

Construction and study of the electrostatic T -matrix and polarizability for particles with spheroidal and spherical layer boundaries

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The electrostatic problem for a multilayered particle having spherical or non-confocal spheroidal layer boundaries arranged in an arbitrary order are solved. Scalar potentials in the vicinity of the layer boundaries are expanded using spherical or spheroidal bases, respectively, and the potential expansions in each layer are matched using expansions of spheroidal functions in terms of spherical functions and vice versa. Using the extended boundary condition method with a spheroidal basis (two schemes of this method are employed), infinite systems of linear algebraic equations for the expansion coefficients of the unknown fields are obtained. The solution to the problem is formulated in terms of a T -matrix, which is convenient for considering particle ensembles. It is shown that the resulting T -matrix must be symmetric, which is important for controlling the accuracy of computations. In this work, the Rayleigh approximation, which is well-known for confocal spheroids, is extended to a significantly wider class of particles, taking into account that the T -matrix element T_{11} is directly proportional to the particle's polarizability. In the particular case of two-layer spheroidal particles with a spherical core, considered previously, the performed calculations demonstrate the high efficiency of the proposed solution to the electrostatic problem.

Keywords: Rayleigh approximation, electrostatics, layered spheroids, T -matrix, particle polarizability.

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Introduction

Scattering of light with particles, whose size is much smaller than the wavelength of the incident radiation is an important part of many optical studies. Simple Rayleigh–Hans approximation [1,2] of homogeneous ellipsoidal particles has been a convenient tool for many decades for similar applications of the light scattering theory. This approximation was confirmed many times [3]. Based on the electrostatic approximation, where the wave number is zero, the Rayleigh–Hans approximation included ellipsoids with a shell, where the layer boundaries are cofocused [4].

Electrostatic approximation also developed based on the needs of current research [5]. In particular, using the extended boundary condition method (EBCM) with a spherical basis and an apparatus of T -matrices [6], developed in articles of the group of Professor E.C. Le [3], paper [7] proposed the improved electrostatic approximation.

However, the application of such approach even to lamellar spheroids is rather limited. In paper [8] it was shown that when EBCM is used with the spherical basis, the task may only be resolved if the condition $a_i/b_i \geq \sqrt{2} + 1$ is met for all surfaces, except for the nucleus, with any ratio of half axes. The tools that enabled to obtain this result are described in articles [9,10] for spherical

and spheroidal bases accordingly. In paper [11] this statement was confirmed numerically for two-layer cofocus spheroids, the obvious solution for which is provided by the application of the spheroidal basis. Therefore, to solve electrostatic problems for multi-layer particles with non-cofocus spheroidal boundaries of layers, use the approach that accounts for the problem geometry to the maximum degree.

This paper is dedicated to the solution of electrostatic problems for multi-layer particles with spherical and spheroidal boundaries of the layers arranged randomly. Previously the private cases of lamellar spheroids were considered [12], as well as of two-layer spheroids with the spherical nucleus [13]. Please find below the uniform approach to building T -matrix and polarizability for any above multi-layer particles, which includes two patterns for production of surface integral equations in implementation of the EBCM method. The specific moment of the suggested solution is presentation of the field near the layer boundary in the form of expansion either by spheroidal or by spherical harmonics, which enables to account for the particle geometry to the maximum degree. After algebraization of the system of integral equations, they are solved in an explicit form in virtue of accounting for the geometry of each interface. Then two expansions

are linked for a single layer. As a result the transition matrices linking the corresponding expansions are infinite-dimensional matrices, except for the case of cofocus layer boundaries. Therefore, the built T -matrix has the simplest form. An important role is played by the main element T_{11} , which with a precision of up to multiplier $\frac{4\pi}{3}$ matches the particle polarizability. Then the properties of T -matrix are considered, namely its symmetricity and dependence of elements on the size of the lamellar particle. The link is analyzed between T -matrices built in spherical and spheroidal bases. Symmetricity of the considered T -matrices, and also the coincidence of polarizability (or main element T_{11}), calculated for different patterns, serves as a good test for the validity of the conducted numerical calculations. The analysis of the results of numerical calculations produced previously for private cases, shows high efficiency of this approach compared to the standard EBCM regardless of the degree of extension or flatness of the layers in the considered particles.

Problem setup

Electrostatic problems are usually solved using scalar potentials Φ , the gradient of which provides the intensity of the electric field [4,14]:

$$\mathbf{E} = -\nabla\Phi. \quad (1)$$

Let us first formulate the problem for the case of a two-layer particle. Let us introduce two indices for potentials Φ_i^j , where the upper $j = 1, 2, 3$ indicates the field potential outside the particle, inside the shell and nucleus accordingly. The lower index takes value $i = 1$ for the regular part that has no features in the origin of coordinates, and value $i = 2$ for the irregular part, which reduces to zero in the infinity and has a feature in the origin of coordinates.

Let us specify the field potential outside the particle as $\Phi^1 = \Phi_1^1 + \Phi_2^1$, where Φ_1^1 — potential of the constant external field, Φ_2^1 — potential of the „scattered“ field, arising due to the particle polarization. The field potential inside the shell is also presented as the sum of two summands: $\Phi^2 = \Phi_1^2 + \Phi_2^2$, where the first part (Φ_1^2) corresponds to the regular field, and the second one (Φ_2^2) — to the irregular one. Finally, for physical considerations, the field inside the nucleus is described only by a regular component: $\Phi^3 = \Phi_1^3$, since it has no features ($\Phi_2^3 \equiv 0$).

From the Maxwell's equations it follows that scalar potentials Φ_i^j are the solutions of the Laplace equation

$$\Delta\Phi_i^j = 0, \quad (2)$$

since in the electrostatics the wave number is $k = 0$. It is feasible to use the Rayleigh approximation based on the solution to the electrostatic problem in consideration of the wave processes, if the particle size is much smaller than the radiation wavelength. In the electrostatics the boundary conditions consist in the continuity of the

tangential components of the electric field intensity $\mathbf{E}$ and normal components of the electric induction $\mathbf{D} = \epsilon\mathbf{E}$ on the interface surfaces [4,14]. For the potentials the corresponding conditions may be written as follows:

$$\left\{ \begin{array}{l} \Phi_1^j + \Phi_2^j = \Phi_1^{j+1} + \Phi_2^{j+1}, \\ \epsilon_j \frac{\partial(\Phi_1^j + \Phi_2^j)}{\partial n_j} = \epsilon_{j+1} \frac{\partial(\Phi_1^{j+1} + \Phi_2^{j+1})}{\partial n_j}, \end{array} \right\}_{\mathbf{r} \in S_j} \quad (3)$$

where $j = 1, 2$. Here $\frac{\partial}{\partial n_j}$ — derivatives along the external normal lines to the surfaces of particle S_1 and nucleus S_2 , and $\epsilon_j, \epsilon_{j+1}$ — dielectric permittivities of j th and $(j+1)$ th media, which are separated by surface S_j .

Equations (1)–(3) are problem setup in a differential form. It may be used, for example, to solve the problem by the method of variable separation [15]. At the same time another approach is used more often, which is formulated as the system of surface integral equations and is called the extended boundary condition method (EBCM) [8].

The foundation for the problem setup in the form of surface integral equations are the ratios similar to the Stratton–Chu identical relations [16]. For regular (Φ_1^j, Φ_1^{j+1}) and irregular (Φ_2^j, Φ_2^{j+1}) potentials these relations related to the surface S_j , may be written as follows:

$$\int_{S_j} \left\{ \Phi_1^{j,(j+1)}(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n_j'} - \frac{\partial \Phi_1^{j,(j+1)}(\mathbf{r}')}{\partial n_j'} G(\mathbf{r}, \mathbf{r}') \right\} ds' = \begin{cases} -\Phi_1^{j,(j+1)}(\mathbf{r}), & \mathbf{r} \in D_j, \\ 0, & \mathbf{r} \in R^3 \setminus \bar{D}_j, \end{cases} \quad (4)$$

$$\int_{S_j} \left\{ \Phi_2^{j,(j+1)}(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n_j'} - \frac{\partial \Phi_2^{j,(j+1)}(\mathbf{r}')}{\partial n_j'} G(\mathbf{r}, \mathbf{r}') \right\} ds' = \begin{cases} 0, & \mathbf{r} \in D_j, \\ \Phi_2^{j,(j+1)}(\mathbf{r}), & \mathbf{r} \in R^3 \setminus \bar{D}_j, \end{cases} \quad (5)$$

where D_j — area occupied by particle ($j = 1$) or nucleus ($j = 2$), $\bar{D}_j$ — closure D_j . Besides, both above and further one and the same Green's function is used in the Laplace scalar equation

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}, \quad (6)$$

where $\mathbf{r}, \mathbf{r}'$ — radii-vectors of observation and integration points.

Summing up the equation (4) for potential Φ_1^j and equation (5) for potential Φ_2^j , we get

$$\int_{S_j} \left\{ (\Phi_1^j(\mathbf{r}') + \Phi_2^j(\mathbf{r}')) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n_j'} - \frac{\partial (\Phi_1^j(\mathbf{r}') + \Phi_2^j(\mathbf{r}'))}{\partial n_j'} G(\mathbf{r}, \mathbf{r}') \right\} ds' \quad (7)$$

$$= \begin{cases} -\Phi_1^j(\mathbf{r}), & \mathbf{r} \in D_j, \\ \Phi_2^j(\mathbf{r}), & \mathbf{r} \in R^3 \setminus \bar{D}_j \end{cases}$$

and taking into account the boundary conditions (3) on the surface S_j , we get the following equations:

$$\left(\frac{\varepsilon_{j+1}}{\varepsilon_j} - 1 \right) \int_{S_j} \left\{ \frac{\partial (\Phi_1^{j+1}(\mathbf{r}') + \Phi_2^{j+1}(\mathbf{r}'))}{\partial n_j'} G(\mathbf{r}, \mathbf{r}') \right\} ds' \quad (8)$$

$$= \begin{cases} -\Phi_1^{j+1}(\mathbf{r}) + \Phi_1^j(\mathbf{r}), & \mathbf{r} \in D_j, \\ -\Phi_2^{j+1}(\mathbf{r}) - \Phi_2^j(\mathbf{r}), & \mathbf{r} \in R^3 \setminus \bar{D}_j. \end{cases}$$

Now, to solve the electrostatic problem, there are two equations related to surface of particle S_1 , i.e. at $j = 1$, and two equations related to surface of nucleus S_2 , i.e. at $j = 2$, to determine four unknown potentials. Note that in the latter case the corresponding equations are simplified, since the potential is $\Phi_2^3 \equiv 0$.

There is another pattern to build the system of integral equations based on the exclusion of the field potentials inside the particle and inside the nucleus in the left part. First in the equations for the potentials Φ_1^{j+1} , Φ_2^{j+1} , similar to equations (7), we will exclude these potentials with account of boundary conditions (3). As a result we get the equations for potentials Φ_1^j , Φ_2^j :

$$\int_{S_j} \left\{ (\Phi_1^j(\mathbf{r}') + \Phi_2^j(\mathbf{r}')) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n_j'} - \frac{\varepsilon_j}{\varepsilon_{j+1}} \frac{\partial (\Phi_1^j(\mathbf{r}') + \Phi_2^j(\mathbf{r}'))}{\partial n_j'} G(\mathbf{r}, \mathbf{r}') \right\} ds' \quad (9)$$

$$= \begin{cases} -\Phi_1^{j+1}(\mathbf{r}), & \mathbf{r} \in D_j, \\ \Phi_2^{j+1}(\mathbf{r}), & \mathbf{r} \in R^3 \setminus \bar{D}_j. \end{cases}$$

Further we multiply the equations by $\frac{\varepsilon_{j+1}}{\varepsilon_j}$ and apply identical relations (7) as we did above. As a result we get

$$\left(\frac{\varepsilon_{j+1}}{\varepsilon_j} - 1 \right) \int_{S_j} \left\{ (\Phi_1^j(\mathbf{r}') + \Phi_2^j(\mathbf{r}')) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n_j'} \right\} ds' \quad (10)$$

$$= \begin{cases} -\frac{\varepsilon_{j+1}}{\varepsilon_j} \Phi_1^{j+1}(\mathbf{r}) + \Phi_1^j(\mathbf{r}), & \mathbf{r} \in D_j, \\ \frac{\varepsilon_{j+1}}{\varepsilon_j} \Phi_2^{j+1}(\mathbf{r}) - \Phi_2^j(\mathbf{r}), & \mathbf{r} \in R^3 \setminus \bar{D}_j. \end{cases}$$

Now there are two equations related to the surface of particle S_1 , relative to the potentials Φ_2^1 , Φ_1^1 , Φ_2^2 (the potential of the external field Φ_1^1 is known), and one equation related to the surface of the nucleus S_2 beyond the surface, relative to the potentials Φ_1^1 , Φ_2^2 (potential $\Phi_2^3 \equiv 0$). Therefore, to determine three unknown potentials, three integral equations are used. In this case it is sufficient to consider the fields beyond the nucleus. The fourth equation may be used to determine the potential inside the nucleus Φ_1^3 . Generalization of the problem set over the three-layer particles results in the increase of the range of values j by 1, namely $j = 1, 2, 3$ in formulae (3)–(10), starting from the external surface of the particle. For the n -layer particle in the specified formulae $j = 1, 2, \dots, n$.

Spheroidal and spherical bases

The considered multi-layer particles may have both spherical and spheroidal surfaces of the shell and nucleus, but these surfaces must have the single center and axis of symmetry. The further used spherical (r, θ, φ) and spheroidal (ξ, η, φ) systems of coordinates are related to the Cartesian system (x, y, z) in a standard manner [17]:

$$x = r \sin \theta \cos \varphi = \frac{d}{2} [(\xi^2 - f)(1 - \eta^2)]^{1/2} \cos \varphi,$$

$$y = r \sin \theta \sin \varphi = \frac{d}{2} [(\xi^2 - f)(1 - \eta^2)]^{1/2} \sin \varphi,$$

$$z = r \cos \theta = \frac{d}{2} \xi \eta, \quad (11)$$

where d — focus distance. Parameter $f = 1$ for the extended spheroidal coordinates, and $\xi \in [1, \infty)$, $\eta \in [-1, 1]$ and $\varphi \in [0, 2\pi)$; $f = -1$ for the flattened spheroidal coordinates, and $\xi \in [0, \infty)$, $\eta \in [-1, 1]$ and $\varphi \in [0, 2\pi)$.

The below bases consist of harmonics [14], which are solutions to the Laplace equation (2):

a) in the spherical system

$$\Upsilon_{ml}^{(1)}(\mathbf{r}) = r^l v_{ml}(\theta, \varphi),$$

$$\Upsilon_{ml}^{(2)}(\mathbf{r}) = \frac{1}{2l+1} r^{-(l+1)} v_{ml}(\theta, \varphi),$$

$$v_{ml}(\theta, \varphi) = \frac{v_{mle}(\theta, \varphi)}{v_{mlo}(\theta, \varphi)} = \bar{P}_l^m(\cos \theta) \sqrt{\frac{2 - \delta_m^0}{2\pi}} \cos m\varphi, \quad (12)$$

b) in the extended spheroidal system

$$\Psi_{ml}^{(1)}(d, \mathbf{r}) = \Psi_{ml}^{(1)}(d, \xi, \eta, \varphi)$$

$$\Psi_{ml}^{(2)}(d, \mathbf{r}) = \Psi_{ml}^{(2)}(d, \xi, \eta, \varphi)$$

$$= \left(\frac{d}{2} \right)^l P_l^m(\xi) \psi_{ml}(\eta, \varphi)$$

$$= \frac{(l-m)!}{(l+m)!} \left(\frac{d}{2} \right)^{-(l+1)} Q_l^m(\xi) \psi_{ml}(\eta, \varphi)$$

$$\psi_{ml}(\eta, \varphi) = \frac{\psi_{mle}(\eta, \varphi)}{\psi_{mlo}(\eta, \varphi)}$$

$$= \bar{P}_l^m(\eta) \sqrt{\frac{2 - \delta_m^0}{2\pi}} \cos m\varphi, \quad (13)$$

c) in the flattened spheroidal system

$$\begin{aligned}\Psi_{ml}^{(1)}(-id, \mathbf{r}) &= \Psi_{ml}^{(1)}(-id, \xi, \eta, \varphi) \\ \Psi_{ml}^{(2)}(-id, \mathbf{r}) &= \Psi_{ml}^{(2)}(-id, \xi, \eta, \varphi) \\ &= \left(\frac{-id}{2}\right)^l P_l^m(i\xi) Q_l^m(i\xi) \psi_{ml}(\eta, \varphi), \\ \psi_{ml}(\eta\varphi) &= \frac{\psi_{mle}(\eta, \varphi)}{\psi_{mlo}(\eta, \varphi)} = \bar{P}_l^m(\eta) \sqrt{\frac{2 - \delta_m^0}{2\pi}} \cos m\varphi, \quad (14)\end{aligned}$$

where $l \geq m \geq 0$ — non-negative integer numbers, Kronecker symbol $\delta_m^0 = 1$ or 0 , when $m = 0$ and $m \neq 0$ accordingly, $P_l^m(\eta)$ — associated Legendre functions of type 1, and the corresponding normalized functions are

$$\begin{aligned}\bar{P}_l^m(\eta) &= \sqrt{\frac{(2l+1)(l-m)!}{2(l+m)!}} P_l^m(\eta) \\ &= N_{ml}^{-1} P_l^m(\eta).\end{aligned} \quad (15)$$

For the functions that depend on the angular coordinate η , the second linearly independent solutions $Q_l^m(\eta)$ are not considered, since they have features at $\eta = \pm 1$ and are not suitable for physical considerations.

The transition from the extended spheroidal harmonics to the spherical ones is carried out with the help of the limit transition $d \rightarrow 0$, $\xi \rightarrow \infty$, and $\frac{d}{2}\xi \rightarrow r$. For the angular functions $d \rightarrow 0$ is sufficient, and $\eta \rightarrow \cos \theta$. In general for the spheroidal harmonics this limit transition provides the following asymptotics:

$$\begin{aligned}\Psi_{mn}^{(1)}(d, \mathbf{r}) &= C_{mn,1} \Upsilon_{ml}^{(1)}(\mathbf{r}) \left(1 + O\left(\frac{1}{\xi}\right)\right) \\ &= C_{mn,1} \Upsilon_{ml}^{(1)}(\mathbf{r}) (1 + O(d)), \quad (16) \\ \Psi_{mn}^{(2)}(d, \mathbf{r}) &= C_{mn,2} \Upsilon_{ml}^{(2)}(\mathbf{r}) \left(1 + O\left(\frac{1}{\xi}\right)\right) \\ &= C_{mn,2} \Upsilon_{ml}^{(2)}(\mathbf{r}) (1 + O(d)), \quad (17)\end{aligned}$$

which with a precision of up to a constant corresponds to the spherical basis.

Spheroidal angular functions ψ_{ml} form a complete orthonormal system in the space $L_2(\Omega)$ with a square metric:

$$\int_0^{2\pi} \int_{-1}^1 \psi_{mn}(\eta, \varphi) \psi_{\mu\nu}(\eta, \varphi) d\eta d\varphi = \delta_m^\mu \delta_n^\nu, \quad (18)$$

i.e. on the surface of any coordinate spheroid Ω . Spherical angular functions (12) also form a complete orthonormal system in the space $L_2(\Omega)$ with a quadratic metric (18) in any coordinate sphere when substituting $\eta \rightarrow \cos \theta$.

For the functions that depend on the radial coordinates, the following definition is used, which differs by a constant multiplier from the one specified in [14]:

$$P_l^m(\xi) = (\xi^2 - 1)^{\frac{m}{2}} \frac{d^m P_l(\xi)}{d\xi^m},$$

$$Q_l^m(\xi) = (\xi^2 - 1)^{\frac{m}{2}} \frac{d^m Q_l(\xi)}{d\xi^m}. \quad (19)$$

To analyze the relations between the spherical and spheroidal harmonics, use the corresponding vectors $\Upsilon_m^{(i)}(\mathbf{r}) = \{\Upsilon_{ml}^{(i)}(\mathbf{r})\}$ and $\Psi_m^{(i)}(d, \mathbf{r}) = \{\Psi_{ml}^{(i)}(d, \mathbf{r})\}$ from ratios (12), (13). Now for the spherical and spheroidal harmonics of the 1st and 2nd types you can record the matrix equations [12]:

$$\Psi_m^{(i)}(d, \mathbf{r}) = \nabla_m^{(i)}(d) \Upsilon_m^{(i)}(\mathbf{r}), \quad (20)$$

where the transition matrix elements are defined from the ratios

$$\begin{aligned}\nabla_{nl,m}^{(1)}(d) &= (-1)^{(n-l)/2} \left(\frac{d}{2}\right)^{(n-l)} \\ &\frac{(n+l-1)!!}{(n-l)!!(l-m)!} \frac{(2n+1)N_{mn}}{(2l+1)N_{ml}}, \quad n \geq l, \quad (21)\end{aligned}$$

$$\begin{aligned}\nabla_{nl,m}^{(2)}(d) &= \left(\frac{d}{2}\right)^{(l-n)} \frac{(l-m)!}{(l-n)!!(n+l+1)!!} \\ &\frac{(2l+1)N_{ml}}{N_{mn}}, \quad n \leq l. \quad (22)\end{aligned}$$

Since the matrix $\nabla^{(1)}(d)$ links the polynomials of the degree not exceeding n -th, it is lower triangular, i.e. $\nabla_{nl,m}^{(1)}(d) = 0$ at $n < l$. It should be noted that the summation in the formula (20) is conducted using indices, the parity of which matches the corresponding parity of the index in the left part of the equation.

For the spheroidal harmonics of the 2nd type the transition matrices are upper triangular, since for them the estimates (17) are true in the infinity.

In case of a limit transition $d \rightarrow 0$, $\xi \rightarrow \infty$ from the spheroidal harmonics to the spherical ones (16), (17) the transition matrices (20–22) become diagonal:

$$\begin{aligned}\nabla_{nl,m}^{(1)}(0) &= \frac{(2n-1)!!}{(n-m)!} \delta_n^l = C_{mn,1} \delta_n^l, \\ \nabla_{nl,m}^{(2)}(0) &= \frac{(n-m)!}{(2n-1)!!} \delta_n^l = C_{mn,2} \delta_n^l.\end{aligned} \quad (23)$$

Thus, the diagonal elements of the matrices correspond to constants $C_{mn,1}$, $C_{mn,2}$ from formulae (16), (17). Note that the product of these constants is equal to one: $C_{mn,1} C_{mn,2} = 1$.

In paper [12] it is shown that for the transition matrices of the 1st and 2nd types the following ratio is true:

$$\nabla_m^{(1)}(d) \nabla_m^{(2),T}(d) = I, \quad (24)$$

which makes it possible to find the inverse matrices

$$\begin{aligned}\left(\nabla_m^{(1)}(d)\right)^{-1} &= \nabla_m^{(2),T}(d), \\ \left(\nabla_m^{(2)}(d)\right)^{-1} &= \nabla_m^{(1),T}(d),\end{aligned} \quad (25)$$

where T means transposition of the matrix.

If it becomes necessary to find the transition matrices from one spheroidal basis to the other linked to the non-cofocus spheroidal system of coordinates, then you first need to approach the spherical basis, and then another spheroidal one.

$$\begin{aligned}\Psi_m^{(1)}(d_1, \mathbf{r}) &= \nabla_m^{(1)}(d_1) \Upsilon_m^{(1)}(\mathbf{r}) = \nabla_m^{(1)}(d_1) \left(\nabla_m^{(1)}(d_2) \right)^{-1} \\ &\times \Psi_m^{(1)}(d_2, \mathbf{r}) = \Delta_m^{(1)}(d_1, d_2) \Psi_m^{(1)}(d_2, \mathbf{r}).\end{aligned}\quad (26)$$

Taking into account ratios (25), we get

$$\begin{aligned}\Delta_m^{(1)}(d_1, d_2) &= \nabla_m^{(1)}(d_1) \nabla_m^{(2)T}(d_2), \\ \Delta_m^{(2)}(d_1, d_2) &= \nabla_m^{(2)}(d_1) \nabla_m^{(1)T}(d_2).\end{aligned}\quad (27)$$

Taking into account ratios (25) and (27), you can find inverse matrices:

$$\begin{aligned}\left(\Delta_m^{(1)}(d_1, d_2) \right)^{-1} &= \Delta_m^{(1)}(d_2, d_1) \\ &= \nabla_m^{(1)}(d_2) \nabla_m^{(2)T}(d_1) = \Delta_m^{(2),T}(d_1, d_2)\end{aligned}\quad (28)$$

and similarly

$$\begin{aligned}\left(\Delta_m^{(2)}(d_1, d_2) \right)^{-1} &= \Delta_m^{(2)}(d_2, d_1) \\ &= \nabla_m^{(2)}(d_2) \nabla_m^{(1)T}(d_1) = \Delta_m^{(1),T}(d_1, d_2).\end{aligned}\quad (29)$$

Therefore, for inversion of the spheroidal transition matrices, it is sufficient to either switch arguments or to transpose the similar matrix of another type.

Further it will be required to expand the Green's function in the spherical system of coordinates [14]:

$$\begin{aligned}G(\mathbf{r}, \mathbf{r}') &= \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} (2 - \delta_{0m}) \frac{1}{2l+1} (r_{<})^l (r_{>})^{-(l+1)} \\ &\times \bar{P}_l^m(\cos \theta) \bar{P}_l^m(\cos \theta') \cos m(\varphi - \varphi') \\ &= \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} \Upsilon_{ml}^{(1)}(\mathbf{r}_{<}) \Upsilon_{ml}^{(2)}(\mathbf{r}_{>}),\end{aligned}\quad (30)$$

where $r_{<} = \min(r, r')$, $r_{>} = \max(r, r')$.

Similarly, the expansion of the Green's function for the Laplace equation in the extended spheroidal system of coordinates with account of ratios (13) has the following appearance [14]

$$\begin{aligned}G(\mathbf{r}, \mathbf{r}') &= \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} = \frac{1}{2\pi} \frac{2}{d} \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} (2 - \delta_0^m) \\ &\times \frac{(l-m)!}{(l+m)!} P_l^m(\xi_{<}) Q_l^m(\xi_{>}) \bar{P}_l^m(\eta) \bar{P}_l^m(\eta') \cos m(\varphi - \varphi') \\ &= \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} \Psi_{ml}^{(1)}(d, \mathbf{r}_{<}) \Psi_{ml}^{(2)}(d, \mathbf{r}_{>}),\end{aligned}\quad (31)$$

where $\xi_{<} = \min(\xi, \xi')$, $\xi_{>} = \max(\xi, \xi')$.

In case of flattened spheroidal coordinates and harmonics, you need to do the standard substitution of $\xi \rightarrow i\xi$, $d \rightarrow -id$. As a result the formulae (21), (22) will have the following appearance

$$\begin{aligned}\nabla_{nl,m}^{(1)}(-id) &= \left(\frac{d}{2} \right)^{(n-l)} \frac{(n+l-1)!!}{(n-l)!!(l-m)!} \frac{(2n+1)N_{mn}}{(2l+1)N_{ml}}, \\ n &\geq l,\end{aligned}\quad (32)$$

$$\begin{aligned}\nabla_{nl,m}^{(2)}(-id) &= \left(\frac{-id}{2} \right)^{(l-n)} \frac{(l-m)!}{(l-n)!!(n+l+1)!!} \\ &\times \frac{(2l+1)N_{ml}}{N_{mn}}, \quad n \leq l.\end{aligned}\quad (33)$$

Other ratios remain true for the above substitution. For example, the link between the extended and flattened harmonics is recorded as follows:

$$\begin{aligned}\Psi_m^{(1)}(d_1, \mathbf{r}) &= \nabla_m^{(1)}(d_1) \Upsilon_m^{(1)}(\mathbf{r}) \\ &= \nabla_m^{(1)}(d_1) \left(\nabla_m^{(1)}(-id_2) \right)^{-1} \Psi_m^{(1)}(-id_2, \mathbf{r}) \\ &= \Delta_m^{(1)}(d_1, -id_2) \Psi_m^{(1)}(-id_2, \mathbf{r}),\end{aligned}\quad (34)$$

besides

$$\begin{aligned}\Delta_m^{(1)}(d_1, -id_2) &= \nabla_m^{(1)}(d_1) \nabla_m^{(2)T}(-id_2), \\ \Delta_m^{(2)}(d_1, -id_2) &= \nabla_m^{(2)}(d_1) \nabla_m^{(1)T}(-id_2).\end{aligned}\quad (35)$$

When other formulae are made, act in the same manner.

Expansion of potentials by harmonics

With account of the maximum accounting for the geometry of the particle, we will present the field potentials within the vicinity of the spheroidal boundary as expansions by spheroidal harmonics, and in the vicinity of the spherical surface, the potentials will be recorded as expansions by spherical harmonics. The linkage of the same potentials inside the particle shell is carried out using links between the spheroidal and spherical harmonics specified in the previous section.

The spheroidal surface in the accordingly selected spheroidal system is coordinate, therefore its equation is

$$\xi = \xi_0, \quad (36)$$

whereas the equation of the spherical surface in the spherical system is recorded as follows:

$$r = r_0. \quad (28)$$

In the electrostatic (as in the wave) problem for the axisymmetric particles you can implement the separation relative to the azimuth angle [18], i.e. the problem is solved independently for each summand of the generalized Fourier series using the azimuth angle φ .

At the vertical orientation of the external field the corresponding potentials do not depend on the azimuth angle, therefore here the index is $m = 0$, and the horizontal orientation uses only members with the index $m = 1$, which will be considered below when summing by index m .

Regular and irregular potentials in the vicinity of the external surface of the particle will be presented as expansions into spheroidal or spherical harmonics.

$$\begin{pmatrix} \Phi_1^1 \\ \Phi_2^1 \end{pmatrix} = \sum_{l=1}^{\infty} \begin{pmatrix} a_{1,ml}^{(1)} \\ b_{1,ml}^{(1)} \end{pmatrix} \begin{pmatrix} \Phi_{ml}^{(1)}(\mathbf{r}) \\ \Phi_{ml}^{(2)}(\mathbf{r}) \end{pmatrix}. \quad (38)$$

where basis functions are either spheroidal harmonics $\Phi_{ml}^{(1)}(\mathbf{r}) = \Psi_{ml}^{(1)}(d, \xi, \eta, \varphi)$, or spherical ones $\Phi_{ml}^{(1)}(\mathbf{r}) = \Upsilon_{ml}^{(1)}(r, \theta, \varphi)$ depending on the type of the surface, and the dash means that with account of the parity properties the summation must only be conducted using odd indices l . Note that here we mean the potentials of the external and „scattered“ emissions. Further the similar designations will be used for the expansions of the potentials in the shell and nucleus, and they do not depend on the refractive index and differ only by the type of radial functions.

If the single external field is directed along or perpendicularly to the axis of symmetry of the spheroid, in the corresponding rows (38) only the coefficients differ from zero

$$a_{01}^{(1)} = -\sqrt{\frac{4\pi}{3}} \quad \text{or} \quad a_{11}^{(1)} = -\sqrt{\frac{4\pi}{3}} \quad (39)$$

accordingly.

For the field inside the shell in the vicinity of the external surface, the corresponding potentials are presented as

$$\begin{pmatrix} \Phi_1^2 \\ \Phi_2^2 \end{pmatrix} = \sum_{l=1}^{\infty} \begin{pmatrix} a_{1,ml}^{(2)} \\ b_{1,ml}^{(2)} \end{pmatrix} \begin{pmatrix} \Phi_{1,ml}^{(1)}(\mathbf{r}) \\ \Phi_{1,ml}^{(2)}(\mathbf{r}) \end{pmatrix}. \quad (40)$$

Field potentials in the external vicinity of the particle nucleus surface may be presented in the form of

$$\begin{pmatrix} \Phi_1^2 \\ \Phi_2^2 \end{pmatrix} = \sum_{l=1}^{\infty} \begin{pmatrix} a_{2,ml}^{(2)} \\ b_{2,ml}^{(2)} \end{pmatrix} \begin{pmatrix} \Phi_{2,ml}^{(1)}(\mathbf{r}) \\ \Phi_{2,ml}^{(2)}(\mathbf{r}) \end{pmatrix}. \quad (41)$$

Different designations for basis functions in the vicinity of the external and internal surfaces of the shell indicate that they can be either spherical or spheroidal harmonics (12)–(14).

Finally, the potential of the field inside the nucleus is

$$\Phi_1^3 = \sum_{l=1}^{\infty} a_{ml}^{(3)} \Phi_{ml}^{(1)}(\mathbf{r}). \quad (42)$$

Let us link the expansions (40) and (41) used inside the shell for the same potentials. Assume that the external surface of the shell is extended spheroidal, and the internal one — spherical. In this case, taking into account equations (20)–(25), we get:

$$\begin{pmatrix} \Phi_1^2 \\ \Phi_2^2 \end{pmatrix} = \sum_{l=1}^{\infty} \begin{pmatrix} a_{1,ml}^{(2)} \\ b_{1,ml}^{(2)} \end{pmatrix} \begin{pmatrix} \Psi_{ml}^{(1)}(\mathbf{r}) \\ \Psi_{ml}^{(2)}(\mathbf{r}) \end{pmatrix}$$

$$\begin{aligned} &= \sum_{l=1}^{\infty} a_{1,ml}^{(2)} \sum_{n=1}^{\infty} \nabla_{ln,m}^{(1)}(d) \Upsilon_{mn}^{(1)}(\mathbf{r}) \\ &= \sum_{n=1}^{\infty} b_{1,ml}^{(2)} \sum_{n=1}^{\infty} \nabla_{ln,m}^{(2)}(d) \Upsilon_{mn}^{(2)}(\mathbf{r}) \\ &= \sum_{n=1}^{\infty} \begin{pmatrix} a_{2,nn}^{(2)} \\ b_{2,nn}^{(2)} \end{pmatrix} \begin{pmatrix} \Upsilon_{nn}^{(1)}(\mathbf{r}) \\ \Upsilon_{nn}^{(2)}(\mathbf{r}) \end{pmatrix}. \end{aligned} \quad (43)$$

In the matrix form with account of the ratios between the extended spheroidal and spherical harmonics (25) we have

$$\begin{pmatrix} \mathbf{a}_2^{(2)} \\ \mathbf{b}_2^{(2)} \end{pmatrix} = \begin{pmatrix} \nabla^{(1)T}(d) & \mathbf{0} \\ \mathbf{0} & \nabla^{(2)T}(d) \end{pmatrix} \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{b}_1^{(2)} \end{pmatrix}. \quad (44)$$

When this matrix equation is reversed with account of the transition matrix properties, we get

$$\begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{b}_1^{(2)} \end{pmatrix} = \begin{pmatrix} \nabla^{(2)}(d) & \mathbf{0} \\ \mathbf{0} & \nabla^{(2)T}(d) \end{pmatrix} \begin{pmatrix} \mathbf{a}_2^{(2)} \\ \mathbf{b}_2^{(2)} \end{pmatrix}, \quad (45)$$

where the vectors of the expansion coefficients and transposed matrices are used.

In case of the external spherical surface and internal spheroidal one after the similar inversions we get

$$\begin{pmatrix} \mathbf{a}_2^{(2)} \\ \mathbf{b}_2^{(2)} \end{pmatrix} = \begin{pmatrix} \nabla^{(2)}(d) & \mathbf{0} \\ \mathbf{0} & \nabla^{(1)}(d) \end{pmatrix} \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{b}_1^{(2)} \end{pmatrix} \quad (46)$$

and

$$\begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{b}_1^{(2)} \end{pmatrix} = \begin{pmatrix} \nabla^{(1)T}(d) & \mathbf{0} \\ \mathbf{0} & \nabla^{(2)T}(d) \end{pmatrix} \begin{pmatrix} \mathbf{a}_2^{(2)} \\ \mathbf{b}_2^{(2)} \end{pmatrix}, \quad (47)$$

i.e. compared to the previous case the transition matrices switched places.

If the external surface of the considered layer is flattened spheroidal, then in ratios (43)–(45) it is sufficient to substitute $d \rightarrow -id$. Generalizing the obtained result over all possible cases, we get

$$\begin{pmatrix} \mathbf{a}_2^{(2)} \\ \mathbf{b}_2^{(2)} \end{pmatrix} = \begin{pmatrix} \tilde{\nabla}_m^{(1)T}(s_1, s_2) & \mathbf{0} \\ \mathbf{0} & \tilde{\nabla}^{(2)T}(s_1, s_2) \end{pmatrix} \begin{pmatrix} \mathbf{a}_1^{(2)} \\ \mathbf{b}_1^{(2)} \end{pmatrix}, \quad (48)$$

where uniform designations are used for the extended spheroids $s_i = d_i$, for the flattened spheroids $s_i = -id_i$, for the layer of spheroidal — spherical surface $\tilde{\Delta}_m^{(1)T}(s_1, 0) = \nabla^{(1)T}(s_1)$ and, finally, spherical — spheroidal surface $\tilde{\Delta}_m^{(1)T}(0, s_2) = \nabla^{(2)T}(s_2)$.

Building and research of T -matrices

Let us consider the electrostatic problem in more detail for the two-layer particles of different types, using two patterns of the EBCM method. To conclude this section, we will study the properties of the built T -matrices.

EBCM method: 1st pattern

For algebraization of integral equations on the particle surface, let us substitute to the system (8) the expansions of the potentials and the Green's function in the system of coordinates corresponding to the external surface. As a result, in virtue of orthogonality of the basis functions we get

$$\begin{cases} \mathbf{a}^{(1)} &= A_{21}^{(1)} \mathbf{a}_1^{(2)} + A_{22}^{(1)} \mathbf{b}_1^{(2)}, \\ \mathbf{b}^{(1)} &= A_{11}^{(1)} \mathbf{a}_1^{(2)} + A_{12}^{(1)} \mathbf{b}_1^{(2)}. \end{cases} \quad (50)$$

Similarly, for the particle nucleus, using the expansions of the potential and the Green's function, we get:

$$\begin{cases} \mathbf{a}_2^{(2)} = A_{21}^{(2)} \mathbf{a}^{(3)}, \\ \mathbf{b}_2^{(2)} = A_{11}^{(2)} \mathbf{a}^{(3)}. \end{cases} \quad (51)$$

Above they introduced vectors $\mathbf{a}_i^{(j)} = \{a_{ml}^j\}_m^\infty$, $\mathbf{b}_i^{(j)} = \{b_{ml}^j\}_m^\infty$ and diagonal matrices

$$\begin{aligned} A_{21}^{(1)} &= \{\delta_l^n + (\varepsilon_2 - 1)L_{nn,m}^{1,21}\}_m^\infty, \\ A_{22}^{(1)} &= \{(\varepsilon_2 - 1)L_{nn,m}^{1,22}\}_m^\infty, \\ A_{11}^{(1)} &= \{-(\varepsilon_2 - 1)L_{nn,m}^{1,11}\}_m^\infty, \\ A_{12}^{(1)} &= \{\delta_l^n - (\varepsilon_2 - 1)L_{nn,m}^{1,12}\}_m^\infty. \end{aligned} \quad (52)$$

Here a designation is used for the integrals

$$L_{nl,m}^{1,ki} = \int_{S_1} \Phi_{mn}^{(k)}(\bar{r}) \frac{\partial \Phi_{ml}^{(i)}(\bar{r})}{\partial n} ds. \quad (53)$$

These matrices are diagonal in virtue of orthogonality of angular functions (18). Note that for the system (50) you can use the determinant

$$\det = A_{21}^{(1)} A_{12}^{(1)} - A_{11}^{(1)} A_{22}^{(1)} = \varepsilon_2, \quad (54)$$

since the properties of the system with diagonal matrices are similar to the properties of the ordinary systems with linear algebraic equations. For the second system, substituting the corresponding harmonics and integrating on the nucleus surface, we get similar matrices.

Finally with account of ratios (12)–(14) the matrix elements for various types of surfaces may easily be found in an explicit form. For example, for different external surfaces of the two-layer particle we get:

a) extended spheroidal

$$\begin{aligned} (A_{21}^{(1)})_{nm} &= 1 + (\varepsilon_2 - 1) \frac{(n-m)!}{(n+m)!} \\ &\quad ((\xi_0)^2 - 1) P_n^{m'}(\xi_0) Q_n^m(\xi_0), \end{aligned}$$

$$\begin{aligned} (A_{22}^{(1)})_{nm} &= (\varepsilon_2 - 1) \left(\frac{(n-m)!}{(n+m)!} \right)^2 \left(\frac{d}{2} \right)^{-(2n+1)} \\ &\quad \times ((\xi_0)^2 - 1) Q_n^{m'}(\xi_0) Q_n^m(\xi_0), \end{aligned}$$

$$\begin{aligned} (A_{11}^{(1)})_{nm} &= -(\varepsilon_2 - 1) \left(\frac{d}{2} \right)^{2n+1} ((\xi_0)^2 - 1) P_n^{m'}(\xi_0) P_n^m(\xi_0), \\ (A_{12}^{(1)})_{nm} &= 1 - (\varepsilon_2 - 1) \frac{(n-m)!}{(n+m)!} ((\xi_0)^2 - 1) Q_n^{m'}(\xi_0) P_n^m(\xi_0), \end{aligned} \quad (55)$$

b) flattened spheroidal

$$\begin{aligned} (A_{21}^{(1)})_{nm} &= 1 - (\varepsilon_2 - 1) \left(\frac{(n-m)!}{(n+m)!} \right) \\ &\quad ((\xi_0)^2 + 1) P_n^{m'}(i\xi_0) Q_n^m(i\xi_0), \\ (A_{22}^{(1)})_{nm} &= -(\varepsilon_2 - 1) \left(\frac{(n-m)!}{(n+m)!} \right)^2 \left(\frac{-id}{2} \right)^{-(2n+1)} \\ &\quad \times ((\xi_0)^2 + 1) Q_n^{m'}(i\xi_0) Q_n^m(i\xi_0), \\ (A_{11}^{(1)})_{nm} &= (\varepsilon_2 - 1) \left(\frac{-id}{2} \right)^{2n+1} \\ &\quad \times ((\xi_0)^2 + 1) P_n^{m'}(i\xi_0) P_n^m(i\xi_0), \\ (A_{12}^{(1)})_{nm} &= 1 + (\varepsilon_2 - 1) \frac{(n-m)!}{(n+m)!} \\ &\quad \times ((\xi_0)^2 + 1) Q_n^{m'}(i\xi_0) P_n^m(i\xi_0), \end{aligned} \quad (56)$$

c) spherical

$$\begin{aligned} (A_{21}^{(1)})_{nm} &= 1 + (\varepsilon_2 - 1) \frac{n}{2n+1}, \\ (A_{22}^{(1)})_{nm} &= -(\varepsilon_2 - 1) \frac{n+1}{2n+1} (r_0)^{-(2n+1)}, \\ (A_{11}^{(1)})_{nm} &= -(\varepsilon_2 - 1) \frac{n}{2n+1} (r_0)^{2n+1}, \\ (A_{12}^{(1)})_{nm} &= 1 + (\varepsilon_2 - 1) \frac{n+1}{2n+1}. \end{aligned} \quad (57)$$

Above for the testing in all three cases you can use equation (54). Matrix elements for the nucleus may easily be found from the above ratios, selecting only two out of four matrices.

The general system may be recorded as

$$\begin{cases} \mathbf{a}^{(1)} = A_1 \mathbf{a}^{(3)}, \\ \mathbf{b}^{(1)} = A_2 \mathbf{a}^{(3)}, \end{cases} \quad (58)$$

where matrices A_1 and A_2 meet the ratio

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_{21}^{(1)} & A_{22}^{(1)} \\ A_{11}^{(1)} & A_{12}^{(1)} \end{pmatrix} \begin{pmatrix} \tilde{\Delta}_m^{(2)}(s_1, s_2) & 0 \\ 0 & \tilde{\Delta}_m^{(1)}(s_1, s_2) \end{pmatrix} \begin{pmatrix} A_{21}^{(2)} \\ A_{11}^{(2)} \end{pmatrix} \quad (59)$$

and are equal to

$$\begin{aligned} A_1 &= \left(A_{21}^{(1)} \tilde{\Delta}^{(2)} A_{21}^{(2)} + A_{22}^{(1)} \tilde{\Delta}^{(1)} A_{11}^{(2)} \right), \\ A_2 &= \left(A_{11}^{(1)} \tilde{\Delta}^{(2)} A_{21}^{(2)} + A_{12}^{(1)} \tilde{\Delta}^{(1)} A_{11}^{(2)} \right). \end{aligned} \quad (60)$$

Now it is not difficult to find the „scattered“ emission potential expansion coefficients, namely

$$\mathbf{b}^{(1)} = A_2(A_1)^{-1} \mathbf{a}^{(1)} = T \mathbf{a}^{(1)}. \quad (61)$$

Previously a link was found between T -matrix and polarizability of particle [18]. For example, if the external field is oriented perpendicularly and in parallel to the axis of rotation of the particle, we have

$$\begin{aligned} \alpha_x = \alpha_y &= -\frac{4\pi}{3} \frac{b_{11}^{(1)}}{a_{11}^{(1)}} = -\frac{4\pi}{3} T_{11}, \\ \alpha_z &= -\frac{4\pi}{3} \frac{b_{01}^{(1)}}{a_{01}^{(1)}} = -\frac{4\pi}{3} T_{11}. \end{aligned} \quad (62)$$

Therefore, only one element of this matrix is required for the Rayleigh approximation — T_{11} . To completely solve the electrostatic problem, the entire T -matrix is required.

EBCM method: 2nd pattern

Here the infinite linear algebraic equation system (ILAES) for the unknown coefficients is obtained in a similar manner from the system (10)

$$\begin{cases} \varepsilon_2 \mathbf{a}_1^{(2)} = B_{21}^{(1)} \mathbf{a}^{(1)} + B_{22}^{(1)} \mathbf{b}^{(1)}, \\ \varepsilon_2 \mathbf{b}_1^{(2)} = B_{11}^{(1)} \mathbf{a}^{(1)} + B_{12}^{(1)} \mathbf{b}^{(1)}. \end{cases} \quad (63)$$

Note that this system may be obtained for solution (50) with account of determinant (54). Similarly for the particle nucleus the expansions of the potential and the Green's function are used, we get:

$$\begin{cases} \varepsilon_3 \mathbf{a}_1^{(3)} = B_{21}^{(2)} \mathbf{a}_2^{(2)} + B_{22}^{(2)} \mathbf{b}_2^{(2)}, \\ 0 = B_{11}^{(2)} \mathbf{a}_2^{(2)} + B_{12}^{(2)} \mathbf{b}_2^{(2)}, \end{cases} \quad (64)$$

where the first equation of the system (64) is only useful to determine the field coefficients inside the nucleus $\mathbf{a}_1^{(3)}$. Above they introduced matrices

$$\begin{aligned} B_{21}^{(1)} &= \left\{ \delta_l^n - (\varepsilon_2 - 1) K_{nl,m}^{1,21} \right\}_m^\infty = A_{12}^{(1)}, \\ B_{22}^{(1)} &= \left\{ (\varepsilon_2 - 1) K_{nl,m}^{1,22} \right\}_m^\infty = -A_{22}^{(1)}, \\ B_{11}^{(1)} &= \left\{ -(\varepsilon_2 - 1) K_{nl,m}^{1,11} \right\}_m^\infty = -A_{11}^{(1)}, \\ B_{12}^{(1)} &= \left\{ \delta_l^n - (\varepsilon_2 - 1) K_{nl,m}^{1,12} \right\}_m^\infty = A_{21}^{(1)}. \end{aligned} \quad (65)$$

Here a designation is used for the integrals

$$K_{nl,m}^{1,ki} = \int_{S_1} \frac{\partial \Phi_{mn}^{(i)}(\mathbf{r})}{\partial n} \Phi_{ml}^{(k)}(\mathbf{r}) d\mathbf{s}. \quad (66)$$

Note that the elements of K and L matrices are linked to each other as follows: $K_{nl}^{11,m} = L_{ln}^{11,m}$, $K_{nl}^{22,m} = L_{ln}^{22,m}$ and $K_{nl}^{12,m} = L_{ln}^{21,m}$, $K_{nl}^{12,m} = L_{ln}^{21,m}$. For the second system, substituting the corresponding harmonics and integrating on the nucleus surface, we get similar matrices.

Now the general system with account of equations (63)–(65) and (44) may be recorded as

$$\begin{pmatrix} -A_{11}^{(2)} & A_{21}^{(2)} \\ 0 & \nabla^{(2)T} T(d) \end{pmatrix} \begin{pmatrix} \nabla^{(1)T}(d) & 0 \\ 0 & \nabla^{(2)} T(d) \end{pmatrix} \begin{pmatrix} A_{12}^{(1)} & -A_{22}^{(1)} \\ -A_{11}^{(1)} & A_{21}^{(1)} \end{pmatrix} \begin{pmatrix} \mathbf{a}^{(1)} \\ \mathbf{b}^{(1)} \end{pmatrix} = 0. \quad (67)$$

From equation (67) you can find T -matrix, which is received within the second pattern:

$$\mathbf{b}^{(1)} = (B_2)^{-1} B_1 \mathbf{a}^{(1)} = T_1 \mathbf{a}^{(1)}, \quad (68)$$

where

$$\begin{aligned} B_1 &= \left(A_{11}^{(2)} \nabla^{(1)T} A_{12}^{(1)} + A_{21}^{(2)} \nabla^{(2)T} A_{11}^{(1)} \right), \\ B_2 &= \left(A_{11}^{(2)} \nabla^{(1)T} A_{22}^{(1)} + A_{21}^{(2)} \nabla^{(2)T} A_{21}^{(1)} \right). \end{aligned} \quad (69)$$

Comparing ratios (60) and (69), it is easy to notice that $B_1 = A_2^T$ and $B_2 = A_1^T$. Therefore, $T_{11} = T_1^T$, where T_1 corresponds to T -matrix within the first pattern. Since in both cases it is the same T -matrix, it is symmetrical:

$$T = A_2(A_1)^{-1} = (B_2)^{-1} B_1 = (A_2(A_1)^{-1})^T. \quad (70)$$

It is important to note that the property of symmetry of T -matrix may be effectively used to verify the results of numerical calculations. For example, direct calculations show that in the first pattern with the substantial influence of errors, i.e. when the pattern becomes numerically unstable, the errors result in a strong increase of the elements in the first column by absolute value, whereas the behavior of the elements of the first line has a different trend — they differ much less from the correct results. For the second pattern the picture is different — the elements of the first line grow significantly. The behavior of the elements of the T -matrix for the wave problem of scattering is approximately the same, besides, the elements of the first column are directly related to the expansion coefficients of the scattered field.

When the problem is solved for the multi-layer particle of matrix A_{ik} , B_{ik} , and the transition matrices $\tilde{\Delta}_m^{(1)}(s_1, s_2)$, $\tilde{\Delta}_m^{(2)}(s_1, s_2)$ are found for each layer in the same manner as for the external layer of the two-layer particle. After that it is easy to find matrices A_1 , A_2 as the result of the

matrix product similarly to (59), but for all available layers. T -matrix and polarizability of the multi-layer particle are determined using formulae (61) and (62) accordingly. The 2nd pattern of the EBCM method for the multi-layer particle works in the same manner as for the two-layer particle, therefore symmetricity of the T -matrix in this case does not cause doubts.

Let us consider the dependence of the elements of the T -matrix on the particle size. For this purpose, apart from the source particle, we will solve the electrostatic problem for such particle with the similarity coefficient ν , in which the focus distance is $\tilde{d} = \nu d$, the ratio of half axes is $\tilde{a}/\tilde{b} = a/b$ and the value of the radial coordinate is $\tilde{\xi}_0 = \xi_0$. The nucleus of the second particle will have the radius $\tilde{r}_0 = \nu r_0$. As a result the shape of the second particle and the shape of its nucleus will be the same as in the source particle, but their dimensions will change in accordance with the similarity coefficient.

Taking into account the geometry of two particles, the corresponding geometric factors (see (53) and (55)) are linked as follows:

$$\begin{aligned}\tilde{L}_{nn,m}^{1,21} &= L_{nn,m}^{1,21}, & \tilde{L}_{nn,m}^{1,22} &= \nu^{-(2n+1)} L_{nn,m}^{1,22}, \\ \tilde{L}_{nn,m}^{1,11} &= \nu^{(2n+1)} L_{nn,m}^{1,11}, & \tilde{L}_{nn,m}^{1,12} &= L_{nn,m}^{1,12},\end{aligned}\quad (71)$$

where $i = 1, 2$, at the same time outside the integral elements at different indices n, l are equal to zero. For the balls with the spheroidal nucleus the equations (71) are also true, which can be easily seen from ratios (57).

For the diagonal matrices that appear in the systems (50) and (51), with account of ratios (71) we get

$$\begin{aligned}\tilde{A}_{21}^{(i)} &= A_{21}^{(i)}, & (\tilde{A}_{22}^{(i)})_{nn} &= \nu^{-(2n+1)} (A_{22}^{(i)})_{nn}, \\ (\tilde{A}_{11}^{(i)})_{nn} &= \nu^{(2n+1)} (A_{11}^{(i)})_{nn}, & \tilde{A}_{12}^{(i)} &= A_{12}^{(i)},\end{aligned}\quad (72)$$

where $i = 1, 2$ and again these equations are preserved for the balls with the spheroidal ball.

For the elements of the triangular matrices $\nabla^{(1)}$ and $\nabla^{(2)}$, linked to the linkage of expansions inside the shell, we have

$$\begin{aligned}\tilde{\nabla}_{nl}^{(1)} &= \nu^{(n-l)} \nabla_{nl}^{(1)}, \\ \tilde{\nabla}_{nl}^{(2)} &= \nu^{(l-n)} \nabla_{nl}^{(1)}.\end{aligned}\quad (73)$$

This result follows from equations (21), (22). For the element of inverse matrices, the corresponding ratios may easily be found from (27)–(29).

Now let us turn to the system (58) and matrices A_1 and A_2 , which are used to build T -matrix (61):

$$\begin{cases} \tilde{\mathbf{a}}^{(1)} = \tilde{A}_1 \tilde{\mathbf{a}}^{(3)}, \\ \tilde{\mathbf{b}}^{(1)} = \tilde{A}_2 \tilde{\mathbf{a}}^{(3)}, \end{cases}\quad (74)$$

where

$$\begin{aligned}\tilde{A}_1 &= \left(\tilde{A}_{21}^{(1)} \tilde{\nabla}^{(2)} \tilde{A}_{21}^{(2)} + \tilde{A}_{22}^{(1)} \tilde{\nabla}^{(1)} \tilde{A}_{11}^{(2)} \right), \\ \tilde{A}_2 &= \left(\tilde{A}_{11}^{(1)} \tilde{\nabla}^{(2)} \tilde{A}_{21}^{(2)} + \tilde{A}_{12}^{(1)} \tilde{\nabla}^{(1)} \tilde{A}_{11}^{(2)} \right).\end{aligned}\quad (75)$$

Taking into account the ratios (71)–(73), you can find the dependence of $\tilde{A}_1$ and $\tilde{A}_2$ matrix elements on the similarity coefficient:

$$\begin{aligned}(\tilde{A}_1)_{nl} &= \nu^{(l-n)} (A_1)_{nl}, \\ (\tilde{A}_2)_{nl} &= \nu^{(n-l)} (A_2)_{nl}.\end{aligned}\quad (76)$$

The found ratios (76) make it possible to record the system (74) as follows:

$$\begin{cases} \nu^n \tilde{a}_n^{(1)} = (A_1)_{nl} (\nu^l \tilde{a}_l^{(3)}), \\ \nu^{-(n+1)} \tilde{b}_n^{(1)} = (A_2)_{nl} (\nu^l \tilde{a}_l^{(3)}). \end{cases}\quad (77)$$

From these equations it follows that vectors $\nu^n \tilde{a}_n^{(1)}$ and $\nu^{-(n+1)} \tilde{b}_n^{(1)}$ are linked with T -matrix:

$$\nu^{-(n+1)} \tilde{b}_n^{(1)} = (T)_{nl} (\nu^l \tilde{a}_l^{(3)}). \quad (78)$$

On the other hand,

$$\tilde{\mathbf{b}}^{(1)} = \tilde{T} \tilde{\mathbf{a}}^{(1)}. \quad (79)$$

Comparing the obtained ratios (78) and (79), we find the link between the elements of T -matrices for two similar particles:

$$(\tilde{T})_{nl} = \nu^{(n+l+1)} (T)_{nl}. \quad (80)$$

In particular, polarizability depends on the particle volume.

$$\tilde{\alpha} = \nu^3 \alpha. \quad (81)$$

Note that this ratio is true for all components of the tensor, which is polarizability (62). The proportionality coefficient depends on the shape and internal structure of the particle, and on the dielectric permittivities of the nucleus and shells, but does not depend on its size.

T -matrix links the vectors of expansion coefficients of the external and „scattered“ fields in (38). If the basis functions are spheroidal harmonics $\Phi_{ml}^{(i)}(\mathbf{r}) = \Psi_{ml}^{(i)}(d, \xi, \eta, \varphi)$, we deal with the spheroidal T -matrix:

$$\mathbf{b}^{(1)} = T \mathbf{a}^{(1)}. \quad (82)$$

In case of a spherical basis $\Phi_{ml}^{(i)}(\mathbf{r}) = Y_{ml}^{(1)}(r, \theta, \varphi)$ we obtain a spherical T -matrix:

$$\tilde{\mathbf{b}}^{(1)} = \tilde{T} \tilde{\mathbf{a}}^{(1)}. \quad (83)$$

The spheroidal and spherical bases are linked to each other by a transition matrix:

$$\tilde{\mathbf{a}}^{(1)} = \nabla^{(1)T}(d) \mathbf{a}^{(1)} \quad (84)$$

and

$$\tilde{\mathbf{b}}^{(1)} = \nabla^{(2)T}(d) \mathbf{b}^{(1)}. \quad (85)$$

Now you can record the link between T -matrices:

$$\tilde{T} = \nabla^{(2)T}(d) T \nabla^{(2)}(d) \quad (86)$$

and

$$T = \nabla^{(1)}(d)\hat{T}\nabla^{(1)T}(d). \quad (87)$$

Here it should be noted that the symmetricity of matrix data may serve as a good criterion of correct numerical calculations. Besides, let us specify another property $\tilde{T}_{11} = T_{11}$, since these elements with a precision of up to one and the same constant are equal to the particle polarizability.

Conclusions

1. The solution to the electrostatic problem is built for the multi-layer particle with spheroidal and spherical boundaries of layers, besides, they may alternate randomly. The used method of expanded boundary conditions will make it possible in an explicit form to solve the corresponding infinite algebraic systems for the unknown expansion coefficients of field potentials for the most suitable bases accordingly for the spherical or spheroidal surfaces. Then for each layer with different types of boundaries the corresponding field expansions are linked. The main member T_{11} of the built T -matrix with a precision of up to constant $4\pi/3$ coincides with the particle polarizability, which enables to estimate its optical properties.

2. Bonds were found between spheroidal and spherical T -matrices, which are produced when spheroidal or spherical bases are used accordingly. The symmetricity of T -matrix is proven, and dependence is also found between its elements and the size of the multi-layer particle. In particular, the polarizability of the particle linearly depends on its volume.

3. Previous numerical calculations showed high efficiency of the proposed solution in the private cases of the spheroid with a spherical or spheroidal nucleus. In particular, it was shown that for a spheroidal particle with the ratio of half axes $a/b \geq \sqrt{2} + 1$ and any nucleus it was feasible to use this approach, since the direct use of the widely known spherical basis yields wrong results.

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