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Grey solitons in the ultracold fermions at the full spin polarization

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A minimal coupling quantum hydrodynamic model of spin-1/2 fermions at the full spin polarization corresponding to a nonlinear Schrödinger equation is considered. The nonlinearity is primarily caused by the Fermi pressure. It provides an effective repulsion between fermions. However, there is the additional contribution of the short-range interaction appearing in the third order by the interaction radius. It leads to the modification of the pressure contribution. Solitons are considered for the infinite medium with no restriction on the amplitude of the wave. The Fermi pressure leads to the soliton in form of the area of decreased concentration. However, the center of solution corresponding to the area of minimal concentration has nonzero value of concentration. Therefore, the grey soliton is found. Soliton exist if the speed of its propagation is below the Fermi velocity.

Keywords: degenerate fermions, hydrodynamics, non-linear Schrödinger equation, dark soliton, Fermi pressure.

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1. Introduction

Solitons are fundamental nonlinear structures existing in various physical systems including the quantum gases [1–10]. Vortexes, shock waves, and skyrmions are among main nonlinear objects considered in atomic quantum gases. Solitons and other nonlinear phenomena are well studied in the bosonic atoms experimentally and theoretically. The theoretical approach is mainly based on the mean-field nonlinear Schrödinger equation called the Gross-Pitaevskii equation which describes the bright and dark solitons in the Bose-Einstein condensates (BECs), where the nonlinearity is caused by the interparticle interaction [11]. The form of solitons is related to the sign of interaction, so the bright and dark solitons appears in the attractive and repulsive BECs, correspondingly. For instance, the experimental study of the dark solitons in BECs is presented in Ref. [12]. Majority of study of fermionic gases are focused on the superfluid phase or BCS state (Bardeen-Cooper-Schrieffer state), where pairs of fermionic atoms with opposite spins and momentum form the Cooper pairs and demonstrate the boson-like behavior [13]. The dark solitons in superfluid Fermi gases are considered in Refs. [14–16], particularly, the Bogoliubov-de Gennes equations are used in [14]. A heavy soliton in a fermionic superfluid is experimentally observed in Ref. [17]. Solitons in the superfluid Fermi gases are considered in terms of the nonlocal generalization of the Ginzburg-Landau model [18], following Ref. [19]. Here, the basic and fundamental solitons are considered in degenerate fermionic atoms with the full spin polarization. The spin-1/2 atoms are chosen, but the same analysis is correct for the fermions with higher spins. The presented theoretical work is based on the quantum hydrodynamics which is straightforwardly

derived from the microscopic many-particle Schrödinger equation (from the full quantum theory). The minimal coupling model of fermions is composed of two hydrodynamic equations: the continuity and Euler equation. These hydrodynamic equations allow to obtain the corresponding nonlinear Schrödinger equation, where the nonlinearity is mainly caused by the Fermi pressure. It is nonlinearity of fractional degree $7/3$. However, the interaction between fermions gives the additional nonlinearity [20].

Unpolarized fermions are mostly discussed in literature regime for fermions [21–23]. If we have system of spin-1/2 Fermi atoms with equal population of the spin-up state and the spin-down state we can observe interesting phases of matter. They are the Bose-Einstein condensate of molecules, crossover superfluid, and the BCS state. The spin orbit coupling can also be engineered in the unpolarized fermions.

If we consider bosons being in the Bose-Einstein condensate state it can be described by the Gross-Pitaevskii equation. The interaction appears in the first order by the interaction radius in term of the hydrodynamic derivation of the Gross-Pitaevskii equation. Or it can be interpreted as the s-wave scattering in terms of the scattering theory. For the polarized fermions, there is no contribution of the interaction in the first order by the interaction radius (FOIR), due to the antisymmetry of the wave function. Hence, main selfaction of the fermion fluid comes from the Fermi pressure. However, the additional contribution of the interaction appearing in the third order by the interaction radius (TOIR) can be derived [20]. This contribution can be interpreted via the p -wave scattering in terms of the scattering theory [24–26].

The spin polarized fermions is the system where all fermions occupy the single spin state. This systems shows

rather avaricious phase. Nevertheless, it also demonstrates some interesting fundamental nonlinear phenomena. To some extent the system of polarized degenerate fermions can be described by the effective macroscopic single particle wave function $\Phi(\mathbf{r}, t)$. This possibility follows from the quantum hydrodynamic equations restricted by the particle density and the momentum density evolution. However, complete description of polarized fermions requires the momentum current evolution, which is the kinetic pressure of polarized fermions. The pressure evolution equation gives more accurate value of the speed of sound [20,23,27]. Nevertheless, the minimal coupling model based on the particle density and the momentum density shows good qualitative description of fermions.

This paper is organized as follows. In Sec. II the quantum hydrodynamics is presented in two regimes: the mean-field approximation and up to the TOIR approximation. In Sec. III solution of the hydrodynamic equations in the one-dimensional regime in the form of the grey soliton is obtained by the Sagdeev potential method. In Sec. IV a brief summary of obtained results is presented.

2. Quantum hydrodynamic equations

Here, we present two quantum hydrodynamic models for the degenerate fermions being in the same spin state (the regime of the full spin polarization). The first model is obtained in the FOIR approximation, where the interaction gives the zero contribution. The second model contains the contribution of the interaction in the TOIR approximation.

2.1. First order by the interaction radius: A minimal coupling hydrodynamic model for the full spin polarization

Nonzero contribution of the interaction in the first order by the interaction radius exists for nonpolarized or the partially polarized systems of fermions. However, the fully polarized fermions have zero contribution of interaction in this case.

In all regimes we have same form of the continuity equation:

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0, \quad (1)$$

In the FOIR approximation we also have the Euler (momentum balance) equation

$$m n (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} - \frac{\hbar^2}{2m} n \nabla \frac{\Delta \sqrt{n}}{\sqrt{n}} + \nabla p = -n \nabla V_{ext}, \quad (2)$$

where p is the Fermi pressure

$$p = \frac{(6\pi^2)^{\frac{2}{3}} \hbar^2 n^{\frac{5}{3}}}{5m^2}. \quad (3)$$

Minimal coupling assumes the application of the continuity and Euler equation with no account of the pressure evolution, but application of the equation of state for the

reduction of the pressure evolution to the concentration evolution.

Equations (1)–(3) correspond to the nonlinear Schrödinger equation for fermions at the potential velocity field [20]:

$$i\hbar \partial_t \Phi = \left(-\frac{\hbar^2 \nabla^2}{2m} + \frac{(6\pi^2 n)^{\frac{2}{3}} \hbar^2}{2m} + V_{ext} \right) \Phi, \quad (4)$$

where $n = |\Phi|^2$. The effective macroscopic wave function Φ is defined via the hydrodynamic wave functions $n(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$:

$$\Phi(\mathbf{r}, t) = \sqrt{n} e^{im\phi/\hbar}, \quad (5)$$

where $\mathbf{v} = \nabla \phi$. The contribution of the Fermi pressure (3) is presented by the second term on the right-hand side of equation (4).

Equation similar to NLSE (4) are used in literature [28–35]. Different forms have different justifications. Equation (4) is justified via the quantum hydrodynamics. Moreover, the partial or full spin polarization is not included there.

Equations (1) and (2) are applied below to consider the possibility of solitons in the systems of neutral atomic degenerate fermions. To complete the description of model we present the hydrodynamics containing the contribution of the interaction between fermions with the same spin polarization. Absence of the interaction in equations (2) and (4) shows that the equilibrium condition cannot be reached in such systems. However, we have interaction between fermions which is presented below. It provides the additional transfer of the momentum and a mechanism of reaching of the equilibrium state.

2.2. Hydrodynamic equations and nonlinear Schrödinger equation for fermions with the interaction included up to the TOIR approximation

Derivation of the macroscopic equations by the many-particle quantum hydrodynamics method [36–40] shows that the hydrodynamic equations appear first. Next, in some simplified regimes the nonlinear Schrödinger equation can be found [20,36].

The nonlinear Schrödinger equation can be derived in the chosen approximation [20]:

$$i\hbar \partial_t \Phi = \left(-\frac{\hbar^2 \nabla^2}{2m} + \frac{(6\pi^2 n)^{\frac{2}{3}} \hbar^2}{2m} + V_{ext} - \frac{4}{5} g_2 (6\pi^2)^{\frac{2}{3}} n^{\frac{5}{3}} \right) \Phi, \quad (6)$$

the additional term caused by the interaction is the last term in equation (6). The additional term is obtained in the TOIR approximation. It contains the interaction constant g_2 which is defined via the potential of interatomic interaction U :

$$g_2 = \int r^2 U(r) d\mathbf{r}. \quad (7)$$

The positive interaction constant decreases the pressure. However, the model is obtained in the weak interaction

limit. So, the contribution of interaction should be small in compare with the Fermi pressure.

Let us present the corresponding hydrodynamic equations. The continuity equation has same form (1). The Euler equation contains the additional term

$$\begin{aligned} mn(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} - \frac{\hbar^2}{2m} n \nabla \frac{\Delta \sqrt{n}}{\sqrt{n}} + \nabla p \\ = -n \nabla V_{ext} + \frac{5m^2}{2\hbar^2} g_2 \nabla(np), \end{aligned} \quad (8)$$

which is the last term in equation (8). The interaction term appear via the kinetic pressure p [20]. The gradient of the kinetic pressure itself also presented by the last term on the left-hand side of equation (8). In this paper we use the equation of state in form of the Fermi pressure (3). Hence, the Euler equation (8) is truncated and its final form corresponds to the nonlinear Schrödinger equation (6).

The model presented above is obtained for the fermions with the full spin polarization. Hydrodynamic model of degenerate spin-1/2 fermions with the partial spin polarization in the mean-field approximation for the interaction between fermions with different spin projections [41].

3. Large amplitude grey solitons

Let us present the analysis of equations for the solitons obtained up to the TOIR approximation. Let us consider solitons in uniform infinite medium. Hence, the symmetry of the system allows to consider the nonlinear waves with the plane wave front. The wave appears as the one dimensional solution. We consider the wave propagation in the arbitrary direction and choose the cartesian coordinates with axis Ox in the direction of the wave propagation. We seek the stationary solutions of the nonlinear equations. We consider the steady state in the comoving frame. Hence, all hydrodynamic functions depend on $\eta = x - ut$ and u , where the parameter u is the constant velocity of the nonlinear solution. The perturbations vanish at $\eta \rightarrow \pm\infty$.

3.1. One dimensional limit of hydrodynamic equations

For the nonlinear plane waves we have the following simplified continuity equation

$$-u\partial_\eta n + \partial_\eta(nv^x) = 0, \quad (9)$$

where the time derivative ∂_t is replaced by $-u\partial_\eta$ in accordance with the variable η introduced for the stationary solution. Similar simplification is made for the Euler

equation

$$\begin{aligned} -umn\partial_\eta v^x + mnv^x\partial_\eta v^x - \frac{\hbar^2}{2m} n\partial_\eta \frac{\partial_\eta^2 \sqrt{n}}{\sqrt{n}} \\ = -\frac{(6\pi^2)^{2/3} \hbar^2}{2m^2} n\partial_\eta n^{2/3} + g_2 \frac{4(6\pi^2)^{2/3}}{5} n\partial_\eta n^{5/3}, \end{aligned} \quad (10)$$

where the terms placed on the right-hand side are represented via construction $n\partial_\eta n^a$ useful for the further transformations, with a is the arbitrary degree.

The one dimensional continuity equation (9) can be integrated

$$n(v^x - u) = -un_0, \quad (11)$$

where the boundary conditions $n(\eta \rightarrow \pm\infty) = n_0$, and $v^x(\eta \rightarrow \pm\infty) = 0$ are used. Equation (11) allows to express the velocity field via the concentration

$$v^x = \frac{u(n - n_0)}{n}. \quad (12)$$

All terms in the Euler equation (10) are proportional to the concentration, so we can drop it. Next, the Euler equation can be integrated. As the result we find

$$\begin{aligned} m\left(\frac{1}{2}v_x^2 - uv^x\right) - \frac{\hbar^2}{2m} \frac{\partial_\eta^2 \sqrt{n}}{\sqrt{n}} + \frac{(6\pi^2)^{2/3} \hbar^2}{2m^2} n^{2/3} \\ - g_2 \frac{4(6\pi^2)^{2/3}}{5} n^{5/3} = \frac{(6\pi^2)^{2/3} \hbar^2}{2m^2} n_0^{2/3} - g_2 \frac{4(6\pi^2)^{2/3}}{5} n_0^{5/3}, \end{aligned} \quad (13)$$

where all terms existing in equation (10) are placed on the left-hand side while the right-hand side contains the result of application of the boundary conditions.

We substitute the velocity field (12) in the integrated Euler equation (13). Moreover, we see that equation (13) contains the second derivative on \sqrt{n} . It shows that we should find solution relatively \sqrt{n} . Equation (13) can be integrated to obtain the „energy integral“ in the following manner

$$\frac{1}{2}(\partial_\eta \sqrt{n})^2 + \tilde{V}_{eff}(\sqrt{n}) = 0, \quad (14)$$

where the first term can be considered as the effective kinetic energy of soliton, while $\tilde{V}_{eff}(\sqrt{n})$ is the effective potential energy called the Sagdeev potential [42–47]. The Sagdeev potential $\tilde{V}_{eff}(\sqrt{n})$ appears in the following form

$$\begin{aligned} \tilde{V}_{eff}(\sqrt{n}) = \frac{1}{2} (6\pi^2)^{2/3} \left(1 + \frac{8mg_2}{5\hbar^2} n_0\right) (n - n_0) \\ + \frac{m^2 u^2}{2\hbar^2} \left(n + \frac{n_0^2}{n} - 2n_0\right) - \frac{3}{10} (6\pi^2)^{2/3} \left(n^{5/3} + \frac{mg_2}{\hbar^2} n^{8/3}\right) \\ - \frac{3}{10} (6\pi^2)^{2/3} \left(n_0^{5/3} + \frac{mg_2}{\hbar^2} n_0^{8/3}\right). \end{aligned} \quad (15)$$

Equations can be solved for parameter \sqrt{n} , but the traditional form of the presentation of the results including the zero value of the effective potential and its first derivative

on parameter \sqrt{n} requires to consider dependence on $\sqrt{n} - \sqrt{n_0}$. Let us to choose the dimensionless form of the chosen parameter $\Delta \equiv (\sqrt{n} - \sqrt{n_0})/\sqrt{n_0}$ for the further analysis. Moreover, the coordinate in the comoving frame η can be presented in the dimensionless form as well $\xi = \eta n_0^{1/3}$.

3.2. Grey soliton in the mean-field approximation

The mean-field approximation corresponds to the FOIR limit of the hydrodynamic equations. Formally, it can be obtained from equations (14) and (15) at $g_2 = 0$. we discuss the Sagdeev potential in the dimensionless form.

The dimensionless form of the Sagdeev potential $V(\Delta) \equiv \tilde{V}_{eff}(\sqrt{n})n_0^{-5/3}$ presented in the mean-field approximation has the following form

$$V(\Delta) = \frac{1}{2} \alpha^2 \left[(\Delta + 1)^2 + \frac{1}{(\Delta + 1)^2} \right] + \frac{1}{2} (\Delta + 1)^2 - \frac{3}{10} (\Delta + 1)^{10/3} - \left[\alpha^2 + \frac{1}{5} \right], \quad (16)$$

where $\alpha \equiv u/v_{Fe}$. The dimensionless Sagdeev potential (16) is a part of the following dimensionless equation $(1/2)(\partial_\xi \Delta)^2 + V(\Delta) = 0$.

Fig. 1 shows the single illustration of the Sagdeev potential in the mean-field regime (16). Value $\Delta_0 \neq 0$ corresponding to $V(\Delta_0) = 0$ shows the amplitude of the soliton. First, we see that Δ_0 is negative. Hence, there is the decrease of concentration in the soliton $n < n_0$. However, Δ_0 does not reach value -1 . Consequently, the concentration of particles in the soliton is always nonzero $n > 0$. The soliton appears as the area of decreased concentration, which is above the zero value at the center of soliton. Such solitons are called the gray soliton. While the dark soliton is the limiting case of the grey soliton with the zero concentration in its center.

Fig. 2 shows that the increase of the speed of the soliton propagation up to the Fermi velocity decreases the amplitude of the soliton down to the zero value at $\alpha \approx 0.6$. Moreover, no solution exists at $\alpha > 0.6$. Obtained behavior shows similarity to the dark soliton in the BECs, where the speed of soliton propagation is limited by the Landau critical velocity [48].

Dimensionless velocity α is the single parameter in the mean-field approximation. This dependence is discussed. Further analysis of the properties of soliton can be made in the TOIR approximation.

Presented here soliton solution for the spin polarized fermions. The spin-0 BECs demonstrate two fundamental solitons in the mean-field regime. Moreover, the spin-0 BEC show the beyond mean-field bright soliton in the repulsive BEC regime. The boson-boson and boson-fermion mixtures show some additional soliton related effects. Particularly, the boson-fermion mixture of the spin-0 BEC and spin-polarized spin-1/2 fermions is considered in Ref. [49], where focus is

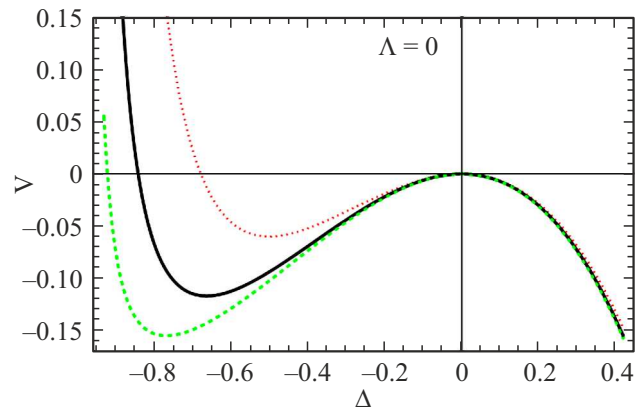


Figure 1. The Sagdeev potential $V = V(\Delta)$ (16) is demonstrated for the noninteracting limit for three different values of the dimensionless speed of the soliton $\alpha = 0.2$ (the upper red dotted line), $\alpha = 0.1$ (the middle black continuous line), $\alpha = 0.05$ (the lower green dashed line).

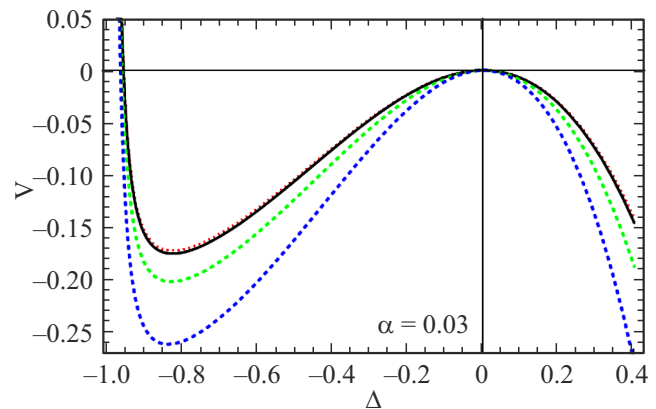


Figure 2. The Sagdeev potential $V = V(\Delta)$ (17) is demonstrated at the account of the interaction up to TOIR approximation. The Sagdeev potential $V = V(\Delta)$ is demonstrated for the fixed speed of perturbation $\alpha = 0.03$ for different values of the dimensionless interaction constant $\Lambda = 0$ (the upper red dotted line), $\Lambda = 0.01$ (the second from above black continuous line), $\Lambda = 0.1$ (the third from above green dashed line), $\Lambda = 0.3$ (the lower blue dashed line).

made on the modification of properties of the beyond mean-field bright soliton existing in the repulsive BECs under influence of the fermions. Hence, there is no direct relation between the grey soliton given here and the fermion part of the relation demonstrated in Ref. [49].

3.3. Generalization of the grey soliton solution up to the TOIR

Complete expression of the dimensionless form of the Sagdeev potential $V(\Delta) \equiv \tilde{V}_{eff}(\sqrt{n})n_0^{-5/3}$ obtained from the expression (15) derived up to the TOIR approximation can

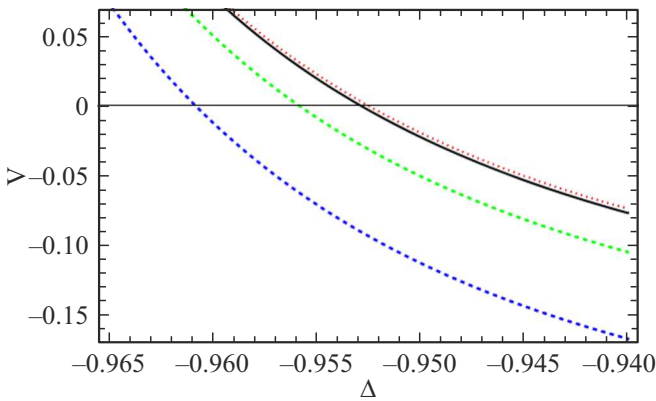


Figure 3. A part of the Sagdeev potential $V = V(\Delta)$ (17) is demonstrated. It is a part of Fig. 2 which corresponds to the point of crossing of the Sagdeev potential of the line of zero potential.

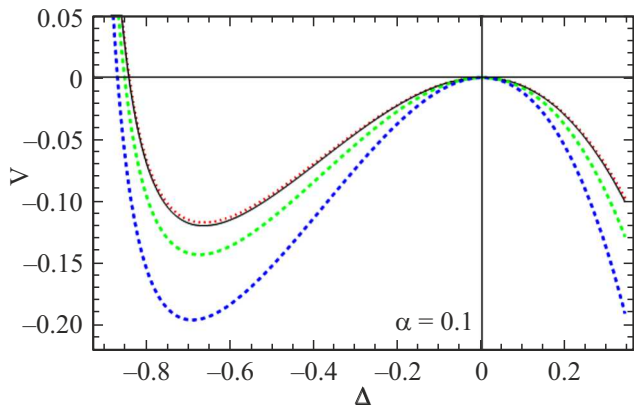


Figure 4. This figure is similar to Fig. 2, but this figure is obtained for the larger speed of the soliton. The Sagdeev potential $V = V(\Delta)$ (17) is demonstrated at the account of the interaction up to TOIR approximation. The Sagdeev potential $V = V(\Delta)$ is demonstrated for the fixed speed of perturbation $\alpha = 0.1$ for different values of the dimensionless interaction constant $\Lambda = 0$ (the upper red dotted line), $\Lambda = 0.01$ (the second from above black continuous line), $\Lambda = 0.1$ (the third from above green dashed line), $\Lambda = 0.3$ (the lower blue dashed line).

be written in the following form

$$\begin{aligned}
 V(\Delta) = & \frac{1}{2} \alpha^2 \left[(\Delta + 1)^2 + \frac{1}{(\Delta + 1)^2} \right] + \frac{1}{2} (\Delta + 1)^2 (1 + \Lambda) \\
 & - \frac{3}{10} (\Delta + 1)^{10/3} - \frac{3}{16} \Lambda (\Delta + 1)^{16/3} \\
 & - \left[\alpha^2 + \frac{1}{2} (1 + \Lambda) - \frac{3}{10} - \frac{3}{16} \Lambda \right]. \quad (17)
 \end{aligned}$$

Additional terms in compare with the FOIR approximation (16) are proportional to Λ .

Contribution of the short-range interaction obtained in the TOIR approximation in the Sagdeev potential is demonstrated in Figs. 2, 3, and 4. Figs. 2 and 3 are obtained for the relatively small speed of soliton $\alpha = 0.03$. In this case, there

is small modification of the amplitude of soliton under the change of the interaction constant. So, this modification is demonstrated in Fig. 3. Fig. 4 presents the Sagdeev potential for the same values of the interaction constant, but it is obtained for the larger velocity $\alpha = 0.1$. The contribution of interaction is larger in this velocity regime. Further increase of the velocity $\alpha \rightarrow 1$ gives large modification of the amplitude under influence of the interaction. This limit is not presented in figures since it is beyond the area of applicability of the model, which corresponds to the weak interaction regime.

4. Conclusion

Grey soliton has been found in the system of weakly interacting fermions being in quantum states with the same spin projection. It has been found from the quantum hydrodynamic equations corresponding to the nonlinear Schrödinger equation. The soliton has been found and studied within the Sagdeev potential method. The form of the Sagdeev potential allows to find the amplitude and width of the soliton. Particularly, it has been found that the concentration is decreased, but it does not reach the zero value. Thus, the soliton is classified as the grey soliton. The change of the grey soliton parameters at the modification of the speed of the soliton and the interaction constant has been analyzed. We have concluded that the polarized fermions demonstrate the existence of one kind of soliton, i.e. the soliton of the partial rarification called the grey soliton. It is in contrast with the Bose-Einstein condensate, where two kinds of solitons are possible for the uniform medium. The forms of solitons for the bosons depend on the sign of the interaction between the Bose atoms. The dark (bright) soliton corresponds to the repulsive (attractive) interaction. The degenerate fermions are mainly affected by the Fermi pressure which provides the effective repulsion. So the dark/grey solitons is the possible structure.

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Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study, which is a purely theoretical one.

Conflict of interest

The author declares that there is no conflict of interest.

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