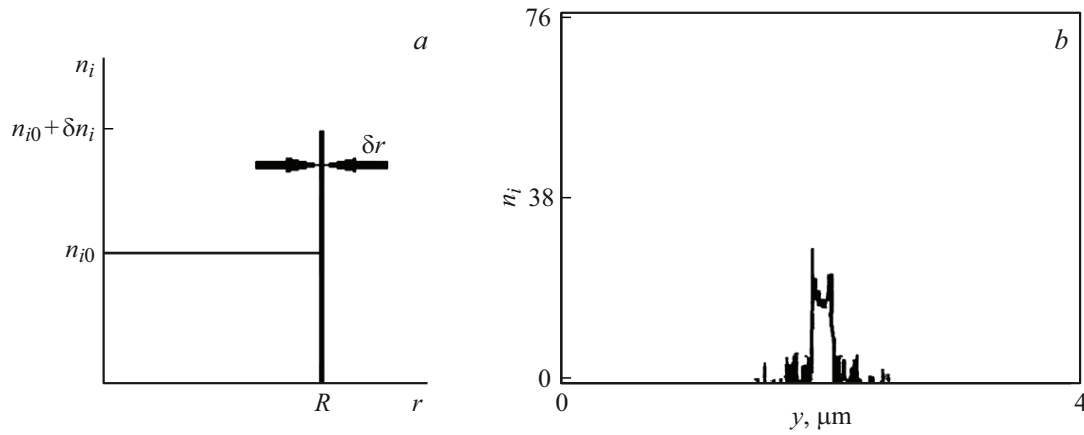


Figure 6. Density profile (n_i) of flying filament ions: (a) model, density in arbitrary units; (b) numerical count, density in units 10^{22} cm^{-3} , $t = 80 \text{ fs}$.

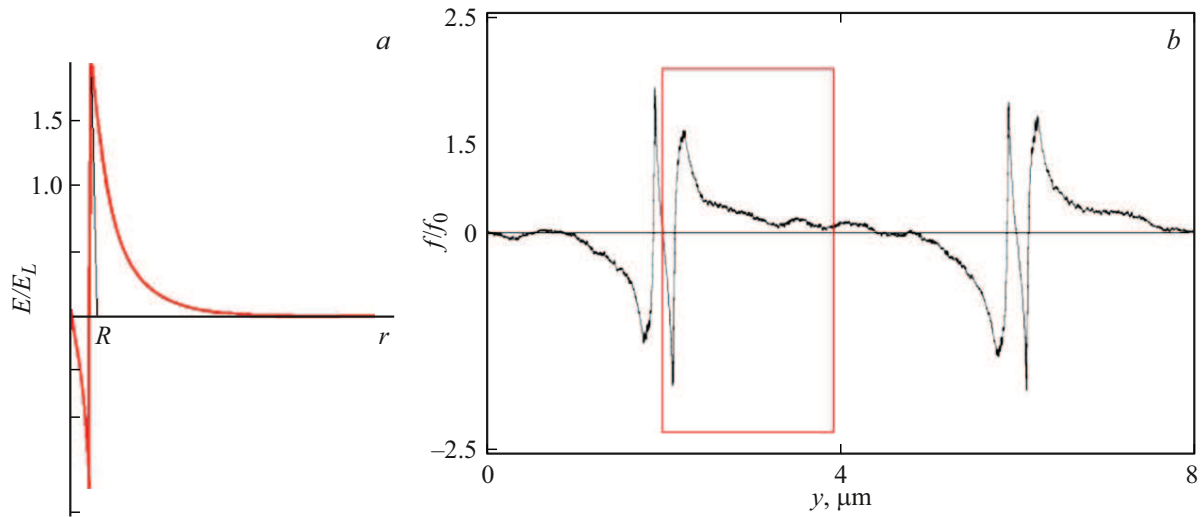


Figure 7. (a) Analytical field of the filament (in relative units) at $\frac{\delta n_i \delta r}{n_{i0} r_D} > 1$. (b) The estimated field of the golden filament (in relative units) with radius of 100 nm, radiated with the laser impulse with intensity of 10^{19} W/cm^2 and duration of 30 fs, the field is provided at the moment of time 30 fs after the completion of the exposure of the laser impulse at the filament.

2. Equation for time profile of amplified USI

To calculate the USI amplification coefficient with the specified time profile, let us solve the wave equation for the vector potential of USI with the bunch electron current density

$$j_{\perp}(x, y, t) = en_e c \int_0^{L_e} dx_{0j} \int_1^{\infty} d\gamma_{0j} f_{\varepsilon}(\gamma_{0j}) \times \frac{(p_{j\perp}(t, y_{0j}, x_{0j}, \gamma_{0j}) - a(x, t)) \delta(ct - ct(x, y_{0j}, x_{0j}, \gamma_{0j}))}{\varepsilon_j(x, y_{0j}, x_{0j}, \gamma_{0j}) - K_E \Phi(x_j, y_j)}, \quad (\text{A1})$$

where functions $p_{j\perp}(t, y_{0j}, x_{0j}, \gamma_{0j})$, $\varepsilon_j(x, y_{0j}, x_{0j}, \gamma_{0j})$, $t(x, y_{0j}, x_{0j}, \gamma_{0j})$, are taken from the solution of the system (6). The expansion (A1) into Taylor series using small parameters K_E/ε_j , $p_{j\perp}/\varepsilon_j$ results in the following

structure of the current density in the wave equation for the vector potential of USI:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{c^2 \partial t^2} \right) a(x, t) = - \frac{4\pi e^2 n_{e0}}{m_e c^2 L_{ey}} \int_{-L_{ey}/2}^{L_{ey}/2} dy_{0j} \int_0^{L_e} dx_{0j} \int_1^{\infty} d\gamma_{0j} f_{\varepsilon}(\gamma_{0j}) \times \left(\frac{c K_E}{\gamma_{0j} L_e} \int_0^t dt \frac{\partial \Phi(x_{0j} + ct(1 - 0.5\gamma_{0j}^{-2}), y_{0j})}{\partial y_{0j}} - \frac{a(x, t)}{L_e \gamma_{0j}} + \frac{K_E}{\gamma_{0j}} \int_0^x dx \frac{\partial \Phi(x + x_{0j}, y_{0j})}{\partial y_{0j}} \frac{\partial}{\partial x} a(x + x_{0j}, t) \right). \quad (\text{A2})$$

The first summand in the brackets of the right part describes the spontaneous radiation of the electron in the undulator [10], the second one — takes into account the effect of USI dispersion in the electron bunch ($\omega_x^2 \rightarrow \omega_x^2 \rightarrow \omega_{pe}^2$), the third one is responsible for the process of the forced radiation and amplification of the USI field, it is considered in this paper. The wave equation (A2) is integral (the current integrally depends on the field), and to determine the space-time structure of the USI field, must be solved with the initial condition (11) at the inlet to the undulator.

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