03,10,11

Non-Hermitian trace in solving the problem of carrier motion in graphene in crossed fields

© E.L. Rumyantsev, A.V. Germanenko

Ural Federal University named after the first President of Russia B.N. Yeltsin
Yekaterinburg, Russia

E-mail: alexander.germanenko@urfu.ru

Received July 21, 2025
Revised November 19, 2025
Accepted January 21, 2026

The energy spectrum of carriers in graphene in crossed homogeneous electric and magnetic fields is obtained within the framework of an algebraic approach. This consideration is based only on the gradient-invariant commutative properties of kinetic momentum, which eliminates the need to fix a specific functional dependence of the vector-potential. Special attention is paid to the analysis of the non-Hermitian component in an effective Schrodinger-like Hamiltonian, which is used to describe the system. The effects associated with this unusual property are discussed, as well as a technique that allows us to return to the Hermitian description corresponding to the original formulation of the problem.

Keywords: graphene, energy spectrum, crossed electric and magnetic fields.

DOI: 10.61011/PSS.2026.01.63232.207-25

1. Introduction

The spectrum of carriers in graphene in the presence of crossed constant and homogeneous magnetic and electric fields is usually considered within the framework of the Fock-Feynman-Gel-Mann approach [1,2], which is due to the formal similarity of the relativistic Dirac Hamiltonian and the graphene Hamiltonian. The solution of Dirac-like equations in graphene, as a rule, is reduced to solving second-order differential equations similar to Schrodinger's using the so-called „squaring" operation [3,4]. However, such an approach requires fixing a specific calibration of the potential vector, which contradicts the principle of gradient invariance. In this regard, the so-called algebraic approach based on the generally accepted gradient-invariant commutation properties of kinetic impulses is of particular importance [5]

$$[\pi_x, \pi_y] = i/l_B^2 \tag{1}$$

depending only on the intensity of the magnetic field B. In this expression, $\widehat{\mathbf{n}} = \widehat{\mathbf{p}} - e\mathbf{A}$, $l_B = \sqrt{1/|eB|}$ is the magnetic length. In the following, we use the unit system $\hbar = c = 1$. The spectrum of the problem and the wave functions in the framework of this consideration can be obtained without fixing a specific functional type of vector potential. As an example of the application of this approach, let us recall how it works when solving the well-known problem of electron motion in a 2D graphene film in the presence of a static homogeneous transverse magnetic field. In this regard, using commutation relations for kinetic impulses, it is convenient to switch to lowering (annihilating) and rising (generating) Bose operators $[a, a^+] = 1$

$$a^+ = \frac{l_B}{\sqrt{2}}(\widehat{\pi}_x - i\widehat{\pi}_y), \quad a = \frac{l_B}{\sqrt{2}}(\widehat{\pi}_x + i\widehat{\pi}_y), \tag{2}$$

kp is the graphene Hamiltonian in the vicinity of the point K of the Brillouin zone in these operators has the form

$$\widehat{H} = v_F\sigma_x\widehat{\pi}_x + v_F\sigma_y\widehat{\pi}_y = \begin{pmatrix} 0 & \frac{v_F\sqrt{2}}{l_B}a^+ \\ \frac{v_F\sqrt{2}}{l_B}a & 0 \end{pmatrix}. \tag{3}$$

The eigenvalues ε and the eigenvectors $|\varphi_{\pm,n}\rangle$ of the problem are easily calculated

$$\varepsilon = \pm\frac{v_F\sqrt{2n}}{l_B}\,|\varphi_{\pm,n}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} |n\rangle \\ \pm|n-1\rangle \end{pmatrix}, \tag{4}$$

where $a|n\rangle = \sqrt{n}|n-1\rangle$. As can be seen, at no stage of the calculations in the Dirac ket-bra representation did we need to use a specific type of vector potential, which is necessary for an analytical solution in the Schrodinger coordinate representation. As will be shown later, the problem of motion in crossed constant and homogeneous magnetic and electric fields can also be solved in a similar way without choosing a specific gauge of vector potential. A variant of such a consideration, proposed in Ref. [6], cannot be considered as an example of the claimed algebraic approach, since it is based from the very beginning on the choice of a fixed linear Landau gauge $\mathbf{A} = (-yB, 0)$. Such a consideration contradicts the very idea of an algebraic approach based only on commutation relations depending on the intensity of fields. It should be noted that the consideration proposed by the authors seems to us unreasonably complicated. In this article, we will show that there is an easy-to-solve and interpretable algebraic approach for solving this problem.

2. Non-Hermite trace in the problem of crossed fields

The Hamiltonian describing the motion of carriers in graphene in the presence of crossed homogeneous, constant electric ($E_x \neq 0$) and magnetic (perpendicular to the graphene plane $H_z \neq 0$) fields in the vicinity of the point K has the form

$$\widehat{H} = \begin{pmatrix} -eEx & v_F\widehat{\pi}^+ \\ v_F\widehat{\pi}^- & -eEx \end{pmatrix}, \tag{5}$$

where $\widehat{\pi}^\pm = \widehat{\pi}_x \mp i\widehat{\pi}_y$. The solution of equation $\widehat{H}|\Psi\rangle - \varepsilon|\Psi\rangle = 0$ will be obtained using the approach described in the monograph [7], since the problem under consideration is formally mathematically similar to the relativistic Dirac problem. For this purpose, we define the following operators

$$\widehat{H}_+ = \begin{pmatrix} -(eEx+\varepsilon) & v_F\widehat{\pi}^+ \\ v_F\widehat{\pi}^- & -(eEx+\varepsilon) \end{pmatrix},$$

$$\widehat{H}_- = \begin{pmatrix} eEx+\varepsilon & v_F\widehat{\pi}^+ \\ v_F\widehat{\pi}^- & eEx+\varepsilon \end{pmatrix}. \tag{6}$$

The eigenvalues ε and the eigenvectors $|\Psi\rangle$ are the solutions of the equation

$$\widehat{H}_+|\Psi\rangle = 0. \tag{7}$$

We will search for the wave function in the form (the multiplier l_B/v_F was added for dimensional reasons)

$$|\Psi\rangle = \frac{l_B}{v_F}\widehat{H}_-|\Psi\rangle. \tag{8}$$

$|\Psi\rangle$ and ε are the solutions of an equation similar to Schrodinger's equation

$$\widehat{H}_+\widehat{H}_-|\Psi\rangle = 0. \tag{9}$$

An important property of the $\widehat{H}_+\widehat{H}_-$ operator is that it acts separately in the coordinate space and in the space of pseudo-spin graphene variables

$$\widehat{H}_+\widehat{H}_- = [v_F^2(\widehat{\pi}_x^2 + \widehat{\pi}_y^2) - (eEx+\varepsilon)^2]I + \widehat{H}_S, \tag{10}$$

$$\widehat{H}_S = -\frac{v_F^2}{l_B^2}[\sigma_z + i\delta\sigma_x],$$

where $\delta = v_d/v_F$, $v_d = E/B$ is an analog of the drift velocity in the classical problem. A similar technique of „squaring" of the initial system of equations is used mainly to solve the relativistic Dirac problem of electron motion in an electromagnetic field [1–4]. Our solution to the equation (9) starts by diagonalizing the operator $\widehat{H}_S$ by rotating in the pseudo-spin space. It should be noted that in the case of the Dirac problem, a similar operator operates in real spin space. The real eigenvalues $\widehat{H}_S$ have the form

$$\lambda_\pm = \pm\frac{v_F^2}{l_B^2}\mu, \tag{11}$$

where $\mu = \sqrt{1-\delta^2}$. The eigenstates of $\widehat{H}_S$ have the form

$$\widehat{H}_S|r_\pm\rangle = \lambda_\pm|r_\pm\rangle,\ |r_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \dfrac{-i\delta}{\sqrt{1+\mu}} \\ \sqrt{1+\mu} \end{pmatrix},$$

$$|r_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1+\mu} \\ \dfrac{i\delta}{\sqrt{1+\mu}} \end{pmatrix}. \tag{12}$$

The remaining Schrodinger part of the pseudo-Hamiltontoniaon $\widehat{H}_+\widehat{H}_-$ is easily diagonalized because, as will be shown below, this problem is mathematically equivalent to the problem of a quantum oscillator in the presence of a constant driving force. At this stage, it is necessary to note an unusual property of the operator $\widehat{H}_+\widehat{H}_-$ used, which for some reason was not noticed in any of the articles known to us using the considered „squaring" technique. We emphasize that this problem also exists for the relativistic Dirac equation. The fact is that the operator $\widehat{H}_+\widehat{H}_-$ is essentially **non-Hermite**, due to the presence of the pseudo-spin Hamiltonian $\widehat{H}_S \neq \widehat{H}_S^\dagger$. A Hamiltonian of this type is unitarily equivalent to the well-known non-Hermitian Hamiltonian, which is used to demonstrate unusual consequences of non-Hermiticity [8,9]. The authors of Ref. [6], who proposed their own version of „squaring", did not avoid the appearance of this feature. We emphasize that $\widehat{H}_S$ becomes Hermitian in the absence of an external electric field.

Despite these properties related to the non-Hermiticity of the pseudo-Hamiltonian, we will continue our consideration based on the fact that the eigenvalues of this pseudo-spin problem are nevertheless real. Accordingly, we will look for a solution to the equation (9) in the form $|\Phi_\pm\rangle = |r_\pm\rangle|\varphi_\pm\rangle$, where $|\varphi_\pm\rangle$ satisfies the equation

$$\left[v_F^2(\widehat{\pi}_x^2 + \widehat{\pi}_y^2) - (eEx+\varepsilon)^2 + \lambda_\pm\right]|\varphi_\pm\rangle = 0. \tag{13}$$

3. Pseudo-moment operators

At this stage of consideration, it is convenient to use the following representation of the coordinate operator $\widehat{\mathbf{r}}$ [10]

$$\widehat{\mathbf{r}} = \widehat{\boldsymbol{\rho}} + \widehat{\mathbf{R}}, \tag{14}$$

where $\widehat{\boldsymbol{\rho}}$ describes „fast" cyclotron rotation, and the vector $\widehat{\mathbf{R}}$, when viewed classically, defines the center of rotation (guiding center vector). A similar separation is used to describe the behavior of plasma in electromagnetic fields that vary slightly both in space and time. The division of the coordinate operator into two components having a well-defined physical meaning has also been repeatedly used and is used in the quantum description of the behavior of an electron in a magnetic field. It first appeared in Ref. [11]. This article presents the corresponding commutation relations for the $\widehat{\mathbf{R}}$ operators defining the location of the cyclotron orbit (guiding center vector), which ensure commutation of coordinate operators. Since cyclotron motion is convenient to represent in terms of kinetic moment operators

$$\widehat{\boldsymbol{\rho}} = l_B^2(-\widehat{\pi}_y, \widehat{\pi}_x). \tag{15}$$

The vector operators of the center of rotation $\widehat{\mathbf{R}}$ can be defined by analogy with (15) through the introduction of so-called pseudo-moments [12]

$$\widehat{\mathbf{k}} = \widehat{\boldsymbol{\pi}} - e[\mathbf{r} \times \mathbf{B}]. \tag{16}$$

Because at the same time

$$\widehat{\mathbf{R}} = l_B^2[\widehat{\mathbf{k}} \times \mathbf{B}]/|\mathbf{B}|, \tag{17}$$

It can be shown using the results from Ref. [11] that the pseudo-moment operators defined in this way obey the following commutation relations

$$[\widehat{k}_x, \widehat{k}_y] = -i/l_B^2, \quad [\widehat{k}_i, \widehat{\pi}_j] = 0. \tag{18}$$

An analog of this partition, for example, is used in Ref. [13] in the description of relativistic Landau levels, where the corresponding operators are called cyclotron and magnetron ones. It should be emphasized that, as follows from their definition (16), the pseudo-moment operators $\widehat{\mathbf{k}}$ are strictly gradient-invariant. Indeed, kinetic moments are gradient invariant by construction, and coordinate operators are insensitive to gradient transformations by definition. In these operators, equation (13) takes the form

$$\left[v_F^2\widehat{\pi}_x^2 + v_F^2\mu^2\widehat{\pi}_y^2 + 2\widehat{\varepsilon}v_d\widehat{\pi}_y - \widehat{\varepsilon}^2 + \lambda_\pm\right]|\varepsilon_\pm\rangle = 0, \tag{19}$$

where $\widehat{\varepsilon} = \varepsilon + v_d\widehat{k}_y$. Since $\widehat{k}_y$ commutes with $\widehat{H}_+\widehat{H}_-$, the solution of equation(19) can be found in the form

$$|\varphi_\pm\rangle = |\psi_\pm\rangle_a|k_y\rangle_b. \tag{20}$$

In this expression, $|k_y\rangle_b$ is a proper function of the operator $\widehat{k}_y$. Using the given commutation relations (18) for pseudo-pulses, we define the corresponding Bose operators b and b^+ ($[b, b^+] = 1$)

$$b = \frac{l_B}{\sqrt{2}}(\widehat{k}_y + i\widehat{k}_x), \;\; b^+ = \frac{l_B}{\sqrt{2}}(\widehat{k}_y - i\widehat{k}_x), \;\; b|0\rangle_b = 0. \tag{21}$$

Following Refs. [14–16], let us consider the state

$$|k_y\rangle_b = \frac{e^{l_B^2k_y^2/2}}{(2\pi)^{1/4}}\sqrt{2}l_B\int_{-\infty}^{\infty} dk_x|\sqrt{2}kl_B\rangle_b$$
$$= \frac{e^{l_B^2k_y^2/2}\pi^{1/4}}{2^{3/4}}\,e^{-(\sqrt{2}k_yl_B-\widehat{b}^+)^2/2}|0\rangle_b, \tag{22}$$

where $k = k_y + ik_x$, $|\sqrt{2}kl_B\rangle$ is a coherent state $b|\sqrt{2}kl_B\rangle = \sqrt{2}kl_B|\sqrt{2}kl_B\rangle$. It is easy to establish the validity of the following commutation relation

$$be^{-(\sqrt{2}k_yl_B-\widehat{b}^+)^2/2} = e^{-(\sqrt{2}k_yl_B-\widehat{b}^+)^2/2}\left[\widehat{b} + \sqrt{2}k_yl_B - \widehat{b}^+\right]. \tag{23}$$

Using (23), it is easy to show that (22) is the desired eigenvector of the operator $\widehat{k}_y$

$$\widehat{k}_y|k_y\rangle_b = k_y|k_y\rangle_b. \tag{24}$$

The states $|k_y\rangle_b$ are normalized to the Dirac delta function

$$\langle k_y'|k_y\rangle_b = \delta(k_y' - k_y). \tag{25}$$

This condition is a consequence of the following integral equality [16]

$$\iint_{-\infty}^{\infty}\frac{dydy'}{\pi}\langle z'/z\rangle = \iint_{-\infty}^{\infty}dydy'\exp -\frac{1}{2}\left(|z|^2+|z'|^2+2z'^*z\right)$$
$$= (2)^{-1/2}\pi^{3/2}e^{-x^2/2}\delta(x - x'), \tag{26}$$

where $z = x + iy$. Using the obtained eigenvectors $|k_y\rangle_b$, the operator $\widehat{\varepsilon}$ in equation (19) can be considered as c-number $\widehat{\varepsilon} \to \varepsilon = \varepsilon + v_dk_y$. In this case, the equation defining the wave function $|\psi_\pm\rangle_a$ (19) takes the form

$$\left[v_F^2\widehat{\pi}_x^2 + v_F^2\mu^2\widehat{\pi}_y^2 + 2\varepsilon v_d\widehat{\pi}_y - \varepsilon^2 + \lambda_\pm\right]|\psi_\pm\rangle_a = 0. \tag{27}$$

4. 1D mapping

At this stage of the consideration, we will make use of the effective 1D property of equation (27). The existence of an exact correspondence of the description of the 2D quantum Hall system to the effective 1D system is discussed, for example, in the work [17]. A similar „designing“ is carried out by defining the quasi-coordinate operators $\widehat{Q}$ and the quasi-moment $\widehat{P}$

$$\widehat{Q} = -l_B^2\widehat{\pi}_y, \quad \widehat{P} = \widehat{\pi}_x. \tag{28}$$

These operators obey the standard coordinate-momentum cjmmutation relation $[\widehat{Q}, \widehat{P}] = i$. Using the introduced operators (28), we define the corresponding Bose creation/annihilation operators $\widehat{c}, \widehat{c}^+$

$$\widehat{c} = \frac{l_B}{\sqrt{2\mu}}\left[\widehat{P} - i\,\frac{\mu\widehat{Q}}{l_B^2}\right], \widehat{c}^+ = \frac{l_B}{\sqrt{2\mu}}\left[\widehat{P} + i\,\frac{\mu\widehat{Q}}{l_B^2}\right], [\widehat{c}, \widehat{c}^+] = 1. \tag{29}$$

In the following, we will need to relate the operators $\widehat{c}$ diagonalizing non-Hermite Hamiltonians to the operators $\widehat{a}$ defined in (2)

$$\widehat{c} = \frac{l_B}{\sqrt{2\mu}}[\widehat{\pi}_x + i\mu\widehat{\pi}_y]$$
$$= \frac{1}{2\sqrt{\mu}}[(1 + \mu)\widehat{a} + (1 - \mu)\widehat{a}^+] = u\widehat{a} + v\widehat{a}^+, \tag{30}$$

where

$$u = \cosh\varphi = (1 + \mu)/2\sqrt{\mu}, \; v = \sinh\varphi = (1 - \mu)/2\sqrt{\mu}.$$

With this redefinition, the desired solution of equation (27) becomes equivalent to the solution of the 1D Schrodinger problem of the behavior of a quantum oscillator when an external constant force is applied.

$$\left[\frac{v_F^2\mu}{l_B^2}\,(2\hat{c}^+\hat{c}+1)+i2\varepsilon\,\frac{v_d}{l_B\sqrt{2\mu}}\,(\hat{c}^+-\hat{c})-\varepsilon^2+\lambda_\pm\right]\times|\Psi_\pm\rangle_c=0. \quad (31)$$

The eigenvectors $|\Psi_\pm\rangle_c$, which are the solution of (27), first appeared in Ref. [19] under the name „semi-coherent states", or as they were named later, „generalized coherent states" [20]. To define them, we define the shift operator

$$D(\alpha_\varepsilon)=\exp[\alpha_\varepsilon\hat{c}^+-\alpha_\varepsilon^*\hat{c}],\ \ \alpha_\varepsilon=-i2\,\frac{\delta l_B\varepsilon}{(2\mu)^{3/2}v_F}. \quad (32)$$

Using (32), the generalized coherent state, which is the solution of equation (31), can be represented as [20]

$$|\Psi_\pm\rangle=|n,\alpha_\varepsilon\rangle_c=D(\alpha_\varepsilon)|n\rangle_c. \quad (33)$$

Using

$$|\Phi\rangle_{\pm,n,k_y}=|r_\pm\rangle|n,\alpha_\varepsilon\rangle_c|k_y\rangle_b, \quad (34)$$

the desired spectrum of the problem is determined from the equation

$$\mu^3\,\frac{v_F^2}{l_B^2}\,(2n+1\pm1)-\varepsilon_n^2=0. \quad (35)$$

Here $\varepsilon_n=\varepsilon^{e,h}_{\pm,n}+v_dk_y$. The final answer

$$\varepsilon^{e,h}_{\pm,n}=-v_dk_y+s\mu^{3/2}\frac{v_F}{l_B}\sqrt{2n+1\pm1}. \quad (36)$$

In this expression, $s=+1(c)$ refers to the conduction band, and $s=-1(h)$ refers to the valence band. The obtained results coincides with the answers obtained in completely different approaches, including the quasi-classical consideration [21], the method of pseudo-Lorentz transformations [22–24], as well as an algebraic solution [6]. The algebraic consideration we have proposed seems simpler in comparison with Ref. [6]. It should be noted that the authors in this paper use a fixed functional dependence of the vector potential (Landau calibration), which contradicts the very idea of an algebraic approach to solving this problem. Let's check whether the received answer matches the known answer obtained in the extreme case of the absence of an electric field ($E=v_d=\delta=0$, $\mu=1$, $\hat{c}=\hat{a}$). In this case, the non-Hermitian pseudo-spin part of the Hamiltonian becomes Hermitian and the expression for the eigenvectors $|r_\pm\rangle$ is simplified.

$$|r_+\rangle=\begin{pmatrix}0\\1\end{pmatrix},\ \ |r_-\rangle=\begin{pmatrix}1\\0\end{pmatrix}. \quad (37)$$

It is surprising that in our consideration we get a twofold degeneracy for a given positive (negative) value of its own energy.

$$\varepsilon^e_{+,n-1}=\frac{v_F}{l_B}\sqrt{2(n-1)+2}=\varepsilon^e_{-,n}=\frac{v_F}{l_B}\sqrt{2n}. \quad (38)$$

Since in this case the Hamiltonian does not depend on pseudo-momenta, which now serve only to lable a degenerate system of eigenvectors belonging to a given Landau level, we can omit the dependence on k_i in expressions for two vectors. Because in this case $\delta=0$ $(\mu=1)$,

$$|\Phi\rangle_{e,+,n-1,k_y}=\begin{pmatrix}0\\1\end{pmatrix}|n-1\rangle_a,\ \ |\Phi\rangle_{e,-,n,k_y}=\begin{pmatrix}1\\0\end{pmatrix}|n\rangle_a. \quad (39)$$

At the same time, the fact that one and only one eigenvector belongs to a given positive (negative) value of its own energy can be easily verified by directly solving the equation $\hat{H}_+|\Psi\rangle=0$ $(E=0)$. We show that in the definition of „proper" eigenvectors $|\Psi\rangle$ in the proposed approach, the operator $\hat{H}_-$, acting according to the definition, becomes crucial (8) $|\Psi\rangle=\hat{H}_-|\Phi\rangle$. Recall that a similar procedure for returning to the Hermitian description is described in the monograph [7] when solving the relativistic Dirac equation. In the presence of only a magnetic field

$$\hat{H}_-=\begin{pmatrix}\varepsilon & v_F\pi^+\\ v_F\pi^- & \varepsilon\end{pmatrix}. \quad (40)$$

Applying the operator $\hat{H}_-$ leads to the result for both vectors (39)

$$|\Psi\rangle_{e,+,n-1,k_y}=\frac{l_B}{v_F}\,\hat{H}_-|r_+\rangle|n-1\rangle_a$$

$$=\sqrt{2}\begin{pmatrix}\sqrt{n} & a^+\\ a & \sqrt{n}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}|n-1\rangle_a=\sqrt{2n}\begin{pmatrix}|n\rangle_a\\ |n-1\rangle_a\end{pmatrix},$$

$$|\Psi\rangle_{e,-,n,k_y}=\frac{l_B}{v_F}\,\hat{H}_-|r_-\rangle|n\rangle_a$$

$$=\sqrt{2}\begin{pmatrix}\sqrt{n} & a^+\\ a & \sqrt{n}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}|n\rangle_a=\sqrt{2n}\begin{pmatrix}|n\rangle_a\\ |n-1\rangle_a\end{pmatrix}. \quad (41)$$

Using $\hat{H}_-$ preserves *status quo*, since in both cases we get one and only one eigenvector belonging to a given (non-degenerate) energy value, as it should be. It is easy to show that a similar „collapse of" eigenvectors is also valid for negative energy values. Let us prove that such fictitious degeneracy is absent in the general case in the presence of an electric field. Let us consider the unnormalized eigenvectors $|\Psi\rangle_{+,n-1,k_y}$ and $|\Psi\rangle_{-,n,k_y}$, which belong to the same electronic energy level $\varepsilon_n=v_F\mu^{3/2}\sqrt{2n}/l_B-v_dk_y$

$$|\Phi\rangle_{+,n-1,k_y}=D(\alpha_{\varepsilon_n})|r_+\rangle|n-1\rangle_c|k_y\rangle_b,$$

$$|\Phi\rangle_{-,n,k_y}=D(\alpha_{\varepsilon_n})|r_-\rangle|n\rangle_c|k_y\rangle_b. \quad (42)$$

To prove that there is no real degeneracy in the problem under consideration, after applying an additional operation, we must obtain

$$|\Psi\rangle_{n,k_y}=\frac{l_B}{v_F}\,\hat{H}_-|\Phi\rangle_{+,n-1,k_y}=\frac{l_B}{v_F}\,\hat{H}_-|\Phi\rangle_{-,n,k_y}. \quad (43)$$

where $\widehat{H}_-$ (6) has the form

$$\widehat{H}_-(\varepsilon_n) = \frac{v_F\sqrt{2}}{l_B} \times \begin{pmatrix} -i\,\frac{\delta}{2}\,(\widehat{a}^+ - \widehat{a}) + \frac{l_B}{v_F\sqrt{2}}\,\varepsilon_n & \widehat{a}^+ \\ \widehat{a} & -i\,\frac{\delta}{2}\,(\widehat{a}^+ - \widehat{a}) + \frac{l_B}{v_F\sqrt{2}}\,\varepsilon_n \end{pmatrix}$$

$$= \frac{v_F\sqrt{2}}{l_B} \times \begin{pmatrix} -i\,\frac{\delta}{2}\,(u+v)(\widehat{c}^+ - \widehat{c}) + \frac{l_B}{v_F\sqrt{2}}\,\varepsilon_n & (u\widehat{c}^+ - v\widehat{c}) \\ (u\widehat{c} - v\widehat{c}^+) & -i\,\frac{\delta}{2}\,(u+v)(\widehat{c}^+ - \widehat{c}) + \frac{l_B}{v_F\sqrt{2}}\,\varepsilon_n \end{pmatrix}. \quad (44)$$

Using the expressions obtained for the states $|\Phi\rangle_{\pm,n,k_y}$ (42), it suffices to show for the proof of (43) that

$$D^\dagger(\alpha_{\varepsilon_n})\widehat{H}_- D(\alpha_{\varepsilon_n})|r_+\rangle|n-1\rangle_c = D^\dagger(\alpha_{\varepsilon_n})\widehat{H}_- D(\alpha_{\varepsilon_n})|r_-\rangle|n\rangle_c. \quad (45)$$

Using the definition of the operator $D(\alpha_{\varepsilon_n})$ (32) and the resulting expression (44) for $\widehat{H}_-(\varepsilon_n)$, the operator $D^\dagger(\alpha_{\varepsilon_n})\widehat{H}_- D(\alpha_{\varepsilon_n})$ it can be represented as

$$D^\dagger(\alpha_{\varepsilon_n})\widehat{H}_- D(\alpha_{\varepsilon_n}) = \frac{v_F\sqrt{2n}}{l_B\sqrt{mu}}\,\widehat{\theta} + \frac{v_F}{l_B\sqrt{2\mu}}\,[\widehat{\vartheta}\widehat{c}^+ + \vartheta^\dagger\widehat{c}], \quad (46)$$

$$\widehat{\theta} = \begin{pmatrix} 1 & i\delta \\ -i\delta & 1 \end{pmatrix}, \ \widehat{\vartheta} = \begin{pmatrix} -i\delta & 1+\mu \\ \mu - 1 & -i\delta \end{pmatrix}.$$

The action of the operators $\widehat{\theta}$ and $\widehat{\vartheta}$ on the pseudo-spin vectors $|r_+\rangle$ and $|r_-\rangle$ is given by the equalities

$$\widehat{\theta}|r_+\rangle = \mu|r_+\rangle^* \ \widehat{\theta}|r_-\rangle = \mu|r_-\rangle^* \ \widehat{\vartheta}|r_+\rangle = 2\mu|r_-\rangle^*$$
$$\widehat{\vartheta}|r_-\rangle = 0 \ \widehat{\vartheta}^\dagger|r_-\rangle = 2\mu|r_+\rangle^* \ \widehat{\vartheta}^\dagger|r_+\rangle = 0. \quad (47)$$

Using these equalities, we obtain the desired expression for a non-degenerate wave function

$$|\Psi\rangle_n = \widehat{H}_-|\Phi\rangle_{+,n-1,k_y} = \widehat{H}_-|\Phi\rangle_{-,n,k_y} = \mu\,\frac{v_F\sqrt{2n}}{l_B}\,D(\alpha_{\varepsilon_n})\big[|r_+\rangle^*|n-1\rangle_c + |r_-\rangle^*|n\rangle_c\big]. \quad (48)$$

This is an important, though expected result. The imaginary degeneracy generated by „squaring“ and caused by the presence of a non-Hermitian component in the effective Hamiltonian, which manifests itself in the non-orthogonality of its eigen pseudo-spin vectors, disappears when the operation $\widehat{H}_-$ is taken into account. The need for an additional operation, similar to the one discussed in our work, to obtain „correct“ wave functions is indicated in the monograph [7]. However, since we are talking about solving the corresponding differential equations, this is explained in Ref. [7] by the need to get rid of „extra“ solutions that arise when considering second-order quadratic equations, whereas initially we start by from the first-order differential equations. As follows from the approach proposed in the paper, this is not an entirely accurate statement. The proposed additional operator action is designed to lead to the disappearance of the imaginary twofold degeneracy by ensuring the collapse of the corresponding wave vectors to a single expression and the orthogonality of the resulting eigen-functions.

5. Conclusion

In this paper, we proposed a consistent algebraic approach, independent of the need to choose a specific vector potential gauge for the problem of finding the spectrum of carriers in graphene in the presence of crossed comstant and homogeneous magnetic and electric fields. The solution was obtained within the framework of the Fock-Feynman-Gel-Mann approach, which is due to the formal similarity of the relativistic Dirac Hamiltonian and the graphene **kp** Hamiltonian and is based on the gradient invariance of the commutation relations of operators and pseudo-momenta. Solving a system of Dirac-like equations in graphene usually reduces to solving second-order differential equations similar to Schrodinger's using the „squaring“ operation. However, such a consideration requires fixing a specific functional form of vector potential. The algebraic approach we have considered using pseudo-momentum operators allows us to circumvent the problem of the specific choice of the vector potential gauge, which distinguishes it from the traditional approach to this problem. It is shown that a specific property of the algebraic solution in crossed fields, which passes through the „squaring“ stage, is the appearance of a non-Hermitian component in the pseudo-Schrodinger Hamiltonian acting in the space of pseudo-spin graphene variables. It is noted that a similar problem in this approach arises when solving the relativistic Dirac equation in crossed fields [7]. Since this non-Hermitian Hamiltonian is invariant with respect to the $\mathcal{PT}$ transformation, its eigenvalues remain real, which makes it possible to obtain a real spectrum of the problem, ignoring the absence of Hermiticity. The only legacy of the appearance of non-Hermiticity in the solution is the collapse of the Landau levels as the drift velocity tends to the Fermi velocity of graphene.

Funding

The study was conducted within the framework of the state assignment (project FEUZ-2023-0017).

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] V. Fock. Phys. Zeit. Sowjetunion **12**, 404 (1937).
[2] R.P. Feynman, M. Gell-Mann. Phys. Rev. **109**, 193 (1958).
[3] A.H. MacDonald. Phys. Rev. B **28**, 2235 (1983).
[4] M.M. Nieto, P.L. Taylor. Am. J. Phys. **53**, 234 (1985).
[5] L.D. Landau, E.M. Lifshitz. Quantum Mechanics, Pergamon Press, Oxford (1977) .
[6] N.M.R. Peres, E.V. Castro. J. Phys.: Condens. Matter **19**, 406231 (2007).
[7] V.B. Berestetskii, E.M. Lifshitz, L.P. Pitaevskii. Relativistic Quantum Theory, Pergamon Press (1971).
[8] A. Mostafazadeh. J. Math. Phys. **43**, 3944 (2002) [and citations within].
[9] C.M. Bender. Rep. Prog. Phys. **70**, 947 (2007).
[10] J.R. Cary, A.J. Brizard. Rev. Mod. Phys. **81**, 693 (2009).
[11] M.H. Johnson, B.A. Lippmann. Phys. Rev. **76**, 828 (1949).
[12] G. Konstantinou, K. Moulopoulos. Eur. J. Phys. **37**, 065401 (2016).
[13] D. Jentschura. Phys. Rev. D **108**, 016016 (2023).
[14] H.Y. Fan. Phys. Lett. A **126**, 145 (1987).
[15] H.Y. Fan, L. Fu. J. Phys. A: Math. Gen. **36**, 4987 (2003).
[16] H. Fan, T. Ruan. Comm. Theor. Phys. **3**, 443 (1984).
[17] M. Horsdal, M. Leinaas. Phys. Rev. B **76**, 195321 (2007).
[18] P. Carruthers, M.M. Nieto. Am. J. Phys. **33**, 537 (1965).
[19] M. Boiteux, A. Levelut. J. Phys. A: Math. Nucl. Gen. **6**, 589 (1973).
[20] M.V. Satyanarayana. Phys. Rev. D **32**, 400 (1985).
[21] Z.Z. Alisultanov. Physica B: Cond. Matt. **438**, 41 (2014).
[22] V. Lukose,R. Shankar, G. Baskaran. Phys. Rev. Lett. **98**, 116802 (2007).
[23] A. Shitov, M. Rudner, N. Gu, M. Katsnelson, L. Levitov. Solid State Comm. **149**, 1087 (2009).
[24] Z.Z. Alisultanov. ZhETF **152**, 986 (2017) (in Russian).

Translated by A.Akhtyamov