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Maximum eigenfrequency of axisymmetric disturbancies of rotating fluid

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The stability of ideal incompressible rotating fluid is considered in linear approximation. It is shown, that eigenfrequency square for axisymmetric disturbances is restricted above by module as for stable so unstable flow. The eigenfrequencies can be enumerate by module reduction from zero for maximum eigenfrequency. The eigenfunction numbers by radius are equal (according to Sturm theory) to eigenfrequency number. As illustration, we calculate eigenfrequencies and eigenfunctions for Taylor-Couette flow with different stability properties..

Keywords: stability, Taylor-Couette flow, incompressible fluid.

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Introduction

The problem of the flow stability of an ideal incompressible rotating fluid is a classical problem of hydrodynamics [1–3]. The stability condition for an ideal incompressible rotating fluid with uniform density with respect to axisymmetric perturbations (see expression (8) of sec. 1 herein) was obtained by Rayleigh [4]. It has been demonstrated later in [5] that condition (8) is necessary and sufficient for stability.

It is well known (see, for example, [1,3]) that the square of the natural frequency for axisymmetric perturbations of a rotating fluid is a real number, and accordingly for normal modes (5) the flow is stable if all the squares of the natural frequencies are positive (if condition (8) is fulfilled at all points of the flow), or unstable if at least one value of the square of the natural frequency is negative (when condition (8) is not fulfilled at any point of the flow).

In this paper, it is shown for the first time that the frequency spectrum of a rotating fluid for axisymmetric disturbances is modulo limited for both positive and negative values of the squares of the natural frequencies.

As an example, the natural frequencies and the eigen functions of the cylindrical Couette flow are calculated for three cases: 1) the flows, at each point of which the condition (8) is not fulfilled; 2) the flows, at each point of which the condition (8) is fulfilled; 3) the flows, in part of which the condition (8) is not fulfilled, but in the other part it is fulfilled. It is shown that for both positive and negative values of the squares of the natural frequency, they can be numbered (separately for positive and negative values) as the modulus decreases, starting from zero for the maximum modulo natural frequency. In accordance with Sturm's theory, the number of nodes of the corresponding eigen function along the radius will be equal to the number of the natural frequency.

1. Main equations and problem setting

The motion of an incompressible ideal fluid with uniform density ρ is characterized by the Euler and continuity equations:

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P,$$
$$\mathrm{div}\mathbf{U} = 0, \tag{1}$$

where $\mathbf{U}$ — velocity, P — pressure. A cylindrical coordinate system (r, ϕ, z) is convenient for use for a rotating fluid, where axisymmetric velocity is written as

$$\mathbf{U} = (0, r\Omega(r), 0), \tag{2}$$

where $\Omega(r)$ is the angular rotation velocity which for an ideal fluid is an arbitrary (sufficiently smooth) function of radius that satisfies the equation (1). For a stationary case, the equation (1) for the velocity (2) takes the form of the equation of hydrostatic equilibrium between the centrifugal force and the force generated by the radial pressure gradient [3]:

$$\Omega^2 r = \frac{1}{\rho}\frac{\mathrm{d}P(r)}{\mathrm{d}r}. \tag{3}$$

To avoid misunderstandings, we emphasize that we use an inertial coordinate system in which there is no Coriolis force (see, for example, [6]). Additionally, boundary conditions are needed to complete the formulation of the problem. In the present study, we consider a region that is not limited in vertical axis coordinate z, extending from inner radius $r_{in} \geq 0$ to the outer radius $r_{out} < \infty$ and covering the entire angular sector $0 \leq \phi \leq 2\pi$.

The method of small perturbations is used to study stability in the linear approximation. The solution is then presented as follows:

$$\mathbf{U}+\mathbf{u} = (u_r(t, r, \phi, z), r\Omega(r)+u_\phi(t, r, \phi, z), u_z(t, r, \phi, z)),$$
$$P + p = P(R) + p(t, r, \phi, z), \tag{4}$$

where the values u_r, u_ϕ, u_z are small compared to the typical value of the azimuthal velocity $r\Omega_0$, where $\Omega_0 = 0.5(\Omega(r_{in}) + \Omega(r_{out}))$, and the pressure perturbation gradients p are small compared to the radial gradient of the unperturbed pressure P. Putting expressions (4) into system (1) and retaining only the terms linear in perturbed quantities, one obtains a linear system of equations for perturbed quantities with coefficients that depend on radial coordinate r. only. The solution may then be presented as a sum of normal modes:

$$F = F_{mk}(r)\exp[\mathrm{i}(m\phi + kz + \omega t)], \tag{5}$$

where $F(r)$ is an arbitrary sought-for function. Given the geometry of the problem, axial number k may assume arbitrary real values, azimuthal number m may be an arbitrary integer, and increment ω may be an arbitrary complex number. The expansion in normal modes (5) transforms a three-dimensional problem into a one-dimensional one. If natural frequencies ω for the velocity (2) have only positive imaginary parts, the flow is linearly stable. If at least one natural frequency with a negative imaginary part is present, the flow is unstable.

In this paper, we will consider only axisymmetric perturbations with $m = 0$ and fixed boundaries. In this case, the linear system is reduced to a single second-order equation for radial velocity u_r (indexes m and k (see (5)) are omitted here)

$$\frac{\mathrm{d}^2u_r}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u_r}{\mathrm{d}r} - \frac{u_r}{r^2} - k^2u_r + \frac{k^2}{\omega^2}\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\Omega\right)^2 u_r = 0, \tag{6}$$

and the boundary conditions in this case are expressed as follows:

$$u_r(r_{in}) = u_r(r_{out}) = 0. \tag{7}$$

The equation (6) is absolutely accurate and takes into account all the effects that occur and, together with the boundary conditions (7) compiles the classic Sturm-Liouville problem on the eigen values of a quantityk^2/ω^2 [1]. It is well known that given the boundary conditions (7) the value ω^2 — is real (e.g., see, [1]) and sign ω^2 coincides with sign k^2/ω^2. According to the general theory (see, for example, [1]), all eigen values are positive (which means that the flow is stable) if and only if

$$\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\Omega\right) > 0 \tag{8}$$

for any point of the considered interval. If condition (8) is not fulfilled in some point the flow is considered as unstable. Condition (8) was proved by Rayleigh [4] and bears his name.

2. Results

To prove the limitation of the natural frequency square ω^2, it is sufficient to use the theory of oscillation of solutions of the second degree linear ordinary differential equations with real coefficients. By definition, the solution is non-oscillating on the interval (a, b), where $-\infty < a < b < \infty$, if it has at most one zero in (a, b). For a general second-order equation

$$\frac{\mathrm{d}^2u_r}{\mathrm{d}r^2} + a_1(r)\frac{\mathrm{d}u_r}{\mathrm{d}r} + a_0(r)u_r = 0 \tag{9}$$

the simplest sufficient condition for non-oscillation of the solution on the interval (a, b) is the condition $a_0(r) < 0$ on this interval (see, for example, [7]). Accordingly, the solutions of the equation (6) are non-oscillating and cannot satisfy the boundary conditions (7) if

$$|\omega^2| > \max\left(\frac{k^2\frac{1}{r^3}|\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\Omega\right)^2|}{\frac{1}{r^2} + k^2}\right) \tag{10}$$

in the interval (r_{in}, r_{out}). According to the classical Sturm comparison theorem (see, for example, [8]), the oscillation frequency (i.e., the number of nodes along the radius) eigen functions of the equation (6) increases with increasing eigen value k^2/ω^2. Accordingly, the number of nodes of the eigen function increases with decreasing natural frequency modulus. By numbering the natural frequencies as the modulus decreases, starting from zero for the maximum modulo natural frequency, we obtain that the number of nodes of the corresponding eigen function along the radius in the interval (r_{in}, r_{out}) will be equal to the number of the natural frequency.

As an example, we find the natural frequencies and eigen functions of axisymmetric perturbations for a flow between two coaxial infinitely long rotating cylinders (cylindrical Couette flow). Recall that for an ideal fluid, any sufficiently smooth radius function satisfying the boundary conditions can be chosen as the angular velocity. As such a function, we choose a function that satisfies the equations of motion of a viscous fluid and, accordingly, has a fixed functional form (see, for example, [1])

$$u_\phi(r) = r\Omega(r) = Ar + \frac{B}{r}, \tag{11}$$

where constants A and B are defined by the boundary conditions:

$$A = \Omega_{in}\frac{\mu - \eta^2}{1 - \eta^2}, \quad B = \Omega_{in}R_{in}^2\frac{1 - \mu}{1 - \eta^2}, \tag{12}$$

$$\eta = \frac{r_{in}}{r_{out}}, \quad \mu = \frac{\Omega_{out}}{\Omega_{in}}, \tag{13}$$

r_{in} and r_{out} — radii, Ω_{in} and Ω_{out} — angular velocities of internal and external cylinders.

Let's note that Rayleigh condition (8) for the flow (11) is expressed as:

$$\mu > \eta^2. \tag{14}$$

Accordingly, for flows with $\mu > 0$ (i.e., if the cylinders rotate in the same direction), the flow will be stable at each point

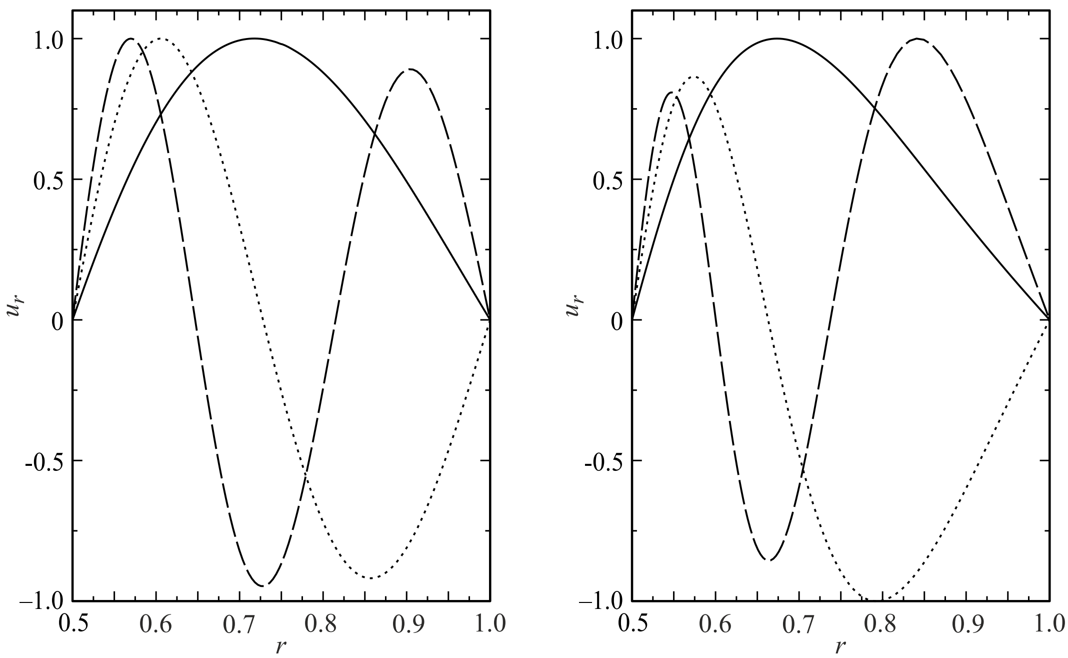

Figure 1. Normalized eigen functions of Couette flow with $\eta = 0.5$ for perturbations with $k = 3$, $m = 0$ (axisymmetric perturbations) for the steady flow (in the left) with $\mu = 0.5$ (see (14)) and $\omega^2 = 0.1556$ (solid curve), $\omega^2 = 0.04679$ (dashed curve), $\omega^2 = 0.02161$ (dotted curve) and unstable flow (in the right) with $\mu = 0$ (non-rotating external cylinder) with $\omega^2 = -0.07678$ (solid curve), $\omega^2 = -0.02085$ (dashed curve), $\omega^2 = -0.009359$ (dotted curve).

(if condition (14) is met) or unstable at each point (if condition (14) is not fulfilled). For the flows with $\mu < 0$ (i.e., for cylinders rotating in different directions), as can be easily seen, the flow will be unstable near the inner cylinder and stable near the outer one.

Before solving equation (6), it is convenient to reduce it to a dimensionless form. Let's take r_{out} as the unit length, and Ω_{in} as a unit angular velocity. Equation (6) with the boundary conditions (7) and dimensionless angular velocity (11) was solved by Runge-Kutta method at fixed parameters μ, η, k and $m = 0$. The shooting method was used. One solution satisfying the left boundary condition $u_r(r_{in}) = 0$, with some trial value of the first derivative at $r = r_{in}$, was consistent at some intermediate point $r_{in} < r_0 < r_{out}$ (usually near the center of the calculated interval) with the second solution satisfying the right boundary condition $u_r(r_{out}) = 0$, with a different trial value of the first derivative at $r = r_{out}$. These solutions can be connected only at certain values of the parameter ω^2 (there are generally infinitely many of these values), which are called the eigen values of the problem, and their corresponding eigen functions are the eigen functions of the problem.

Numerous calculations have been carried out, which have shown that solutions exist only for values $|\omega^2|$ that do not exceed a certain limit value corresponding to the condition (10). Fig. 1 and 2 show the normalized dimensionless eigen functions. The functions are normalized so that their maximum modulo value is one. In all cases, the eigen functions for the three largest modulo squares of the eigen frequencies are presented. It is also easy to verify that in all cases the squares of the maximum natural frequencies satisfy the condition (10).

Figure 1 shows the results for a flow with cylinders rotating in one direction (i.e. $\mu > 0$). For stable flow with $\mu > \eta^2$, all squares of the natural frequencies are positive. Naturally, they will change with the change of parameters. For instance, with the change of the wave number k. The following figure shows the eigen functions for the first three (in magnitude) natural frequencies. It can be seen that the number of nodes of the eigen functions along the radius, according to the Sturm comparison theorem, corresponds to the number of the natural frequency. Similarly, for the unstable flow with $0 < \mu < \eta^2$, all squares of the natural frequencies are negative. There is a modulo-limiting natural frequency, and the number of nodes of the corresponding eigen function increases with the decline of the natural frequency modulus.

Figure 2 shows the results for a flow with cylinders rotating in different directions (with $\mu < 0$). In this case, the squares of the natural frequencies will be both positive and negative, given that the flow is unstable near the inner cylinder and stable near the outer one. This is clearly noticeable by the nature of the eigen functions, which are concentrated to the inner cylinder for the negative squares of the natural frequencies and to the outer cylinder for the positive squares of the natural frequencies. At the same time, the behavior of the natural frequency spectrum corresponds to the above: there is a modulo-limiting frequency, and the number of nodes of the eigen functions

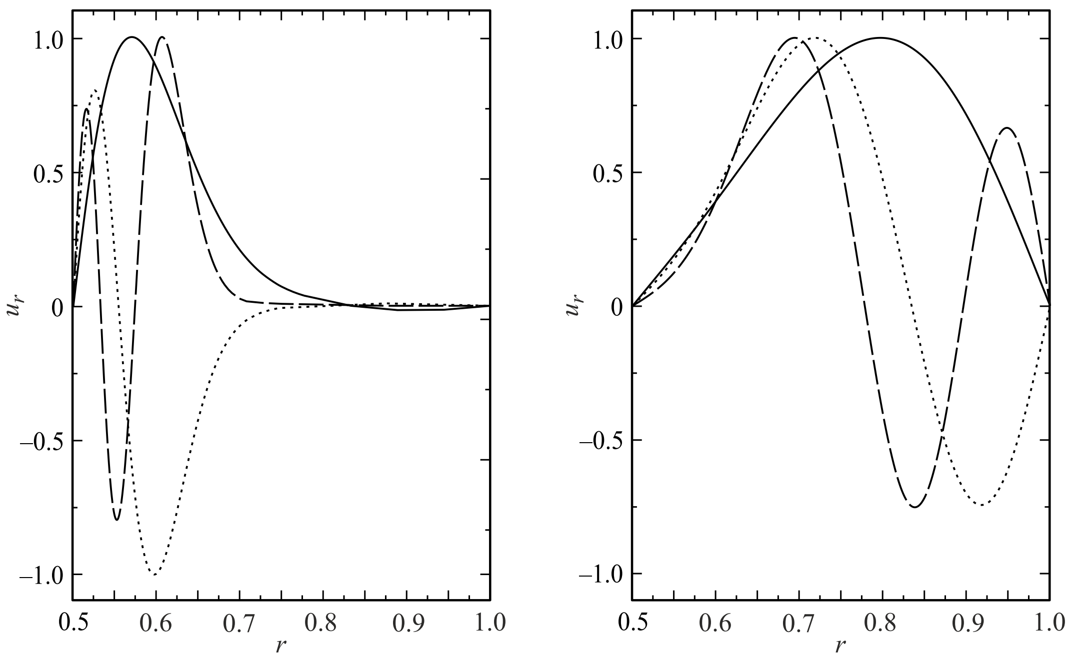

Figure 2. Normalized eigen functions of Couette flow with $\eta = 0.5$ for perturbations with $k = 3$, $m = 0$ (axisymmetric perturbations) for the cylinders rotating in different directions with $\mu = -1$ and with negative squares of the natural frequencies (in the left) $\omega^2 = -0.0674$ (solid curve), $\omega^2 = -0.0131$ (dashed curve), $\omega^2 = -0.00536$ (dotted curve) and with positive squares of the natural frequencies (in the right) $\omega^2 = 0.6$ (solid curve), $\omega^2 = 0.142$ (dashed curve), $\omega^2 = 0.0599$ (dotted curve).

along the radius increases as the natural frequency modulus declines.

Conclusion

Despite an over century-old history of studying the stability of fluid rotation, the problem is still far from its final solution. In this paper, even for the simplest case of a homogeneous ideal fluid, it is shown for the first time that the natural frequencies of axisymmetric perturbations are limited in modulus.

In addition, the natural frequencies can be numbered as their modulus decreases from the maximum modulo frequency which has zero number assigned. In this case, the numerator of the nodes of the eigen function along the radius will correspond (in accordance with Sturm theory) to the number of the natural frequency. The general results are illustrated by calculations for the cylindrical Couette flow.

Note that the findings of this study are substantially based on the results of the theory of oscillation of solutions of second-order ordinary differential equations with real coefficients. It should be emphasized that this theory cannot be extended to both, second-order differential equations with complex coefficients and higher-order differential equations. Despite this, this theory may and has already found (see, for example, [9]) a wide application in various problems of hydrodynamics and magnetic hydrodynamics.

Conflict of interest

The author declares that he has no conflict of interest.

References

[1] S. Chandrasekhar. *Hydrodynamic and Hydromagnetic Stability* (Clarendon Press, Oxford, 1961)
[2] H.P. Greenspan. *The Theory of Rotating Fluids* (Cambridge Univ. Press, Cambridge, 1968).
[3] P.G. Drazin, W.H. Raid. *Hydrodynamic stability* (Cambridge Univ. Press, Cambridge, 1981)
[4] R. Lord, Proc. R. Soc. London A, **93**, 148 (1917). DOI: 10.1098/rspa.1917.0010
[5] J.L. Synge. Trans. R. Soc. Can., **27**, 1 (1933).
[6] L.D. Landau, E.M. Lifshits. *Teoreticheskaya fizika. Hydrodynamics* (Moscow, Nauka, 1986), v. 6, p. 66.
[7] V.A. Kondratiev. UMN, **12** (3), 159 (1957).
[8] C.A. Swanson, *Comparision and oscillation theory of linear differential equation* (Academic Press, NY., London, 1968)
[9] J.P. Goedbloed, S. Poedts. *Principles of Magnetohydrodynamics* (Cambridge Univ. Press, Cambridge 2004)

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