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Solving of the Hufford integral equation for short distances in the radio wave propagation problem

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The article examines the Hufford integral equation and two simplified versions of it. It is shown that the second version of the simplified equation can be used for distances up to approximately 100 km. A method for the analytical solution of the Hufford integral equation for a homogeneous spherical earth is presented. A refined formula for short distances to calculate the attenuation factor is derived.

Keywords: diffraction, attenuation function, attenuation factor, Hufford integral equation, Sprut-N1.

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The problem of quantifying the influence of the Earth's surface on radio waves propagating along the interface between two media remains relevant in the context of development of ground radio navigation systems (RNSs) [1]. The requirements imposed on RNS parameters are constantly becoming more and more stringent at present. For example, the influence of the Earth's surface needs to be taken into account if one is to achieve a 5–10 m positional accuracy for ground RNS Sprut-N1 [1,2].

The introduction of an attenuation factor (AF) is the traditional way to characterize the influence of the underlying surface in radio wave propagation theory [3,4]. The AF for an inhomogeneous spherical path may be calculated by solving the Feinberg integral equation [3,5] numerically, since it offers fine numerical stability. However, this requires calculating the AF over a homogeneous spherical surface (with its parameters chosen arbitrarily) first. Here and elsewhere, a homogeneous surface is understood as a uniformly layered one, since any uniformly layered medium may be substituted with a homogeneous medium with equivalent parameters (see [6]). A normal wave series (Fock's formula) is often used to calculate the AF over homogeneous paths [4], but it converges poorly at short distances. When large distances (approximately 100 km) are considered, it is sufficient to sum only the first few terms. Different ways to calculate the AF at short distances, which have their own advantages and disadvantages, are known; e.g., the formula for the flat Earth model may be used in this case [3]. In the present study, the formula for short distances obtained earlier in [7] is refined with the aim of extending it to larger distances.

Let us consider the Hufford integral equation [8] for the model of a homogeneous spherical Earth

$$V(x) = 1.0 + \alpha\sqrt{x}\int_0^x V(s)\left[\delta + \sin\left(\frac{x-s}{2}\right)\right] \times \frac{\exp\left(j\frac{4\pi a}{\lambda}\left[\sin\left(\frac{s}{2}\right) + \sin\left(\frac{x-s}{2}\right) - \sin\left(\frac{x}{2}\right)\right]\right)}{\sqrt{s(x-s)}}ds, \quad (1)$$

where x is the straight-line distance between the antenna and the observation point that is normalized to spherical surface radius a, $\alpha = [ka/(2\pi)]^{1/2}\exp(j3\pi/4)$, a is the radius of the spherical Earth's surface, k is the wave number in free space, $\delta = (\varepsilon_{rc} - 1)^{1/2}/\varepsilon_{rc}$ is the reduced surface impedance of the underlying surface, and ε_{rc} is the relative complex permittivity.

Distance x between the antenna and the observation point in Eq. (1) is normalized to the radius of the Earth. Therefore, condition $x \ll 1$ is satisfied even at distances of hundreds and thousands of kilometers. This allows one to simplify the equation significantly and find its approximate analytical solution. A simple approximation of the integral equation kernel proposed in [7] specifies that $\sin([x-s]/2) \approx [x-s]/2$ and the exponential is equal to unity. Equation (1) is then written as

$$V(x) = 1.0 + \alpha\sqrt{x}\int_0^x \left[\delta + \frac{x-s}{2}\right]\frac{V(s)}{\sqrt{s(x-s)}}ds. \quad (2)$$

A solution to this equation may be found in the form of a series [9]:

$$W(x) = \sum_{k=0}^{\infty} \alpha^k \varphi_k(x), \quad (3)$$

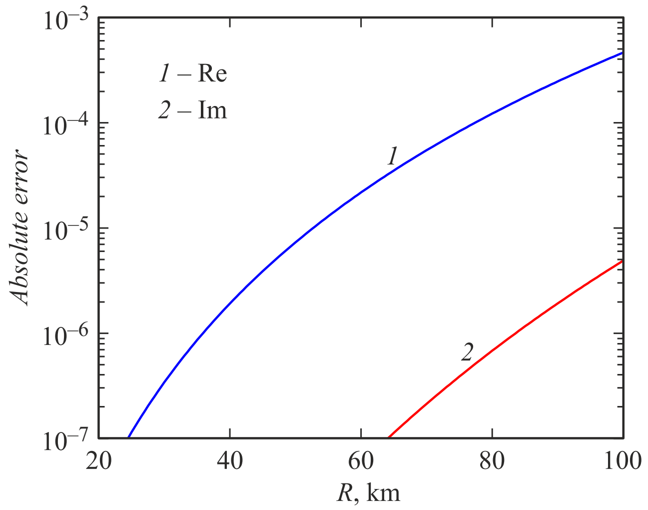

Figure 1. Absolute error of exponential approximation.

$$\varphi_{k+1}(x) = \int_0^x \left[\delta + \frac{x-s}{2}\right] \frac{\varphi_k(x)}{\sqrt{s(x-s)}} ds, \quad \varphi_0(x) = 1. \tag{4}$$

Performing certain transformations and discarding inessential terms, one may obtain the following approximate solution [7]:

$$W_{SM}(R) = y(SR) + \frac{1}{a}\sqrt{\frac{jk}{8S}} \times \left\{R - j\sqrt{\frac{\pi R}{4S}} + \frac{1+2SR}{2S}\left[y(SR) - 1\right]\right\}, \tag{5}$$

where $R = ax$ is the straight-line distance between the antenna and the observation point, $y(\rho)$ is the attenuation function for the flat homogeneous Earth model ($\rho = SR$ is the numerical distance), and $S = ik[1 - (1-\delta^2)^{1/2}]$ is the complex scaling factor that characterizes the influence of the underlying surface.

The results of calculations with formula (5) agree closely with more accurate formulae up to a distance of approximately 50 km; after that, the calculation error increases sharply. The aim of the present study is to find a simple and more accurate approximation of the integral equation kernel and obtain a refined solution.

The approximation used for $\sin([x-s]/2)$ is the same as the one in [7]. Its absolute error is no greater than 10^{-4} at a distance of 1000 km:

$$\sin\left(\frac{x-s}{2}\right) \approx \frac{x-s}{2}. \tag{6}$$

The exponential factor is expanded into a Taylor series, and all terms except the first two are discarded:

$$\sin\left(\frac{s}{2}\right) + \sin\left(\frac{x-s}{2}\right) - \sin\left(\frac{x}{2}\right) \approx \frac{sx(x-s)}{16}. \tag{7}$$

Expanding the exponential into a Taylor series and applying formula (7), we obtain

$$\exp\left(j\frac{4\pi a}{\lambda}\left[\sin\left(\frac{s}{2}\right) + \sin\left(\frac{x-s}{2}\right) - \sin\left(\frac{x}{2}\right)\right]\right) \approx 1 + j\frac{\pi a}{4\lambda} sx(x-s). \tag{8}$$

The modulus of the maximum absolute error of approximation (8) within the integration interval is shown in Fig. 1. Curves *1* and *2* correspond to real and imaginary parts of the error. It follows from the figure that the maximum error of the real part does not exceed 10^{-3} if the observation point is located at distances up to 100 km from the antenna.

Applying the above approximations, we obtain the following „refined“ integral equation:

$$V(x) = 1.0 + \alpha\sqrt{x}\int_0^x \left[\delta + \frac{x-s}{2}\right] \times \left[1 + j\frac{\pi a}{4\lambda} sx(x-s)\right] \frac{V(s)}{\sqrt{s(x-s)}} ds. \tag{9}$$

Similar to the solution of Eq. (2), the solution of this equation may also be found in the form of a series [9]:

$$W(x) = \sum_{k=0}^{\infty} \alpha^k \psi_k(x), \tag{10}$$

$$\psi_{k+1}(x) = \sqrt{x}\int_0^x \left[\delta + \frac{x-s}{2}\right]\left[1 + j\frac{\pi a}{4\lambda} sx(x-s)\right] \times \frac{\psi_k(x)}{\sqrt{s(x-s)}} ds, \qquad \psi_0(x) = 1. \tag{11}$$

It follows from the comparison of formulae (4) and (11) that the terms of series (11) contain the terms of series (4); i.e., $\psi_k(x) = \varphi_k(x) + \psi_k'(x)$. This statement may be proven by induction. The terms of series (11) may then be expressed through formula (4):

$$\psi_{k+1}(x) = \sqrt{x}\int_0^x \left(\delta + \frac{x-s}{2}\right)\left(1 + j\frac{\pi a}{4\lambda} sx(x-s)\right) \times \frac{\varphi_k(x) + \psi_k'(x)}{\sqrt{s(x-s)}} ds = \varphi_{k+1}(x) + \sqrt{x}\left[\int_0^x \left(\delta + \frac{x-s}{2}\right) \times \left(\frac{\psi_k'(x)}{\sqrt{s(x-s)}} + j\frac{\pi a}{4\lambda} x\sqrt{s(x-s)}\left(\varphi(x) + \psi_k'(x)\right)\right) ds\right]. \tag{12}$$

We thus obtain the first three terms of series (10) with formula (12) taken into account:

$$\psi_1(x) = \varphi_1(x) + \frac{\beta}{8} x^3 \varphi_1(x), \tag{13}$$

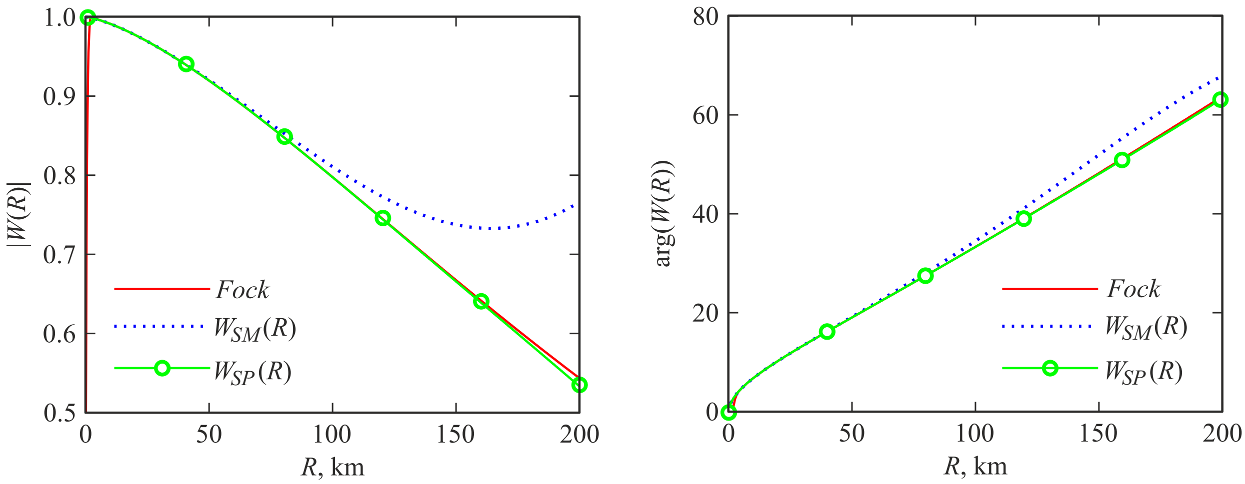

Figure 2. AF above a spherical sea surface.

$$\psi_2(x) = \varphi_2(x) + \frac{4\beta}{21}x^3\left[\varphi_2(x) + \frac{\pi x^3}{165} + \frac{2x^3}{2145}\right.$$

$$\left.\times\left(52\delta^2 + 16\delta x + x^2\right)\right] \approx \varphi_2(x) + \frac{4\beta}{21}x^3\varphi_2(x), \qquad (14)$$

$$\psi_3(x) = \varphi_3(x) + \frac{11\beta}{48}x^3\left[\varphi_3(x) + \pi^2\sqrt{x}x^3\left(\frac{3\delta}{1760} + \frac{41x}{591\,360}\right)\right]$$

$$+ \frac{\pi^2\beta^2x^7\sqrt{x}}{128}\left(\delta^3 + \frac{31\delta^2 x}{84} + \frac{473\delta x^2}{13\,440} + \frac{143x^3}{161\,280}\right)$$

$$\approx \varphi_3(x) + \frac{11\beta}{48}x^3\varphi_3(x), \qquad (15)$$

where $\beta = j\pi a/4\lambda$ and expressions $\varphi_1(x)$–$\varphi_3(x)$ are given by [7]

$$\varphi_1(x) = \pi\sqrt{x}\left(\delta + \frac{x}{4}\right),\ \varphi_2(x) = \pi x\left(2\delta^2 + \frac{2}{3}\delta x + \frac{x^2}{30}\right),$$

$$\varphi_3(x) = \pi^2 x^{3/2}\left(\delta^3 + \frac{3}{8}\delta^2 x + \frac{\delta x^2}{32} + \frac{x^3}{1536}\right).$$

Since solution (10) of refined Eq. (9) contains solution (3) of Eq. (2), the summation of series terms $\varphi_k(x)$ in (10) may be replaced by formula (5):

$$W_{SP}(R) = y(SR) + \frac{1}{a}\sqrt{\frac{jk}{8S}}\left\{R - j\sqrt{\frac{\pi R}{4S}} + \frac{1+2SR}{2S}\left[y(SR) - 1\right]\right\}$$

$$+ \Phi\left(Ra^{-1}\right) + \alpha\psi_1'\left(Ra^{-1}\right) + \alpha^2\psi_2'\left(Ra^{-1}\right) + \alpha^3\psi_3'\left(Ra^{-1}\right), \qquad (16)$$

where R is the straight-line distance (not along the Earth's surface) between the transmitting antenna and the receiving point, a is the Earth's radius, and $\Phi(x)$ is a part of the series terms [7] discarded in derivation of formula (5), which start to exert a noticeable influence at distances up to 100 km:

$$\Phi(x) = \frac{\alpha^2\pi}{30}x^3 + \frac{\alpha^3\pi^2\delta}{32}\sqrt{x}x^3 + \frac{2\alpha^4\pi^2\delta^2}{35}x^4. \qquad (17)$$

The resulting refined formula for short distances (RFSD) (16) is suitable for calculating the attenuation factor only for high-conductivity sea paths with $|\delta| \ll 1$. An example calculation of the AF for a homogeneous spherical sea surface (here and elsewhere, the sea parameters are $\varepsilon_r = 80$, $\sigma = 5$ S/m, and frequency $f = 1.9$ MHz) with the obtained formulae is presented in Fig. 2. The AF dependence calculated using the classical Fock formula [4] is shown in the same figure for comparison. It is evident that formula for short distances (FSD) (5) provides a lower accuracy than RFSD (16). In accordance with the stated purpose, the RFSD may be used for distances of at least 100 km.

To evaluate the accuracy of the obtained formulae, Huffford integral equation (1) and its simplified (2) and refined (9) versions were solved numerically in accordance with the procedure outlined in [10]. Errors for the sea surface were then plotted (Fig. 3) as $|W - W_H|/|W_H| \cdot 100\,\%$, where W_H are the complex AF values obtained by solving Hufford integral equation (1) and W are the complex AF values determined by the method indicated in the legend. Solid curves in the figure correspond to the errors of the numerical solution of Hufford equations (2) and (9), while dashed curves represent FSD (5) and RFSD (16). The cal-

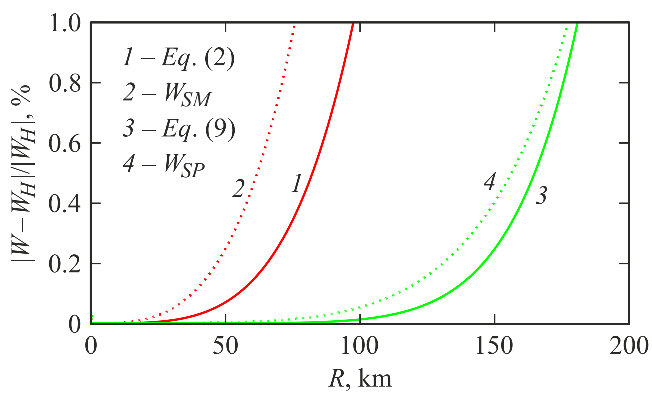

Figure 3. Error of AF calculation over a spherical sea surface.

culation results reveal that the accuracy of the FSD is lower than that provided by simplified Hufford equation (2). This is attributable to the truncation of series terms in the process of derivation: if at least some of discarded terms (17) are added to FSD (5), the two plots start to merge. The refined formula for short distances also falls a little short of the ultimate accuracy, but the error at a distance of 100 km is just 0.05 %, which is sufficient in most cases.

It should also be noted that the refined formula for short distances is not suitable for low-conductivity paths, which include any land areas. The reason for this is the use of a small number of terms (just the first three) of series (10).

To summarize, refined formula for short distances (16):

(1) provides high accuracy for sea paths at distances up to about 100 km;

(2) may be used over low-conductivity paths if a larger number of series (10) terms are taken into account;

(3) allows one to calculate accurately the AF for homogeneous sea paths where the normal wave series (Fock's formula [4]) converges poorly.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] I.V. Kulikovskii, O.R. Adamovich, D.V. Berbenev, Voen. Mysl', No 12, 50 (2023). https://elibrary.ru/ebcise

[2] A.S. Seliverstov, Zap. Gidrogr., **308**, 54 (2019) (in Russian). https://mil.ru/activity/social/media/journal/notes_on_hydrography

[3] E.L. Feinberg, *Rasprostranenie radiovoln vdol' zemnoi poverkhnosti* (Nauka, M., 1999), pp. 167, 281, 284 (in Russian).

[4] V.A. Fock, *Problemy difraktsii i rasprostraneniya elektromagnitnykh voln* (Sov. Radio, M., 2011), p. 249 (in Russian).

[5] M.G. Dembelov, Yu.B. Bashkuev, Fiz. Volnovykh Protsessov Radiotekh. Sist., **11** (1), 89 (2008) (in Russian). https://www.elibrary.ru/item.asp?id=12946241

[6] G.I. Makarov, V.V. Novikov, S.T. Rybachek, *Rasprostranenie elektromagnitnykh voln nad zemnoi poverkhnost'yu* (Nauka, M., 1991), pp. 33–49 (in Russian).

[7] A.A. Senchenko, Yu.P. Salomatov, Izv. Vyssh. Uchebn. Zaved. Fiz., **53** (9/2), 114 (2010) (in Russian). https://www.elibrary.ru/item.asp?id=16036544

[8] G.A. Hufford, Quart. Appl. Math., **9** (4), 391 (1952). DOI: 10.1090/qam/44350

[9] I.S. Berezin, N.P. Zhidkov, *Metody vychislenii* (Fizmatlit, M., 1959), Vol. 2, p. 613 (in Russian).

[10] A.A. Senchenko, Yu.P. Salomatov, Dokl. TUSUR, **26** (2-2), 36 (2012) (in Russian). https://www.elibrary.ru/item.asp?id=18900071

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